Dblcat workshop talk

Tuesday, October 29, 2024 3:44 PM

DOUBLE-CATEGORICAL FRAMEWORKS FOR ALGEBRAIC K-THEORY

How does algebraic K-theory work?

input $\mathcal{C}$ → [K-theory machinery] → $K(\mathcal{C})$ (space/spectrum) → [π_i] → $\pi_i K(\mathcal{C}) =: K_i(\mathcal{C})$ (abelian groups)

Possible inputs:

1. ALGEBRAIC STRUCTURES

i.e. abelian / exact (think R-Mod)

Features:

- zero object
- monos and epis
- (co)kernels, short exact seq's

$0 \to A \overset{i}{\hookrightarrow} B \to \operatorname{coker} i \to 0$

$0 \to \ker p \hookrightarrow B \overset{p}{\twoheadrightarrow} C \to 0$

What does K-theory do? it splits s.e.s.

$K_0\mathcal{C} = \mathbb{Z}[\operatorname{ob}\mathcal{C}] \,/\, [B] = [A] + [C]$ for any $0 \to A \hookrightarrow B \twoheadrightarrow C \to 0$ exact $\iff$

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ 0 & \hookrightarrow & C \end{array}$$

$[A \oplus B] = [A] + [B]$ $\quad 0 \to A \hookrightarrow A\oplus B \to B \to 0$

space?

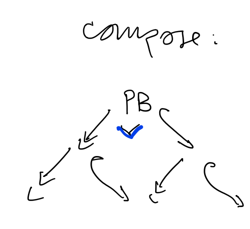
Q-constr.

$\mathcal{C} \rightsquigarrow Q\mathcal{C} \rightsquigarrow K\mathcal{C} = \Omega|NQ\mathcal{C}|$ ⚠ HARD!

ob = ob $\mathcal{C}$

maps $A \to B$: $A \twoheadleftarrow C \hookrightarrow B$

compose:



2. HOMOTOPICAL STR'S

i.e. Waldhausen cats (think: model str w/ all obj cofib; Chain cpx's w/ quasi-isos)

Features:

- zero obj
- cofibrations (~ monos)
- cofiber seq's i.e. $A \hookrightarrow B$, $0 \hookrightarrow B/A$ (~ s.e.s)
- weak equivs

K-theory: splits cofiber seq's + glue along w.e.

$K_0\mathcal{C} = \mathbb{Z}[\operatorname{ob}\mathcal{C}] \,/\, [B] = [A] + [B/A]$, $A \hookrightarrow B \to B/A$ and $[A] = [B]$, $A \overset{\sim}{\to} B$

$S_\bullet$-constr.

$wS_3\mathcal{C}$: cat w/

obj: $0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow A_3$, $0 \hookrightarrow A_{12} \hookrightarrow A_{13}$, $0 \to A_{23}$, 0

maps: nat tr that are pointwise w.e.

$\rightsquigarrow K\mathcal{C} = \Omega|wS_\bullet\mathcal{C}|$ ⚠

Thomason constr.

$T_3\mathcal{C}$ is the cat w/

obj: $A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow A_3$

maps: $A_0' \hookrightarrow A_1' \hookrightarrow A_2' \hookrightarrow A_3'$

$\rightsquigarrow K\mathcal{C} = \Omega|T_\bullet\mathcal{C}|$

OBS: if you squint, you start to see dbl cats!

3. CGW categories [Cambell-Zakharevich] [S-Shapiro]

THE PURPOSE: axiomatize essential features of abelian cats needed for K-theory. Notably: not additive.

How do I turn an abelian cat $\mathcal{A}$ into a dbl cat?

- obj = obj $\mathcal{A}$
- hor mor = $\hookrightarrow$ monos
- ver mor = $(\downarrow$ epis$)^{op}$
- squares = commutative

Defn A CGW cat is a double cat

denote: $\mathcal{M}$ underlying hor cat, $\mathcal{E}$ " ver cat

What properties need to be encoded?

- monos / epis — all maps in $\mathcal{M}$, $\mathcal{E}$ are monic
- zero obj 0 — ∃ initial obj $\varnothing$ for both $\mathcal{M}$, $\mathcal{E}$
- (co)kernels

$$\begin{array}{ccccc} A & \overset{i}{\hookrightarrow} & B & \to & \operatorname{coker} i \\ \downarrow & & \downarrow & & \downarrow \\ A' & \underset{i'}{\hookrightarrow} & B' & \to & \operatorname{coker} i' \end{array}$$

There are functors

coker : $\{\hookrightarrow$ (obj), squares (mor)$\} \to \{\ldots\}$

dual fctr: . . . .

- Every $\hookrightarrow$, $\downarrow$ determines a unique s.e.s.

$0 \to A \hookrightarrow B \twoheadrightarrow C \to 0$ ses $\Updownarrow$

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & C \end{array}$$ bicartesian.

Every hor mor $A \hookrightarrow B$ determines a ! dist. sq

$$\begin{array}{ccc} \varnothing & \hookrightarrow & \operatorname{coker} i \\ \updownarrow & \square & \updownarrow \\ A & \hookrightarrow & B \end{array}$$

$$\begin{array}{ccccc} A & \hookrightarrow & B & \leftarrow & \text{coker} \\ \updownarrow & \square & \updownarrow & & \updownarrow \; * \\ C & \hookrightarrow & D & \leftarrow & \text{coker} \\ \uparrow & & \uparrow & & \\ \ker & \hookrightarrow & \ker & & \end{array}$$

- iso $\iff$ * iso

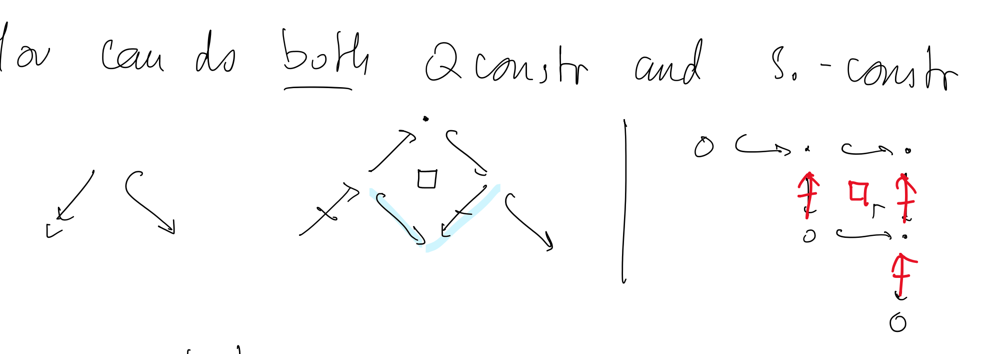
in this case, the square is "distinguished"

What's special about bicartesian sq?

$$\begin{array}{ccccc} \ker f & \overset{\cong}{\hookrightarrow} & \ker g & & \\ \downarrow & & \downarrow & & \\ A & \overset{i}{\hookrightarrow} & B & \twoheadrightarrow & \operatorname{coker} i \\ f\downarrow & BC & \downarrow g & & \downarrow \cong \\ C & \underset{j}{\hookrightarrow} & D & \twoheadrightarrow & \operatorname{coker} j \end{array}$$ iff

What can we do?

You can do both Q constr and $S_\bullet$-constr



More ex's!

Sets: obj = sets

hor maps = ver maps = inclusions $\hookrightarrow$

squares = cartesian

dist sq = bicartesian.

$$\begin{array}{ccc} \varnothing & \hookrightarrow & B\setminus A \ (\text{coker}) \\ \updownarrow & BC & \updownarrow \\ A & \hookrightarrow & B \end{array}$$

"(co)kers are to ab. cats as complements are to sets"

$$\begin{array}{ccccc} A & \hookrightarrow & B & \leftrightarrow & B\setminus A \\ \updownarrow & ? & \updownarrow & & \downarrow \\ C & \hookrightarrow & D & \leftrightarrow & D\setminus C \end{array}$$

Similarly: varieties (smooth), any extensive cat $A \hookrightarrow A \amalg B$.

4. SQUARES CATEGORIES [Campbell-Kuijper-Merling-Zakharevich]

Accomodate ex's where you have 4-term relations.

Defn A squares cat. is a flat double cat $\mathbb{D}$ which has an obj O initial in $\mathcal{H}\mathbb{D}$, $\mathcal{V}\mathbb{D}$ $\quad O \overset{!}{\hookrightarrow} A$, $O \updownarrow A$

So basically... not much!

K-th: $\pi_0(\Omega|N^{\square}\mathbb{D}|)$

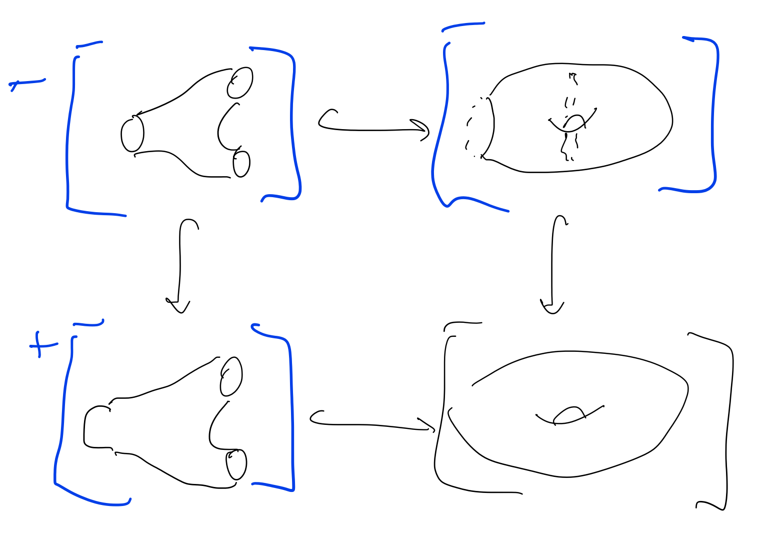
Thm: if ∀ A, B ∃ X

$$\begin{array}{ccc} O & \hookrightarrow & A \\ \updownarrow & \alpha & \updownarrow \\ B & \hookrightarrow & X \end{array}, \qquad \begin{array}{ccc} O & \hookrightarrow & B \\ \updownarrow & & \updownarrow \\ A & \hookrightarrow & X \end{array}$$ then

$K_0\mathcal{C} = \mathbb{Z}[\operatorname{ob}\mathbb{D}] \,/\, [A] + [D] = [B] + [C]$

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \updownarrow & \alpha & \updownarrow \\ C & \hookrightarrow & D \end{array}$$

Ex: Mflds up to cut & paste.



SK-groups

Hoekzema – Merling – Murray – Rovi – Semikina.