DOUBLE CATEGORIES
VERSUS
FACTORIZATION SYSTEMS

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SECOND VIRTUAL WORKSHOP
ON DOUBLE CATEGORIES

October 21, 2024

PLAN OF THE TALK

(I)
- DOUBLE CATEGORIES
- DBL CATS $\leftrightsquigarrow$ FACT. SYSTEMS

(II)
- CODESCENT OBJECTS
- LAX MORPHISM CLASSIFIERS
- PINWHEELS

DEF) A **DOUBLE CATEGORY** X CONSISTS OF

- OBJECTS $a, b, \dots$
- VERTICAL MORPHISMS $$\begin{tikzcd} a \arrow[d, "u"] \\ c \end{tikzcd}$$
- HORIZONTAL MORPHISMS $a \xrightarrow{g} b$
- SQUARES: $$\begin{tikzcd} a \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[d, "v"] \\ c \arrow[r, "h"'] & d \end{tikzcd}$$

HAVE

- HORIZONTAL & VERTICAL COMP.
- HORIZONTAL & VERTICAL IDENTITIES

$$\begin{tikzcd} a \arrow[d, equal] \\ a \end{tikzcd} \qquad a = a \qquad \begin{tikzcd} a \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & b \arrow[d, equal] \\ a \arrow[r, "h"'] & b \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow"] & a \arrow[d, "u"] \\ c \arrow[r, equal] & c \end{tikzcd}$$

MOREOVER, THE MIDDLE-FOUR INTERCHANGE HOLDS:

$$\begin{tikzcd}
a \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[r, "g'"] \arrow[d, "v"] \arrow[dr, phantom, "\Downarrow \beta"] & b' \arrow[d, "v'"] \\
c \arrow[r, "h"'] \arrow[d, "\tilde{u}"'] \arrow[dr, phantom, "\Downarrow \tilde{\alpha}"] & d \arrow[r, "h'"'] \arrow[d, "\tilde{v}"] \arrow[dr, phantom, "\Downarrow \tilde{\beta}"] & d' \arrow[d, "\tilde{v}'"] \\
\tilde{c} \arrow[r, "\tilde{h}"'] & \tilde{d} \arrow[r, "\tilde{h}'"'] & \tilde{d}'
\end{tikzcd}$$

HAS UNIQUE COMPOSITE

SPECIAL CASE: IF $\mathrm{ob}\,X$, $\mathrm{vmor}\,X$, $\mathrm{hmor}\,X \cong *$

X IS COMMUTATIVE MONOID

(ECKMANN-HILTON ARGUMENT)

SPECIAL CASE: A CATEGORY $\mathcal{C}$:

OB: $\mathrm{ob}\,\mathcal{C}$

HMOR: $\mathrm{mor}\,\mathcal{C}$

VMOR: IDENTITIES

SQS: IDENTITIES

SPECIAL CASE: A 2-CATEGORY $\mathcal{K}$:

OB: $\mathrm{ob}\,\mathcal{K}$

HMOR: $\mathrm{mor}\,\mathcal{K}$

VMOR: 1_a, $a \in \mathrm{ob}\,\mathcal{C}$

SQS:

$$\begin{tikzcd}
a \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[d, equal] \\
a \arrow[r, "h"'] & b
\end{tikzcd}$$

$:=$ 2-CELL $\alpha : g \Rightarrow h$ IN $\mathcal{K}$

DEF) A DOUBLE FUNCTOR $F : X \longrightarrow Y$

IS AN ASSIGNMENT

$$X \ni \begin{tikzcd} a \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[d, "v"] \\ c \arrow[r, "h"'] & d \end{tikzcd} \mapsto \begin{tikzcd} Fa \arrow[r, "Fg"] \arrow[d, "Fu"'] \arrow[dr, phantom, "\Downarrow F\alpha"] & Fb \arrow[d, "Fv"] \\ Fc \arrow[r, "Fh"'] & Fd \end{tikzcd} \in Y$$

PRESERVING ALL COMP, ALL IDS

$\Longrightarrow$ HAVE A CATEGORY Dbl

- Cat, 2-Cat EMBED* IN Dbl

* IN MANY WAYS

DOUBLE CATEGORY OF COMMUTATIVE SQS

EXAMPLE LET $\mathcal{C}$ CATEGORY. $\boxed{Sq(\mathcal{C})}$

S.T. OBJECTS : $\mathrm{ob}\,\mathcal{C}$

HMORS : $\mathrm{mor}\,\mathcal{C}$

VMORS : $\mathrm{mor}\,\mathcal{C}$

SQ : $\exists$

$$\begin{tikzcd} a \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[d, "v"] \\ c \arrow[r, "h"'] & d \end{tikzcd} \iff \begin{tikzcd} a \arrow[r, "g"] \arrow[d, "u"'] & b \arrow[d, "v"] \\ c \arrow[r, "h"'] & d \end{tikzcd}$$

COMMUTES IN $\mathcal{C}$

EXAMPLE $\boxed{PbSq(\mathcal{C})} \subseteq Sq(\mathcal{C})$

- SUB-DOUBLE CAT'RY SPANNED BY PULLBACK SQUARES

DOUBLE CATEGORY OF COMMUTATIVE SQS

QUESTION: GIVEN $\mathcal{C}$, WHAT PROPERTIES DOES $Sq(\mathcal{C})$ HAVE?

- IS FLAT:

IF
$$\begin{tikzcd} a \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[d, "v"] \\ c \arrow[r, "h"'] & d \end{tikzcd} \quad \begin{tikzcd} a \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow \beta"] & b \arrow[d, "v"] \\ c \arrow[r, "h"'] & d \end{tikzcd}$$
THEN $\alpha = \beta$

- IS INVARIANT:

$\forall$
$$\begin{tikzcd} a \arrow[d, "\theta"', "\cong"] & b \arrow[d, "\tau", "\cong"'] \\ c \arrow[r, "h"'] & d \end{tikzcd}$$
$\exists!$
$$\begin{tikzcd} a \arrow[r, "\tilde{h}"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Downarrow \lambda"] & b \arrow[d, "\tau"] \\ c \arrow[r, "h"'] & d \end{tikzcd}$$

→ HERE $\tilde{h} := \tau^{-1} h \theta$

DOUBLE CATEGORY OF COMMUTATIVE SQS

QUESTION: WHAT PROPERTIES DISTINGUISH A PULLBACK SQUARE FROM OTHER SQUARES IN $Sq(\mathcal{C})$?

SAY SQUARE λ IN DOUBLE CAT'RY $\mathbb{X}$ IS (BOT-RIGHT) BICARTESIAN IF $\forall \alpha$ $\exists! \rho$ S.T.

$$\begin{tikzcd}
x \arrow[r, "k"] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[d, "v"] \\
c \arrow[r, "h"'] & d
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, dashed, "\dot{\theta}"] \arrow[d, dashed, "\theta"'] \arrow[rr, bend left, "k"] \arrow[dd, bend right, "l"'] \arrow[dr, phantom, "\Downarrow \rho"] & a \arrow[r, "g"] \arrow[d, equal] & b \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, "u"'] & a \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow \lambda"] & b \arrow[d, "v"] \\
c \arrow[r, equal] & c \arrow[r, "h"'] & d
\end{tikzcd}$$

DOUBLE CATEGORY OF COMMUTATIVE SQS

QUESTION: WHAT PROPERTIES DISTINGUISH A PULLBACK SQUARE FROM OTHER SQUARES IN $Sq(\mathcal{C})$?

IN $Sq(\mathcal{C})$ THIS MEANS: $\exists!\theta$ S.T.

$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "l"'] & b \arrow[d, "v"] \\
c \arrow[r, "h"'] & d
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
X \arrow[r, dashed, "\theta"] \arrow[d, dashed, "\theta"'] \arrow[rr, bend left=40, "k", "\shortparallel"'] \arrow[dd, bend right=60, "l"', "="] & a \arrow[r, "g"] \arrow[d, equal] & b \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, "u"'] & a \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & b \arrow[d, "v"] \\
c \arrow[r, equal] & c \arrow[r, "h"'] & d
\end{tikzcd}$$

DOUBLE CATEGORY OF COMMUTATIVE SQS

WE KNOW THAT PULLBACK PROJECTIONS ARE JOINTLY MONIC:

$$\begin{tikzcd} a \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & b \arrow[d, "v"] \\ c \arrow[r, "h"'] & d \end{tikzcd}$$

$\forall\ \theta, \tau : x \longrightarrow a$

$\pi_1 \tau = \pi_1 \theta$ & $\pi_2 \tau = \pi_2 \theta$

$\Longrightarrow \tau = \theta$

QUESTION: HOW TO SAY DOUBLE CAT'LLY?

SAY (π_1, π_2) IS JOINTLY MONIC IN X IF

$\forall$

$$\begin{tikzcd} x \arrow[r, "\theta^\bullet"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Downarrow k_1"] & a \arrow[d, equal] \\ a \arrow[r, equal] & a \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\tau^\bullet"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\Downarrow k_2"] & a \arrow[d, equal] \\ a \arrow[r, equal] & a \end{tikzcd}$$

IF $\pi_1 \tau = \pi_1 \theta$

$\pi_2 \tau^\bullet = \pi_2 \theta^\bullet;$

THEN $\tau = \theta$

$\tau^\bullet = \theta^\bullet$

Strict factorization systems

VERSUS

Double categories

DEF) **STRICT FACT SYSTEM** $(\mathcal{E}, \mathcal{M})$ ON $\mathcal{C}$

$$\forall f \in \mathrm{mor}\,\mathcal{C} \quad \exists!\, e \in \mathcal{E} \quad \exists!\, m \in \mathcal{M}$$

$$\text{W/ } f = me$$

DEF) LET $\begin{cases} (\mathcal{E}', \mathcal{M}') \text{ SFS ON } \mathcal{C}' \\ (\mathcal{E}'', \mathcal{M}'') \text{ SFS ON } \mathcal{C}'' \end{cases}$

MORPHISM OF SFS' $(\mathcal{E}', \mathcal{M}') \longrightarrow (\mathcal{E}'', \mathcal{M}'')$ IS FUNCTOR $F : \mathcal{C}' \to \mathcal{C}''$

S.T. $\begin{cases} F\mathcal{E}' \subseteq \mathcal{E}'' \\ F\mathcal{M}' \subseteq \mathcal{M}'' \end{cases}$

$\rightsquigarrow$ HAVE CATEGORY **$\mathcal{SFS}$**

SPECIAL CASE: IF $\mathcal{C}$ MONOID, THIS IS ZAPPA-SZÉP PRODUCT

EXAMPLE $\mathcal{C} = GL_n(\mathbb{R})$

$\mathcal{E} = \{$ UPPER-TRIANGULAR MAT W/ >0 DIAGONAL $\}$

$\mathcal{M} = \{$ ORTHOGONAL MATRICES $\}$

(QR DECOMPOSITION)

EXAMPLE $\mathcal{C} = \mathcal{A} \times \mathcal{B}$

$\mathcal{E} = \{ (f, 1_b) \mid f \in \mathrm{mor}\,\mathcal{A},\ b \in \mathrm{ob}\,\mathcal{B} \}$

RARE ↗

$\mathcal{M} = \{ (1_a, g) \mid a \in \mathrm{ob}\,\mathcal{B},\ g \in \mathrm{mor}\,\mathcal{A} \}$

$$(f, g) = (1, g) \circ (f, 1)$$

EXAMPLE $\mathcal{C} = \mathbf{Set}$

$\mathcal{E} = \{$ SURJECTIONS $\}$

$\mathcal{M} = \{$ SUBSET INCLUSIONS $\}$

DIGRESSION — SOME FACTS:

FACT: IN SFS $(\mathcal{E}, \mathcal{M})$, NECESSARILY

$\mathcal{E} \cap \mathcal{M} = \{$ IDENTITIES OF $\mathcal{C} \}$

DIGRESSION CONTINUED

FACT: IF WE IDENTIFY

CATEGORIES $\longleftrightarrow$ MONADS IN $\mathrm{Span}(\mathrm{Set})$

THEN:

STRICT FS' $\longleftrightarrow$ DISTRIBUTIVE LAWS

FACT: THEY ARE STRICT ALGEBRAS FOR THE SQUARING 2-MONAD

$$\mathcal{C} \mapsto \mathcal{C}^{2} = \mathrm{Cat}(2, \mathcal{C})$$

FS ⇝ DBL

CONTAIN ALL OBJECTS OF $\mathcal{C}$

ASSUME HAVE CAT'RY $\mathcal{C}$ & TWO WIDE SUBCATEGORIES $\mathcal{E}, \mathcal{M} \subseteq \mathcal{C}$

CONSTRUCT A DOUBLE CAT'RY $\mathbb{D}_{\mathcal{E},\mathcal{M}}$

S.T. **OBJECTS:** ob $\mathcal{C}$

HMORS: mor $\mathcal{M}$

VMORS: mor $\mathcal{E}$

SQUARES: COMMUTATIVE SQUARES

$$\exists \quad \begin{tikzcd} a \arrow[r, "m"] \arrow[d, "e"'] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[d, "e'"] \\ c \arrow[r, "m'"'] & d \end{tikzcd} \iff \begin{tikzcd} a \arrow[r, "m"] \arrow[d, "e"'] \arrow[dr, phantom, "//"] & b \arrow[d, "e'"] \\ c \arrow[r, "m'"'] & d \end{tikzcd} \text{ COMMUTES IN } \mathcal{C}$$

FS ⇝ DBL

GIVEN SFS $(\mathcal{E},\mathcal{M})$ ON $\mathcal{C}$.

QUESTION: WHAT PROPERTIES DOES $D_{\mathcal{E},\mathcal{M}}$ HAVE?

$\forall m, e:$

$$\begin{tikzcd}
a \arrow[r, "m"] \arrow[d, dashed, "e'"'] \arrow[dr, phantom, "\Downarrow \exists!"] & b \arrow[d, "e"] \\
\bullet \arrow[r, dashed, "m'"'] & c
\end{tikzcd}$$

HERE (e',m') IS $(\mathcal{E},\mathcal{M})$-FACTORIZATION OF $e \circ m \in \mathcal{C}$

- THIS PROPERTY FULLY CHARACTERIZES $X \in \mathrm{Dbl}$ THAT ARE OF FORM $D_{\mathcal{E},\mathcal{M}}$ FOR A SFS $(\mathcal{E},\mathcal{M})$ ON $\mathcal{C}$

FS $\rightsquigarrow$ DBL

CALL A DOUBLE CAT X **CODOMAIN-DISCRETE**

IF $\forall$

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[d, "u"] \\ & c \end{tikzcd}$$

$\exists !$

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[d, "u"] \\ \bullet \arrow[r] & c \end{tikzcd}$$

(THE CODOMAIN FUNCTOR $d_0 : X_1 \longrightarrow X_0$ IS A DISCRETE OPFIBRATION)

DENOTE **CoDDiscr** $\subseteq_{FULL}$ Dbl

FACT: $D : SFS \longrightarrow$ CoDDiscr IS AN EQUIVALENCE OF CATS

FS $\leftsquigarrow$ DBL

GIVEN A CODOMAIN-DISCRETE DOUBLE CAT X CONSTRUCT Cnr(X) W/ OB: obX

MOR:
$$\begin{tikzcd} a \arrow[d, "u"'] & \\ b \arrow[r, "g"'] & c \end{tikzcd}$$

IDS:
$$\begin{tikzcd} a \arrow[d, equal] & \\ a \arrow[r, equal] & a \end{tikzcd}$$

COMPOSITION: $(v, h) \circ (u, g) = (\tilde{v} \circ u, h \circ \tilde{g})$

$$\begin{tikzcd} a \arrow[d, "u"'] & & \\ b \arrow[r, "g"] & c \arrow[d, "v"] & \\ & d \arrow[r, "h"] & e \end{tikzcd} \rightsquigarrow \begin{tikzcd} a \arrow[d, "u"'] & & \\ b \arrow[r, "g"] \arrow[d, "\tilde{v}"'] \arrow[dr, phantom, "\Downarrow \exists!"] & c \arrow[d, "v"] & \\ x \arrow[r, "\tilde{g}"'] & d \arrow[r, "h"] & e \end{tikzcd}$$

PROPERTIES OF Cnr(X) (1/2):

① Cnr(X) ADMITS A STRICT FS $(\mathcal{E}_X , \mathcal{M}_X)$:

$$\left\{ \begin{tikzcd} a \arrow[d, "u"'] & \\ b \arrow[r, equal] & b \end{tikzcd} \;\middle|\; u \text{ VMOR} \right\} \qquad \left\{ \begin{tikzcd} a \arrow[d, equal] & \\ a \arrow[r, "g"] & b \end{tikzcd} \;\middle|\; g \text{ HMOR} \right\}$$

PF:

$$\begin{tikzcd} a \arrow[d, "u"'] & \\ b \arrow[r, "g"'] & c \end{tikzcd} \quad = \quad \begin{tikzcd} a \arrow[d, "u"'] & & \\ b \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\exists!"] & b \arrow[d, equal] & \\ b \arrow[r, equal] & b \arrow[r, "g"] & c \end{tikzcd}$$

$\square$

PROPERTIES OF Chr(X) (2/2):

② SQUARES IN X „COMMUTE" IN Chr(X):

$$\begin{tikzcd}
a \arrow[r, "m"] \arrow[d, "e"'] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[d, "e'"] \\
c \arrow[r, "m'"'] & d
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
a \arrow[r, "{(1,m)}"] \arrow[d, "{(e,1)}"'] \arrow[dr, phantom, "//"] & b \arrow[d, "{(e',1)}"] \\
c \arrow[r, "{(1,m')}"'] & d
\end{tikzcd}$$

IN Chr(X)

& Chr(X) IS „UNIVERSAL" W/ THIS PROPERTY

PF:

$$\begin{tikzcd}
a \arrow[d, equal] & & \\
a \arrow[r, "m"] & b \arrow[d, "e'"] & \\
& d \arrow[r, equal] & d
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
a \arrow[d, equal] & & \\
a \arrow[r, "m"] \arrow[d, "e"'] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[d, "e'"] & \\
c \arrow[r, "m"'] & d \arrow[r, equal] & d
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
a \arrow[d, "e"'] & \\
c \arrow[r, "m"'] & d
\end{tikzcd}$$

□

FS ⟷ DBL

HAVE A FUNCTOR $\mathrm{Cnr} : \mathrm{CoDDiscr} \to \mathrm{SFS}$

$$X \mapsto (\mathcal{E}_X, \mathcal{M}_X)$$

THEOREM THE FUNCTOR D IS AN EQUIVALENCE W/ EQUIV. INVERSE Cnr

$$\begin{tikzcd}
\mathrm{SFS} \arrow[r, bend right, "D"'] \arrow[r, phantom, "\simeq"] & \mathrm{CoDDiscr} \arrow[l, bend right, "\mathrm{Cnr}"']
\end{tikzcd}$$

STRICT FACT. SYSTEMS $\simeq$ CODOMAIN-DISCR DOUBLE CATEGORIES

Orthogonal factorization systems

VERSUS

Double categories

DEF) AN ORTHOGONAL FACT. SYSTEM ON $\mathcal{C}$ IS TWO WIDE SUBCATS $\mathcal{E}, \mathcal{M}$

S.T. ① $\forall f \in \mathrm{mor}\,\mathcal{C}$ $\exists e \in \mathcal{E}$ $\exists m \in \mathcal{M}$:

w/ $f = me$

AND THIS FACT IS UNIQUE UP TO ISO:

$$\begin{tikzcd}
 & b \arrow[dd, dashed, "\exists!"] \arrow[dr, "m"] & \\
a \arrow[ur, "e"] \arrow[dr, "e'"'] & & c \\
 & b' \arrow[ur, "m'"'] &
\end{tikzcd}$$

② $\mathcal{E} \cap \mathcal{M} = \{$ ISOMORPHISMS OF $\mathcal{C} \}$

⇝ AGAIN HAVE CAT'RY OFS

EXAMPLE Set, $\mathcal{E} = \{\text{SURJECTIONS}\}$

$\mathcal{M} = \{\text{INJECTIONS}\}$

FACT: ANY STRICT FS $(\mathcal{E}, \mathcal{M})$ ON $\mathcal{C}$ INDUCES ORTHOG. FS $(\tilde{\mathcal{E}}, \tilde{\mathcal{M}})$ ON $\mathcal{C}$

IF WE PUT:

$$\tilde{\mathcal{E}} := \{ ie \mid e \in \mathcal{E}, i \in \mathcal{C} \text{ ISO} \}$$

$$\tilde{\mathcal{M}} := \{ mi \mid m \in \mathcal{M}, i \in \mathcal{C} \text{ ISO} \}$$

FS $\rightsquigarrow$ DBL

LET $(\mathcal{E}, \mathcal{M})$ BE AN ORTHOG. FS ON $\mathcal{C}$.

QUESTION: WHAT PROPERTIES DOES $\mathbb{D}_{\mathcal{E},\mathcal{M}}$ HAVE?

① EVERY

$$\begin{tikzcd}
a \arrow[r, "m"] & b \arrow[d, "e"] \\
 & d
\end{tikzcd}$$

CAN BE FILLED.

$$\begin{tikzcd}
a \arrow[r, "m"] \arrow[d, dashed, "e'"'] \arrow[dr, phantom, "\Downarrow"] & b \arrow[d, "e"] \\
\cdot \arrow[r, dashed, "m'"'] & d
\end{tikzcd}$$

HERE (e', m') IS THE $(\mathcal{E}, \mathcal{M})$-FACTORIZATION OF $e \circ m \in \mathcal{C}$

→ NO LONGER UNIQUE!

FS ⇝ DBL

② EVERY SQUARE IN $\mathbb{D}_{E,M}$ IS (TOP-RIGHT) BICARTESIAN: $\forall \alpha \ \exists! \rho$:

$$\begin{tikzcd} a \arrow[r, "m"] \arrow[d, "\tilde{e}"'] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[d, "e"] \\ x \arrow[r, "\tilde{m}"'] & d \end{tikzcd} = \begin{tikzcd} a \arrow[r, equal] \arrow[d, "e'"'] & a \arrow[r, "m"] \arrow[d, "e'"'] \arrow[dr, phantom, "\Downarrow \lambda"] & b \arrow[d, "e"] \\ c \arrow[r, equal] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Downarrow \rho"] & c \arrow[r, "m'"'] \arrow[d, equal] & d \arrow[d, equal] \\ x \arrow[r, "\theta^{\bullet}"'] & c \arrow[r, "m'"'] & d \end{tikzcd}$$

TRANSLATES TO (SINCE $\theta^{\bullet} = \theta^{-1}$):

$$\begin{tikzcd} & c \arrow[dr, "m'"] \arrow[dd, dashed, "\exists! \theta"] & \\ a \arrow[ur, "e'"] \arrow[dr, "\tilde{e}"'] & & d \\ & x \arrow[ur, "\tilde{m}"'] & \end{tikzcd}$$

INTUITION:

"VERTICALLY OPPOSITE PULLBACK SQS"

FS $\rightsquigarrow$ DBL

③ $D_{\mathcal{E},\mathcal{M}}$ IS INVARIANT:

$\forall \theta, m, \tau$:

$$\begin{tikzcd}
a \arrow[d, "\cong\theta"'] & b \arrow[d, "\tau\cong"] \\
c \arrow[r, "m"'] & d
\end{tikzcd}$$

$\exists! \alpha$:

$$\begin{tikzcd}
a \arrow[r, dashed, "⊛"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Downarrow\alpha"] & b \arrow[d, "\tau"] \\
c \arrow[r, "m"'] & d
\end{tikzcd}$$

⊛ MUST EQUAL $\tau^{-1} \circ m \circ \theta \in \mathrm{mor}\,\mathcal{E}$

DOES BELONG TO $\mathcal{M}$?
YES SINCE τ^{-1}, θ, m DO.

FS $\rightsquigarrow$ DBL

VERTICAL DUAL

④ EVERY $\begin{tikzcd}[cramped] \cdot \arrow[r] \arrow[d] & \cdot \\ \cdot & \end{tikzcd}$ CORNER IN $D^{\sim}_{E,M}$

IS JOINTLY MONIC.

– FOLLOWS FROM THE FACT THAT

$$\begin{tikzcd}
 & c \arrow[rd, "m'"] \arrow[dd, dashed] & \\
a \arrow[ru, "e'"] \arrow[rd, "\tilde{e}"'] & \circledast & d \\
 & x \arrow[ru, "\tilde{m}"'] &
\end{tikzcd}$$

IF θ, τ MAKE $\circledast$ COMMUTE, THEN $\theta = \tau$

FS ⇝ DBL

DEF) CALL X AN ORTHOGONAL FACTORIZATION DOUBLE CATEGORY

IF ① EVERY $\begin{tikzcd}\bullet \arrow[r] & \bullet \arrow[d] \\ & \bullet\end{tikzcd}$ CAN BE FILLED

② EVERY SQUARE IS TOP-RIGHT BICART.

③ X IS INVARIANT $\begin{tikzcd}\bullet \arrow[d, "\cong"'] & \bullet \arrow[d, "\cong"] \\ \bullet \arrow[r] & \bullet\end{tikzcd}$

④ EVERY $\begin{tikzcd}\bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$ JOINTLY MONIC IN X^{Λ}

DENOTE FactDbl $\subseteq_{\text{FULL}}$ Dbl

FACT: $D : \text{OFS} \longrightarrow \text{FactDbl}$

IS AN EQUIVALENCE OF CATS

FS $\leftsquigarrow$ DBL

GIVEN A DOUBLE CAT'RY X SATISFYING ①, ②, CONSTRUCT $\mathrm{Cnr}(X)$ W/ **OB**: $\mathrm{ob}X$

MOR: EQUIVALENCE CLASS OF CORNERS $[e, m]$ **IDS**: $[1_a, 1_a]$

HERE $(e, m) \sim (e', m')$ IF $\exists \beta$:

$$\begin{tikzcd}
a \arrow[r, equal] \arrow[d, "e"'] \arrow[dd, bend right=70, "e'"', "="] & a \arrow[d, "e"] & \\
c \arrow[r, equal] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Downarrow \beta"] & c \arrow[r, "m"] \arrow[d, equal] & b \arrow[d, equal] \\
c' \arrow[r, "\tau"'] \arrow[rr, bend right=50, "m'"', "\|"] & c \arrow[r, "m"'] & b
\end{tikzcd}$$

SUCH β IS
- UNIQUE
- INVERTIBLE

FS $\leftsquigarrow$ DBL

COMPOSITION: $[v, h] \circ [u, g] := [\tilde{v} \circ u, h \circ \tilde{g}]$

CHOOSE:

$$\begin{tikzcd}
a \arrow[d, "u"'] & & \\
b \arrow[r, "g"] \arrow[d, "\tilde{v}"'] \arrow[dr, phantom, "\Downarrow"] & c \arrow[d, "v"] & \\
\cdot \arrow[r, "\tilde{g}"'] & d \arrow[r, "h"] & e
\end{tikzcd}$$

EXAMPLES OF $\mathrm{Cnr}(X)$ (1/3):

$\boxed{\text{EXAMPLE}}$ $\mathcal{C}$ CAT'RY W/ PBS, CONSIDER $\mathrm{PbSq}(\mathcal{C})^{v}$

VERTICAL DUAL

$\mathrm{Cnr}(\mathrm{PbSq}(\mathcal{C})^{v})$ **OB**: ob $\mathcal{C}$

MOR: EQUIV CLASSES

$$\begin{tikzcd}
a & \\
x \arrow[u, "u"] \arrow[r, "g"'] & b
\end{tikzcd}$$

NOTE:

$$\begin{tikzcd}
 & a \arrow[r, equal] & a & \\
 & c \arrow[u, "u"] \arrow[r, equal] & c \arrow[u, "u"'] \arrow[r, "g"] & b \\
 & c' \arrow[u, "\theta"] \arrow[r, "\tau"'] \arrow[uu, bend left=60, "u'"] \arrow[rr, bend right=40, "g'"'] \arrow[ur, phantom, "\Downarrow \beta"] & c \arrow[u, equal] \arrow[r, "g"] & b \arrow[u, equal]
\end{tikzcd}$$

$\Leftrightarrow$ $\theta = \tau$ ISO &

$$\begin{tikzcd}
 & a & \\
 & c \arrow[u, "u"'] \arrow[r, "g"] & b \\
c' \arrow[uur, bend left, "u'"] \arrow[ur, "\theta"', "\cong"] \arrow[urr, bend right, "g'"'] & &
\end{tikzcd}$$

IN $\mathcal{C}$

$\Rightarrow$ $\mathrm{Cnr}(\mathrm{PbSq}(\mathcal{C})^{v}) \cong \mathrm{Span}(\mathcal{C})$

EXAMPLES OF Cnr(X) (2/3):

EXAMPLE $\mathrm{MPbSq}(\mathcal{C}) \subseteq \mathrm{PbSq}(\mathcal{C})$ WHOSE VERTICAL MORS ARE MONOS

$\mathrm{Cnr}(\mathrm{MPbSq}(\mathcal{C})^{N}) \cong \mathrm{Par}(\mathcal{C})$ PARTIAL MAPS IN $\mathcal{C}$

$$\begin{tikzcd}
a \\
a' \arrow[u, hook] \arrow[r] & b
\end{tikzcd}$$

MAP IS TOTAL IF

$$\begin{tikzcd}
a \\
a \arrow[u, equal] \arrow[r] & b
\end{tikzcd}$$

EXAMPLES OF Cnr(X) (3/3):

EXAMPLE HAVE A DOUBLE CAT'RY **BOFIB**

W/ OBJECTS: SMALL CATEGORIES

HMORS: DISCRETE OPFIBS

VMORS: (BIJECTIONS ON OBJECTS)^{op}

SQUARE: COMMUTATIVE SQS

Cnr(BOFIB) $\cong$ Cof CATEGORIES & COFUNCTORS

- EVERY COFUNCTOR CAN BE PRESENTED AS:

$$\begin{tikzcd}
A \\
A' \arrow[u, "\text{B.O.}"] \arrow[r, "\text{DISCR OPFIB}"] & B
\end{tikzcd}$$

FS $\leftsquigarrow$ DBL

PROP LET X FACT. DOUBLE CAT'RY.

THE TWO CLASSES

$$\mathcal{E}_X = \{ [u,1] \mid u \text{ VMOR IN } X \}$$

$$\mathcal{M}_X = \{ [1,h] \mid h \text{ HMOR IN } X \}$$

FORM AN ORTHOGONAL FS ON Cnr(X).

FS $\leftsquigarrow$ DBL

$\boxed{\text{PROP}}$ LET X FACT. DOUBLE CAT'RY. THE TWO CLASSES

$$\mathcal{E}_X = \{[u,1] \mid u \text{ VMOR IN } X\}$$

$$\mathcal{M}_X = \{[1,h] \mid h \text{ HMOR IN } X\}$$

FORM AN ORTHOGONAL FS ON Cnr(X).

PROOF: ① EVERY

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[d] \\ & \bullet \end{tikzcd}$$

CAN BE FILLED

② EVERY SQUARE IS TOP-RIGHT BICART.

→ USED TO CONSTRUCT Cnr(X)

③ X IS INVARIANT

$$\begin{tikzcd} \bullet \arrow[d, "\cong"'] & \bullet \arrow[d, "\cong"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

→ USED TO PROVE $\mathcal{E}_X \cap \mathcal{M}_X = \{\text{ISOS}\}$

④ EVERY

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

JOINTLY MONIC IN X

→ USED TO PROVE $\mathcal{E}_X \perp \mathcal{M}_X$ □

FS $\leftrightsquigarrow$ DBL

THEOREM THE FUNCTOR D IS AN EQUIVALENCE W/ EQUIV. INVERSE Cnr

$$\begin{tikzcd}
\mathcal{OFS} \arrow[r, bend right, "D"'] \arrow[r, phantom, "\simeq"] & \mathsf{FactDbl} \arrow[l, bend right, "\mathrm{Cnr}"']
\end{tikzcd}$$

ORTHOG. FACT. SYSTEMS $\simeq$ FACTORIZATION DOUBLE CATEGORIES

$$\begin{tikzcd}
\mathsf{OFS} \arrow[r, bend right, "D"'] \arrow[r, phantom, "\simeq"] & \mathsf{FactDbl} \arrow[l, bend right, "\mathsf{Chr}"']
\end{tikzcd}$$

EXAMPLES $\mathsf{Par}(\mathcal{C}) \longleftrightarrow \mathsf{MPbSq}(\mathcal{C})$

$\mathsf{Cof} \longleftrightarrow \mathsf{BOFIB}$

NON-EXAMPLES $\mathsf{Span}(\mathcal{C})$ ... $\mathsf{PbSq}(\mathcal{C})^{N}$

LACKS JOINT MONICITY

END OF

PART 1

Codescent objects

VERSUS

Double Categories

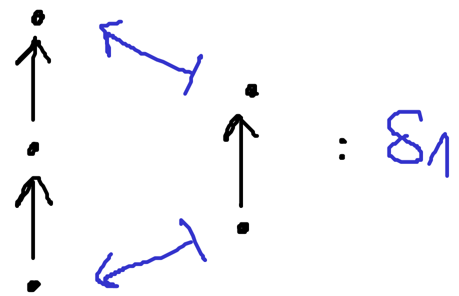
SIMPLICIAL NOTATION

DEF) $[n] := \{0 \to 1 \to \dots \to n\} \in \mathrm{Cat}$

DEFINE SUBCATEGORY OF Cat:

$$\begin{tikzcd}
\Delta_2 := & {[2]} & {[1]} \arrow[l, shift right=3, "\delta_2"'] \arrow[l, "\delta_1" description] \arrow[l, shift left=3, "\delta_0"] \arrow[r, "\sigma_0" description] & {[0]} \arrow[l, shift right=3, "\delta_0"'] \arrow[l, shift left=3, "\delta_0"]
\end{tikzcd}$$

HERE $\delta_j^{-1}(j) = \emptyset$ I.E.



HAVE CANONICAL FUNCTOR $W : \Delta_2 \hookrightarrow \mathrm{Cat}$

DIAGRAM $X : \Delta_2^{op} \to \mathcal{C}$ COHERENCE DATA

$W * X$ IS CALLED THE CODESCENT OBJECT

EXAMPLE $\mathcal{D}$ SMALL CAT'RY,

$$\begin{tikzcd} \mathcal{D}_2 \arrow[r, shift left=3, "\pi_1"] \arrow[r, "\mathrm{comp}"] \arrow[r, shift right=3, "\pi_2"'] & \mathcal{D}_1 \arrow[r, shift left=3, "s"] \arrow[r, shift right=3, "t"'] & \mathcal{D}_0 \arrow[l, "\mathrm{ids}"'] \end{tikzcd}$$

COHERENCE DATA
IN Set

- CAN EMBED IT IN Cat
  VIA Disc : Set ⟶ Cat

EXAMPLE X DOUBLE CATEGORY, GIVES

$$\begin{tikzcd}
X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_1 \arrow[r, shift left=3, "d_1"] \arrow[r, shift right=3, "d_0"'] & X_0 \arrow[l, "s" description]
\end{tikzcd}$$

COH DATA IN $\mathcal{C}at$

- X_0 CAT'RY OF OBJECTS, VMORS
- X_1 CAT'RY OF HMORS, SQUARES

FOR EXAMPLE:

$$\begin{tikzcd}
a \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[d, "v"] & \overset{d_0}{\longmapsto} & b \arrow[d, "v"] \\
c \arrow[r, "h"'] & d & & d
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[d, "u"'] & \overset{s}{\longmapsto} & a \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow 1_u"] & a \arrow[d, "u"] \\
c & & c \arrow[r, equal] & c
\end{tikzcd}$$

CAN PROVE: IF $X : \Delta_2^{op} \to \mathcal{C}at$ DBL CAT,

- A (W-WEIGHTED) COCONE WITH APEX $\mathcal{C}$ IS A PAIR $(F : X_0 \to \mathcal{C}, \xi : h(X) \to \mathcal{C})$ OF FUNCTORS W/ $\mathrm{ob} F = \mathrm{ob}\, \xi$

S.T.
$$\begin{tikzcd} a \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[d, "v"] \\ c \arrow[r, "h"'] & d \end{tikzcd}$$
$\exists$ $\Longrightarrow$
$$\begin{tikzcd} Fa \arrow[r, "\xi(g)"] \arrow[d, "Fu"'] \arrow[dr, phantom, "//"] & Fb \arrow[d, "Fv"] \\ Fc \arrow[r, "\xi(h)"'] & Fd \end{tikzcd}$$
IN $\mathcal{C}$

- IT IS A COLIMIT COCONE IF

$\forall (G, \psi)$ $\exists ! \theta$:
$$\begin{tikzcd} X_0 \arrow[r, "F"] \arrow[dr, bend right, "G"'] & \mathcal{C} \arrow[d, dashed, "\theta"] & h(X) \arrow[l, "\xi"'] \arrow[dl, bend left, "\psi"] \\ & \mathcal{D} & \end{tikzcd}$$

DEFINE A CATEGORY Cocone $\subseteq \mathrm{Cat}^{\bullet \to \bullet \leftarrow \bullet}$

WITH **OBJECTS** $\mathcal{V} \xrightarrow{F} \mathcal{C} \xleftarrow{\xi} \mathcal{H}$

PAIRS OF FUNS S.T. $\begin{cases} \mathrm{ob}\,\mathcal{V} = \mathrm{ob}\,\mathcal{H} \\ \mathrm{ob}\,F = \mathrm{ob}\,\xi \end{cases}$

MORPHISMS $(F', \xi') \longrightarrow (F'', \xi'')$

TRIPLES (V, θ, H) S.T.

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "F"] \arrow[d, "V"'] & \mathcal{C} \arrow[d, "\theta"'] & \mathcal{H} \arrow[l, "\xi"'] \arrow[d, "H"] \\
\mathcal{V}' \arrow[r, "F'"'] & \mathcal{C}' & \mathcal{H}' \arrow[l, "\xi'"]
\end{tikzcd}$$

& $\mathrm{ob}\,V = \mathrm{ob}\,H$

GIVEN A COCONE $\mathcal{V} \xrightarrow{F} \mathcal{C} \xleftarrow{\xi} \mathcal{H}$

CONSTRUCT A DOUBLE CAT'RY $\mathbb{D}_{F,\xi}$

S.T. OBJECTS: $\operatorname{ob} \mathcal{V}$

HMORS: $\operatorname{mor} \mathcal{H}$

VMORS: $\operatorname{mor} \mathcal{V}$

SQUARES:

$$\exists \quad \begin{tikzcd} a \arrow[r, "m"] \arrow[d, "e"'] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[d, "e'"] \\ c \arrow[r, "m'"'] & d \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} Fa \arrow[r, "\xi(g)"] \arrow[d, "Fu"'] \arrow[dr, phantom, "//"] & Fb \arrow[d, "Fv"] \\ Fc \arrow[r, "\xi(h)"'] & Fd \end{tikzcd} \quad \text{IN } \mathcal{C}$$

THM THERE IS AN ADJUNCTION:

$$\begin{tikzcd} \mathrm{Cocone} \arrow[r, bend right, "D"'] \arrow[r, phantom, "\perp"] & \mathrm{Dbl} \arrow[l, bend right, "\mathrm{CoD}(-)"'] \end{tikzcd}$$

– THE LEFT ADJOINT SENDS X TO ITS COLIMIT COCONE

COR THE ADJUNCTION RESTRICTS TO EQUIVALENCES

- S∃FS ≅ CoDDisck
- O∃FS ≅ FactDbl

ALSO: W∃FS ≅ FULL SUBCAT OF Dbl

(*1)

FOR A DOUBLE CAT'RY SATISFYING ①, ②, DENOTE $F_X : v(X) \longrightarrow \mathrm{Chr}(X)$

$$\begin{tikzcd} a \arrow[d, "u"'] \\ b \end{tikzcd} \mapsto [u, 1]$$

$$\xi_X : h(X) \longrightarrow \mathrm{Chr}(X)$$

$$\left(a \xrightarrow{h} b \right) \mapsto [1, h]$$

THE PAIR (F_X, ξ_X) IS THE CODESCENT OBJECT OF X

RECALL $X = \mathrm{PbSq}(\mathcal{C})^{\mathbb{N}}$, $\mathrm{Cnr}(X) = \mathrm{Span}(\mathcal{C})$

THM (UNIVERSAL PROPERTY OF SPANS)

LET $\mathcal{C}$ CAT'RY W/ PBS.

FOR ANY PAIR $\mathcal{C}^{op} \xrightarrow{G} \mathcal{E} \xleftarrow{\psi} \mathcal{C}$

WITH THE PROPERTY THAT

$$\begin{tikzcd}
a \arrow[r, "m"] & b & & Ga \arrow[r, "\psi m"] \arrow[d, "Ge"'] \arrow[dr, phantom, "//"] & Gb \arrow[d, "Ge'"] & \\
c \arrow[u, "e"] \arrow[r, "m'"'] \arrow[u, phantom, "\lrcorner", shift left=4] & d \arrow[u, "e'"'] & \text{IN } \mathcal{C} \quad \Rightarrow & Gc \arrow[r, "\psi m'"'] & Gd & \text{IN } \mathcal{E}
\end{tikzcd}$$

THERE IS A UNIQUE FUN $\mathrm{Span}(\mathcal{C}) \xrightarrow{\theta} \mathcal{E}$

S.T.

$$\theta\left(\begin{tikzcd} a & \\ b \arrow[u, "u"] \arrow[r, "g"] & c \end{tikzcd}\right) = \psi g \circ Gu$$

(✱2)

$$\begin{tikzcd} \mathrm{Cocone} \arrow[r, bend right, "D"'] \arrow[r, phantom, "\perp"] & \mathrm{Dbl} \arrow[l, bend right, "\mathrm{CoD}(-)"'] \end{tikzcd}$$ IS AN IDEMPOTENT ADJUNCTION

AS SUCH, GIVES IDEMPOTENT MONAD

$$\tilde{T} \circlearrowright \mathrm{Dbl}$$

- $\forall X \in \mathrm{Dbl}$, $\tilde{T}X$ IS FLAT

BUT NOT EVERY FLAT DOUBLE CAT IS OF THIS FORM, CONSIDER:

$$\begin{tikzcd}
0 \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\Downarrow"] & 1 \arrow[rr] \arrow[d] \arrow[drr, phantom, "\Downarrow"] & & 2 \arrow[d] \\
& 3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & 4 \arrow[r] \arrow[d] \arrow[ddr, phantom, "\Downarrow"] & 5 \arrow[dd] \\
6 \arrow[r] \arrow[d] \arrow[drr, phantom, "\Downarrow"] & 7 \arrow[r] & 8 \arrow[d] & \\
9 \arrow[rr] & & 10 \arrow[r] & 11
\end{tikzcd}$$

CLEARLY NOT $\tilde{T}X \cong X$

(✳3)

A WORD ON LAX MORPHISM CLASSIFIERS

EVERY STRICT MONCAT $(\mathbb{A}, \otimes, I)$ GIVES A DOUBLE CAT'RY $\mathrm{Res}(\mathbb{A}, \otimes, I)$

OB: $(a_1, \dots, a_n)$, $a_i \in \mathrm{ob}\,\mathbb{A}$

HMOR: PARTIAL EVALUATION

E.G. $(a_1, a_2, a_3) \longrightarrow (a_1 \otimes a_2, a_3, I)$

VMOR: $(f_1, \dots, f_n)$, $f_i \in \mathrm{mor}\,\mathbb{A}$

THE TRANSPOSE OF $\mathrm{Res}(\mathbb{A}, \otimes, I)$ IS A CODOMAIN-DISCR DOUBLE CAT

$\rightsquigarrow$ CAN COMPUTE COLIMIT

DENOTE $\mathrm{Ctr}(\mathcal{A}) := \mathrm{Ctr}(\mathrm{Res}(\mathcal{A}, \otimes, I)^{T})$

OB: $(a_1, .., a_n)$, $a_i \in \mathrm{ob}\mathcal{A}$

MOR:

$$\begin{tikzcd}
(a_1, a_2, a_3) \arrow[r] & (a_1 \otimes a_2, a_3, I) \arrow[d, "{(f_1, f_2, f_3)}"] \\
& (b_1, b_2, b_3)
\end{tikzcd}$$

- ADMITS A STRICT MONOIDAL STRUCTURE
  $(a_1, .., a_n) \boxplus (a'_1, .., a'_n) = (a_1, .., a_n, a'_1, .., a'_n)$
  W/ UNIT $()$

- ADMITS A STRICT FACT. SYSTEM

FACT: IT CLASSIFIES
LAX MONOIDAL FUNCTORS

$\forall (F,F)$ LAX MONOIDAL
$\exists ! F'$ STRICT MONOIDAL

S.T.

$$\begin{tikzcd}
(\mathbb{A},\otimes,I) \arrow[r, squiggly] \arrow[d, "{(F,\bar{F})}"'] & (\mathrm{Cnr}(\mathbb{A}),\boxplus,()) \arrow[dl, dashed, "F'"] \\
(\mathbb{B},\odot,I') &
\end{tikzcd}$$

SAME STORY FOR LAX FUNCTORS,
LAX DOUBLE FUNCTORS,
LAX NAT TRS' (. . .)

(✱ 4)

LET X A GENERAL DOUBLE CAT'RY.

WHAT IS THE FORMULA FOR CoD(X)?

Cnr(X) OB: obX

MOR: EQUIVALENCE CLASSES OF PATHS $[f_1, \dots, f_n]$ W/ f_i EITHER HMOR OR VMOR

CONGRUENCE GENERATED BY:

- $\exists$
$$\begin{tikzcd} a \arrow[r, "m"] \arrow[d, "e"'] \arrow[dr, phantom, "\Downarrow \alpha"] & b \arrow[d, "e'"] \\ c \arrow[r, "m'"'] & d \end{tikzcd}$$
$\Longrightarrow$ $(m, e') \sim (e, m')$
- $(f_1, f_2) \sim (f_2 \circ f_1)$ IF f_i BOTH VMOR OR HMOR
- $(1_a) \sim ()$ IF 1_a HORIZ ID OR VERT. ID

EXAMPLE IF $\mathcal{C}$ CATEGORY REGARDED AS DOUBLE CAT'RY $\mathbb{C}$, $\mathrm{Cnr}(\mathbb{C}) \cong \mathcal{C}$

EXAMPLE $\mathrm{Cnr}(\mathrm{Sq}(\mathcal{C})) \cong \mathcal{C}$

EXAMPLE $\mathrm{Cnr}(\mathrm{Sq}(\mathcal{C})^{\sim}) \cong \mathcal{C}[\mathrm{mor}\,\mathcal{C}^{-1}]$

THOUGHTS, IDEAS

- ARE THERE NON-FLAT DOUBLE CATS WITH BICARTESIAN SQS?
- WHAT IS THE DOUBLE CATEGORICAL COUNTERPART OF SFS $\longrightarrow$ OFS

$$(\mathcal{E},\mathcal{M}) \mapsto (\tilde{\mathcal{E}},\tilde{\mathcal{M}})$$

- GIVEN OFS $(\mathcal{E}, \mathcal{M})$ ON $\mathcal{C}$, DOES $\mathbb{D}_{\mathcal{E},\mathcal{M}}$ HAVE ANY (CO)LIMITS ?

- HOW TO DESCRIBE DOUBLE CATS OF FORM $\mathbb{D}_{\mathcal{E},\mathcal{M}}$ FOR A WFS $(\mathcal{E}, \mathcal{M})$ IN ELEMENTARY TERMS ?

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THANK YOU

FOR YOUR ATTENTION.