Change-of-base for generalized multicategories

Rui Prezado
Universidade de Aveiro

joint work with F. Lucatelli Nunes

Second Virtual Workshop on Double Categories

30 October 2024

Original motivation

Theorem (Lucatelli Nunes 2018; Cottrell, Fujii, Power 2017)

For a suitable cartesian monoidal category $\mathcal{V}$, we have an adjunction

$$\mathcal{V}\text{-}\mathsf{Cat} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{Cat}(\mathcal{V})$$

whose left adjoint is fully faithful.

Enriched-internal dichotomy as change-of-base

We have

$$\mathsf{Set} \underset{\mathcal{V}(1,-)}{\overset{-*1}{\rightleftarrows}} \mathcal{V} \quad (\bot) \qquad \mapsto \qquad \mathcal{V}\text{-}\mathsf{Mat} \underset{\mathcal{V}(1,-)}{\overset{-*1}{\rightleftarrows}} \mathsf{Span}(\mathcal{V}) \quad (\bot)$$

which, in turn, induces the adjunction

$$\mathcal{V}\text{-}\mathsf{Cat} \underset{\mathcal{V}(1,-)}{\overset{-*1}{\rightleftarrows}} \mathsf{Cat}(\mathcal{V}) \quad (\bot)$$

Enriched-internal dichotomy as change-of-base

$$\mathcal{V}\text{-}\mathsf{Cat} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathsf{Cat}(\mathcal{V})$$

- Can we obtain a similar result for generalized multicategories?
- Can we also exhibit this as a change-of-base adjunction?

Vector spaces and multilinear maps

Let $V_1, \ldots, V_n, W$ be vector spaces.

A *multilinear map* $f\colon (V_1, \ldots, V_n) \to W$ consists of a function

$$f\colon V_1 \times \ldots \times V_n \to W$$

that is linear in each component:

$$f(v_1, \ldots, v_j + \lambda w, \ldots, v_n) = f(v_1, \ldots, v_n) + \lambda f(v_1, \ldots, w, \ldots, v_n)$$

Vector spaces and multilinear maps

Let $f\colon (V_1, \ldots, V_n) \to W$ be a multilinear map.

- If $n = 1$, f is a linear map.
- If we have a list of multilinear maps

  $$g_j\colon (U_{j1}, \ldots, U_{jk_j}) \to V_j,$$

  then the function

  $$f \circ (g_1 \times \ldots \times g_n)$$

  defines a multilinear map, which is denoted

  $$f \circ (g_1, \ldots, g_n)\colon (U_{11}, \ldots, U_{1k_1}, \ldots, U_{n1}, \ldots, U_{nk_n}) \to W$$

Multicategory **Vect**

The composition operation satisfies suitable associativity and identity laws.

Hence, vector spaces and multilinear maps form a *multicategory*.

Multicategories

Let X^* be the free monoid generated by a set X. A *multicategory* $\mathcal{M}$ consists of

- a set $\mathcal{M}_0$,
- a span

$$\begin{array}{ccccc} & & \mathcal{M}_1 & & \\ & \swarrow^{d_1} & & \searrow^{d_0} & \\ \mathcal{M}_0^* & & & & \mathcal{M}_0 \end{array}$$

- a function $s_0 \colon \mathcal{M}_0 \to \mathcal{M}_1$ (units),
- a function $d_1 \colon \mathcal{M}_2 \to \mathcal{M}_1$ (composition),

satisfying suitable associativity and identity properties, where

$$\begin{array}{ccc} \mathcal{M}_2 & \longrightarrow & \mathcal{M}_1 \\ \big\downarrow & \ulcorner & \big\downarrow d_1 \\ \mathcal{M}_1^* & \xrightarrow[d_0^*]{} & \mathcal{M}_0^* \end{array}$$

Generalized multicategories

Multicategories generalize categories by allowing the domain to be a finite list of objects.

By abstracting the “shape” of the domain of a morphism, we obtain the notion of *generalized multicategory*.

These “shapes” are modeled by suitable monads.

Topological spaces

Let $\mathcal{U}$ be the ultrafilter monad on Set.

Theorem (Barr, 1970)

A topological space X is characterized by

- its underlying set X,
- a convergence relation $\rightsquigarrow \subseteq \mathcal{U}(X) \times X$: we have $\mathfrak{x} \rightsquigarrow x$ whenever the ultrafilter $\mathfrak{x}$ on X converges to x.
- the convergence relation must satisfy suitable reflexivity and transitivity conditions.

Topological spaces

- Principal ultrafilters: $\mathfrak{p}\colon X \to \mathcal{U}(X)$.
- Flattening ultrafilters of ultrafilters: $\mathfrak{m}\colon \mathcal{U}\mathcal{U}(X) \to \mathcal{U}(X)$.

- Reflexivity: $\mathfrak{p}(x) \rightsquigarrow x$.
- Transitivity: If $\mathfrak{X} \rightsquigarrow^{*} \mathfrak{x}$ and $\mathfrak{x} \rightsquigarrow x$, then $\mathfrak{m}(\mathfrak{X}) \rightsquigarrow x$.

Thus, a topological space is a generalized ordered set.

$$\mathsf{Top} \simeq (\mathcal{U}, 2)\text{-}\mathsf{Cat}$$

Settings for generalized multicategories

- $\mathcal{V}$ category with finite limits.
- Cartesian monad $T = (T, m, e)$ on $\mathcal{V}$.

$$\begin{array}{ccc} TT(x) & \xrightarrow{m_x} & T(x) \\ {\scriptstyle TT(f)}\downarrow & & \downarrow{\scriptstyle T(f)} \\ TT(y) & \xrightarrow[m_y]{} & T(y) \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{e_x} & T(x) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle T(f)} \\ y & \xrightarrow[e_y]{} & T(y) \end{array}$$

Internal T-categories (Burroni 1971, Hermida 2000)

T induces a suitable monad on $\mathsf{Span}(\mathcal{V})$.

The category $\mathsf{Cat}(T, \mathcal{V})$ of internal T-categories is given by a suitable notion of *lax algebras* over T.

Settings for generalized multicategories

- $\mathcal{V}$ distributive monoidal category.
- A suitable lax monad T on $\mathcal{V}$-Mat.

Enriched T-categories (Clementino, Tholen 2003)

The category $(T, \mathcal{V})$-Cat of enriched T-categories is given by a suitable notion of *lax algebras* over T.

Settings for generalized multicategories

Definition (Cruttwell, Shulman 2010)

Let VDbCat be the 2-category of

- virtual double categories,
- functors of virtual double categories,
- vertical transformations between such functors.

A *setting* for generalized multicategories is a monad in VDbCat.

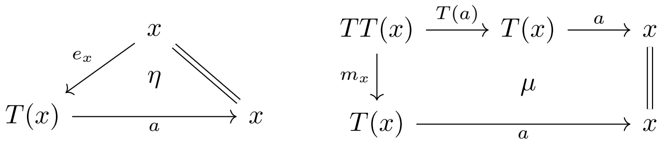
Horizontal lax algebras

Let $T = (T, m, e)$ be a monad on a virtual double category $\mathbb{V}$.

T-monoids (Cruttwell, Shulman 2010)

A *horizontal lax T-algebra* consists of a quadruple

- an object x of $\mathbb{V}$,
- a horizontal morphism $a\colon T(x) \to x$,
- a unit 2-cell η and a multiplication 2-cell μ given by



satisfying suitable identity and associativity conditions.

The problem with of change-of-base

Monad morphisms (Street, 1972)

Let

- T be a monad on $\mathbb{V}$,
- S be a monad on $\mathbb{W}$.

A *monad morphism* $T \to S$ consists of:

- A functor $F\colon \mathbb{V} \to \mathbb{W}$,
- A vertical transformation $\phi\colon SF \to FT$

satisfying suitable properties.

The problem with of change-of-base

Let $(F, \phi)\colon T \to S$ be a monad morphism.

If (x, a, η, μ) is a horizontal lax T-algebra, how do we obtain the horizontal lax S-algebra induced by (F, ϕ)?

$$SF(x) \xrightarrow{\ ?\ } F(x)$$

$$\begin{array}{ccc} SF(x) & & \\ \downarrow{\scriptstyle \phi_x} & & \\ FT(x) & \xrightarrow[F(a)]{} & F(x) \end{array}$$

The solution we propose:

- "Flip" ϕ into a "horizontal transformation".
- Have horizontal composites.

Conjoints and companions

Let $\mathbb{D}$ be a pseudodouble category.

A *conjoint* of a vertical morphism $f\colon x \to y$ in $\mathbb{D}$ consists of

- A horizontal morphism $r\colon y \to x$ in $\mathbb{D}$,
- Unit and counit 2-cells in $\mathbb{D}$

$$\begin{array}{ccc} x & \xrightarrow{1_x} & x \\ {\scriptstyle f}\downarrow & \eta & \Vert \\ y & \xrightarrow[r]{} & x \end{array} \qquad\qquad \begin{array}{ccc} y & \xrightarrow{r} & x \\ \Vert & \epsilon & \downarrow{\scriptstyle f} \\ y & \xrightarrow[1_y]{} & y \end{array}$$

- Satisfying $\epsilon \circ \eta = 1_f$ and $\eta \cdot \epsilon \cong \mathsf{id}_r$.

We write $r = f^*$ and $f \dashv f^*$.

Companions are defined dually (vertical or horizontal), and we write $f_! \dashv f$.

Conjoints and companions

Lemma

If $\mathbb{D}, \mathbb{E}$ are pseudodouble categories, then we have two pseudodouble categories

$$\mathsf{Lax}_{\mathsf{lax}}(\mathbb{D}, \mathbb{E}) \qquad \mathsf{Lax}_{\mathsf{opl}}(\mathbb{D}, \mathbb{E})$$

that have

- lax functors as objects,
- vertical transformations as vertical morphisms,
- lax (respectively, oplax) horizontal transformations as horizontal morphisms,
- generalized modifications as 2-cells.

Conjoints and companions

A pseudodouble category $\mathbb{D}$ is *conjoint-closed* if every vertical morphism has a conjoint. Dually, we have *companion-closed* pseudodouble categories.

Theorem (P., Lucatelli Nunes 2023)

Let $\mathbb{D}$ and $\mathbb{E}$ be pseudodouble categories.

- If $\mathbb{E}$ is conjoint-closed, then $\mathsf{Lax}_{\mathsf{lax}}(\mathbb{D}, \mathbb{E})$ is conjoint-closed.
- If $\mathbb{E}$ is companion-closed, then $\mathsf{Lax}_{\mathsf{opl}}(\mathbb{D}, \mathbb{E})$ is companion-closed.

Conjoints and companions

Hence, if

- $\mathbb{E}$ is an equipment,
- $F, G \colon \mathbb{D} \to \mathbb{E}$ are lax functors,
- $\phi \colon F \to G$ is a vertical transformation,

then we have

- a lax horizontal transformation $\phi^* \colon G \to F$, with $\phi \dashv \phi^*$,
- an oplax horizontal transformation $\phi_! \colon F \to G$, with $\phi_! \dashv \phi$.

We say ϕ has a *strong* conjoint/companion if $\phi^*/\phi_!$ is a strong horizontal transformation.

Change-of-base for horizontal lax algebras

- Ambient 2-category: $\mathsf{Equip}_{\mathsf{lax}}$ of equipments, lax functors, vertical transformations.
- We let T, S be monads on the respective equipments $\mathbb{D}$, $\mathbb{E}$.
- We let $(F, \phi)\colon T \to S$ be a monad morphism, such that ϕ and $T(\phi)$ have strong conjoints.
- We let $(G, \psi)\colon S \to T$ be a monad opmorphism.

Let $\mathbb{H}\mathsf{LaxAlg}(T)$ be the category of horizontal lax T-algebras and respective morphisms.

Theorem (P., Lucatelli Nunes 2023)

The correspondences

$$(x, a, \eta, \mu) \mapsto (F(x), F(a) \cdot \phi_x^*, F_!(\eta), F_!(\mu)) \qquad (y, b, \eta, \mu) \mapsto (G(y), G(b) \cdot \psi_{!y}, G_!(\eta), G_!(\mu))$$

define functors

$$(F, \phi)_! \colon \mathbb{H}\mathsf{LaxAlg}(T) \to \mathbb{H}\mathsf{LaxAlg}(S), \qquad (G, \psi)^* \colon \mathbb{H}\mathsf{LaxAlg}(S) \to \mathbb{H}\mathsf{LaxAlg}(T).$$

Back to our motivation

Let $\mathcal{V}$ be a suitable category, let T be a cartesian monad on $\mathcal{V}$.

Can we obtain an adjunction

$$(?,\mathcal{V})\text{-}\mathsf{Cat} \underset{\mathcal{V}(1,-)}{\overset{-\cdot 1}{\rightleftarrows}} \mathsf{Cat}(T,\mathcal{V}) \qquad \bot$$

with our tools?

Induced lax monad

Let $\mathcal{V}$ be lextensive, T cartesian monad on $\mathcal{V}$.

$$\mathsf{Set} \underset{\mathcal{V}(1,-)}{\overset{-\cdot 1}{\underset{\bot}{\rightleftarrows}}} \mathcal{V} \qquad \text{(in } \mathsf{Cat}\text{)}$$

$$\mathcal{V}\text{-}\mathsf{Mat} \underset{\mathcal{V}(1,-)}{\overset{-\cdot 1}{\underset{\bot}{\rightleftarrows}}} \mathsf{Span}(\mathcal{V}) \qquad \text{(in } \mathsf{Equip}_{\mathsf{lax}}\text{)}$$

Induced lax monad

Let $\mathcal{V}$ be lextensive, T cartesian monad on $\mathcal{V}$.

$$\overline{T} \circlearrowright \mathsf{Set} \underset{\mathcal{V}(1,-)}{\overset{-\cdot 1}{\underset{\perp}{\rightleftarrows}}} \mathcal{V} \circlearrowleft T \qquad \text{(in } \mathsf{Cat}\text{)}$$

$$\overline{T} \circlearrowright \mathcal{V}\text{-}\mathsf{Mat} \underset{\mathcal{V}(1,-)}{\overset{-\cdot 1}{\underset{\perp}{\rightleftarrows}}} \mathsf{Span}(\mathcal{V}) \circlearrowleft T \qquad \text{(in } \mathsf{Equip}_{\mathsf{lax}}\text{)}$$

Generalized enriched-internal dichotomy

Theorem (P., Lucatelli Nunes 2023)

Under suitable* conditions, we have an adjunction

$$(\overline{T}, \mathcal{V})\text{-}\mathsf{Cat} \underset{\mathcal{V}(1,-)}{\overset{-\cdot 1}{\rightleftarrows}} \mathsf{Cat}(T, \mathcal{V}) \qquad (\bot)$$

Theorem (P., Lucatelli Nunes 2023)

If $- \cdot 1 \colon \mathsf{Set} \to \mathcal{V}$ is fully faithful, then so is $- \cdot 1 \colon (\overline{T}, \mathcal{V})\text{-}\mathsf{Cat} \to \mathsf{Cat}(T, \mathcal{V})$.

Thank you!