Virtual double categories as coloured box operads

Universiteit Antwerpen

Lander Hermans, 2024

fwo

Virtual double categories as coloured box operads

(in collaboration with Wendy Lowen and Hoang Dinh Van)

Universiteit Antwerpen

Lander Hermans, 2024

fwo

The Road map

The road map

monoids

The road map

monoids $\subseteq$ categories

The road map

monoids $\subseteq$ categories

$\cap\!|$

operads

The road map

$$\begin{array}{ccc} \text{monoids} & \subseteq & \text{categories} \\ \rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} \\ \text{operads} & \subseteq & \text{multicategories} = \\ & & \text{coloured operads} \end{array}$$

The road map

$$\begin{array}{ccccc}
\text{monoids} & \subseteq & \text{categories} & & \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \\
\text{operads} & \subseteq & \begin{array}{c}\text{multicategories} = \\ \text{coloured operads}\end{array} & \supseteq & \begin{array}{c}\text{monoidal} \\ \text{categories}\end{array}
\end{array}$$

The Road map

$$
\begin{tikzcd}[column sep=small]
\text{monoids} \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] & \text{categories} \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] & & \\
\text{operads} \arrow[r, phantom, "\subseteq"] & \begin{array}{c}\text{multicategories} = \\ \text{coloured operads}\end{array} \arrow[r, phantom, "\supseteq"] & \begin{array}{c}\text{monoidal} \\ \text{categories}\end{array} \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] \\
& & \begin{array}{c}\text{double} \\ \text{categories}\end{array}
\end{tikzcd}
$$

The Road Map

$$
\begin{tikzcd}[column sep=small]
\text{monoids} \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] & \text{categories} \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] & \\
\text{operads} \arrow[r, phantom, "\subseteq"] & \begin{array}{c}\text{multicategories} = \\ \text{coloured operads}\end{array} \arrow[r, phantom, "\supseteq"] \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] & \begin{array}{c}\text{monoidal} \\ \text{categories}\end{array} \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] \\
& \text{virtual double categories} \arrow[r, phantom, "\supseteq"] & \begin{array}{c}\text{double} \\ \text{categories}\end{array}
\end{tikzcd}
$$

The road map

$$
\begin{tikzcd}
\text{monoids} \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, "\subseteq" sloped] & \text{categories} \arrow[d, phantom, "\subseteq" sloped] & \\
\text{operads} \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, "\subseteq" sloped] & \text{multicategories} = \text{coloured operads} \arrow[r, phantom, "\supseteq"] \arrow[d, phantom, "\subseteq" sloped] & \text{monoidal categories} \arrow[d, phantom, "\subseteq" sloped] \\
\boxed{\text{box operads}} \arrow[r, phantom, "\subseteq"] & \text{virtual double categories} \arrow[r, phantom, "\supseteq"] & \text{double categories}
\end{tikzcd}
$$

The Road Map

$$
\begin{tikzcd}
\text{monoids} \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, sloped, "\subseteq"] & \text{categories} \arrow[d, phantom, sloped, "\subseteq"] & \\
\text{operads} \arrow[r, phantom, "\subsetneq"] \arrow[d, phantom, sloped, "\subseteq"] & \begin{array}{c}\text{multicategories} = \\ \text{coloured operads}\end{array} \arrow[d, phantom, sloped, "\subseteq"] & \begin{array}{c}\text{monoidal} \\ \text{categories}\end{array} \arrow[l, phantom, "\supsetneq"] \arrow[d, phantom, sloped, "\subseteq"] \\
\boxed{\text{box operads}} \arrow[r, phantom, "\subsetneq"] & \begin{array}{c}\text{virtual double categories} \\ \text{“=“ coloured box operads}\end{array} & \begin{array}{c}\text{double} \\ \text{categories}\end{array} \arrow[l, phantom, "\supsetneq"]
\end{tikzcd}
$$

A little bit of History

A little bit of History

- Multicategories over a monad - Burroni (1971)

A little bit of History

- Multicategories over a monad - Burroni (1971)
- Fc-multicategories - Leinster (2004)

A little bit of History

- Multicategories over a monad - Burroni (1971)
- Fc-multicategories - Leinster (2004)
- Lax double categories - Dawson, Paré and Pronk (2006)

A little bit of History

- Multicategories over a monad - Burroni (1971)
- Fc-multicategories - Leinster (2004)
- Lax double categories - Dawson, Paré and Pronk (2006)
- Virtual double categories - Crutwell and Shulman (2009)

A little bit of History

- Multicategories over a monad - Burroni (1971)
- Fc-multicategories - Leinster (2004)
- Lax double categories - Dawson, Paré and Pronk (2006)
- Virtual double categories - Crutwell and Shulman (2009)
- Augmented virtual double categories - Koudenburg (2022)

A little bit of History

- Multicategories over a monad - Burroni (1971)
- Fc-multicategories - Leinster (2004)
- Lax double categories - Dawson, Paré and Pronk (2006)
- Virtual double categories - Crutwell and Shulman (2009)
- Augmented virtual double categories - Koudenburg (2022)
- Box operads - Dinh Van, Hermans and Lowen (2023)

Point of view

object of study

object encodes

categories

Point of view	object of study	object encodes
categories	$\mathbf{Gp}$, $\mathbf{Ab}$, $\mathbf{Set}_\Delta$, …	

Point of view

	object of study	object encodes
categories	Gp, Ab, Set_Δ, …	Δ, R (ring), A (k-algebra), …

Point of view

	object of study	object encodes
categories	$\mathbf{Grp}$, $\mathbf{Ab}$, $\mathbf{Set}_\Delta$, …	Δ, $\underset{\text{ring}}{R}$, $\underset{k\text{-algebra}}{A}$, …
multicategories = coloured operads		

Point of view

Point of view	object of study	object encodes
categories	$\mathsf{Grp}, \mathsf{Ab}, \mathsf{Set}_\Delta, \ldots$	$\Delta, \overset{\text{ring}}{R}, \overset{k\text{-algebra}}{A}, \ldots$
multicategories = coloured operads		$\mathsf{Assoc}, \mathsf{Lie}, \mathsf{Op}, \ldots$

Point of view

	object of study	object encodes
categories	$\mathrm{Grp}, \mathrm{Ab}, \mathrm{Set}_\Delta, \ldots$	$\Delta, \overset{\text{ring}}{R}, \overset{k\text{-algebra}}{A}, \ldots$
multicategories = coloured operads	$(\mathrm{Mod}(k), \otimes, k), \ldots$	$\mathrm{Assoc}, \mathrm{Lie}, \mathrm{Op}, \ldots$

Point of view

	object of study	object encodes
categories	$\mathsf{Grp}, \mathsf{Ab}, \mathsf{Set}_\Delta, \ldots$	Δ, R (ring), A (k-algebra), $\ldots$
multicategories = coloured operads	$(\mathsf{Mod}(k), \otimes, k), \ldots$	$\mathsf{Assoc}, \mathsf{Lie}, \mathsf{Op}, \ldots$
virtual double categories = coloured box operads		

Point of view

	object of study	object encodes
categories	$\mathsf{Grp}, \mathsf{Ab}, \mathsf{Set}_\Delta, \ldots$	ring, k-algebra Δ, R, A, …
multicategories = coloured operads	$(\mathsf{Mod}(k), \otimes, k), \ldots$	$\mathsf{Assoc}, \mathsf{Lie}, \mathsf{Op}, \ldots$
virtual double categories = coloured box operads	$\mathsf{Bimod}(\mathcal{V}), \mathsf{Prof}, \mathsf{dFib}, \ldots$	

Point of view

	object of study	object encodes
categories	$\mathrm{Grp}, \mathrm{Ab}, \mathrm{Set}_\Delta, \ldots$	$\Delta, \overset{\text{ring}}{R}, \overset{k\text{-algebra}}{A}, \ldots$
multicategories = coloured operads	$(\mathrm{Mod}(k), \otimes, k), \ldots$	$\mathrm{Assoc}, \mathrm{Lie}, \mathrm{Op}, \ldots$
virtual double categories = coloured box operads	$\mathrm{Bimod}(\mathcal{V}), \mathrm{Prof}, \mathrm{dFib}, \ldots$ $\mathrm{Span}(\mathcal{V})$	

Point of view	object of study	object encodes
categories	Grp, Ab, Set_Δ, ...	Δ, R (ring), A (k-algebra), ...
multicategories = coloured operads	$(\mathsf{Mod}(k), \otimes, k)$, ...	Assoc, Lie, Op, ...
virtual double categories = coloured box operads	$\mathsf{Bimod}(\mathcal{V})$, Prof, dFib, ..., $\mathsf{Span}(\mathcal{V})$	Lax_u, colax_u, Pres_u, ...

Motivation

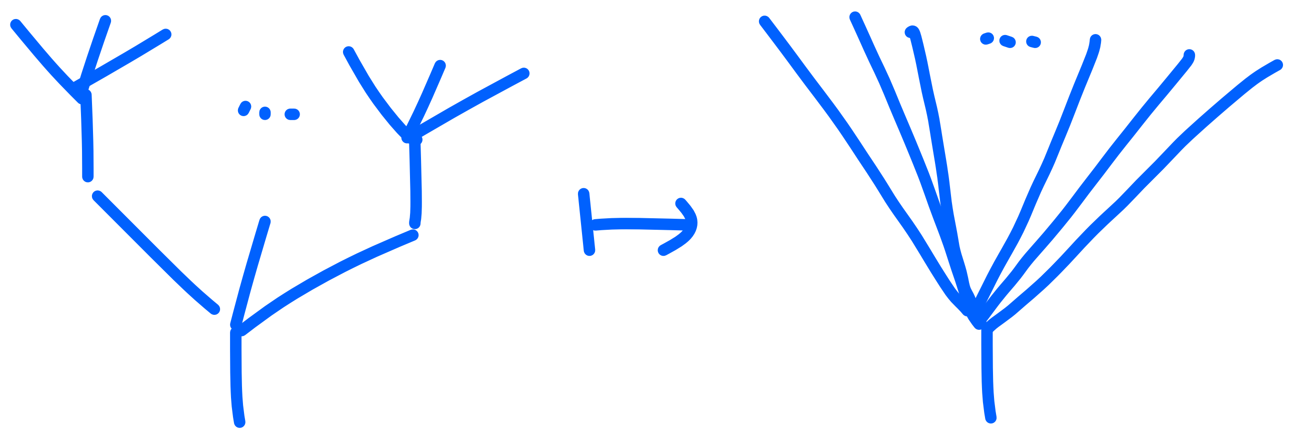
encode prestacks

via coloured box operads

to study

their deformation and homotopy theory

operads	multicategories = coloured operads	monoidal categories
box operads	virtual double categories “=“ coloured box operads	double categories



operads

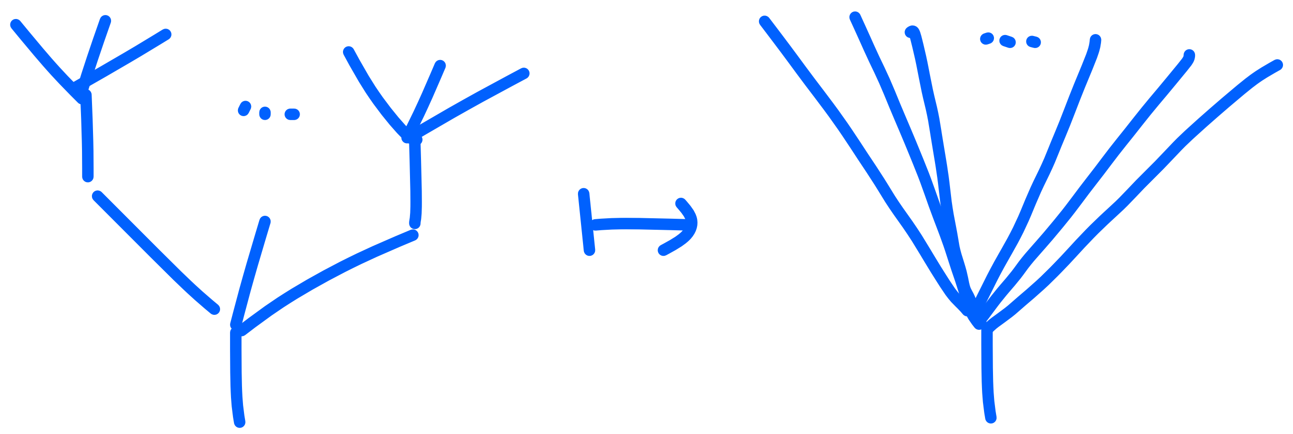
multicategories =
coloured operads

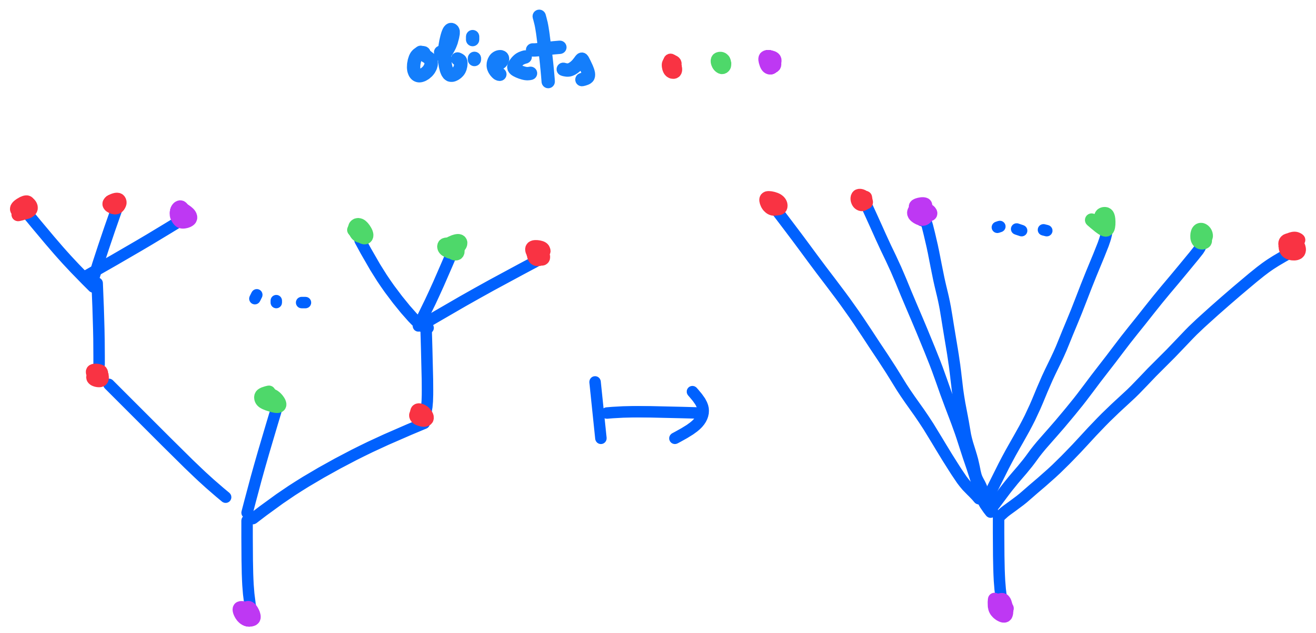
monoidal
categories

box operads

virtual double categories “=“
coloured box operads

double
categories





operads

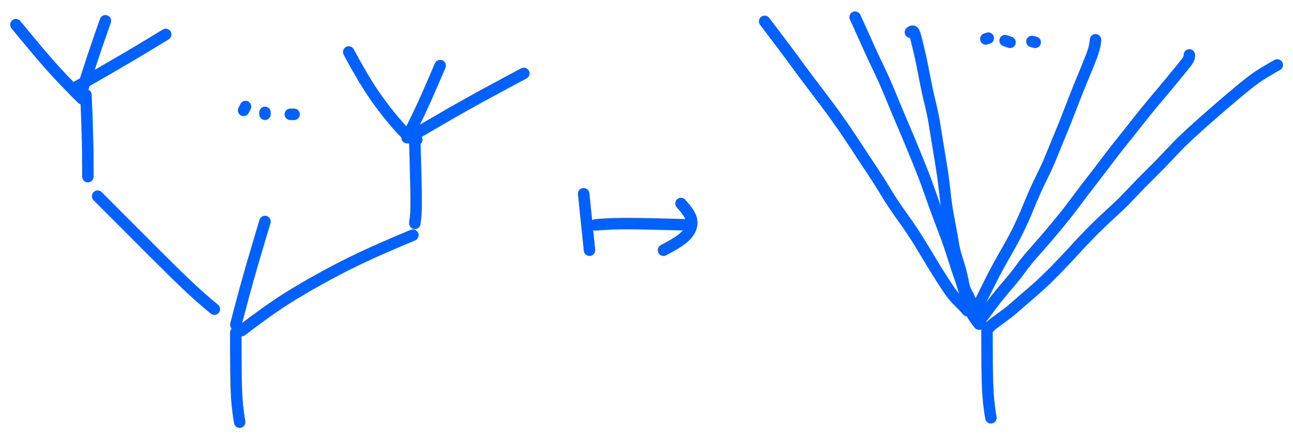
multicategories =
coloured operads

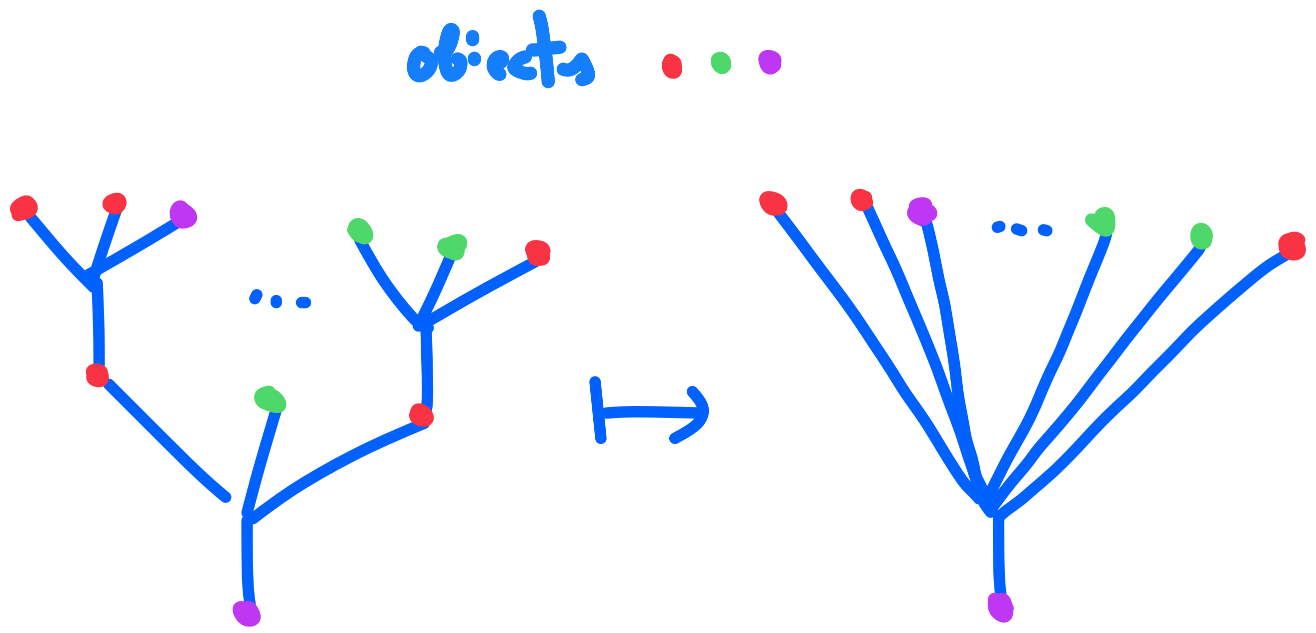
monoidal
categories

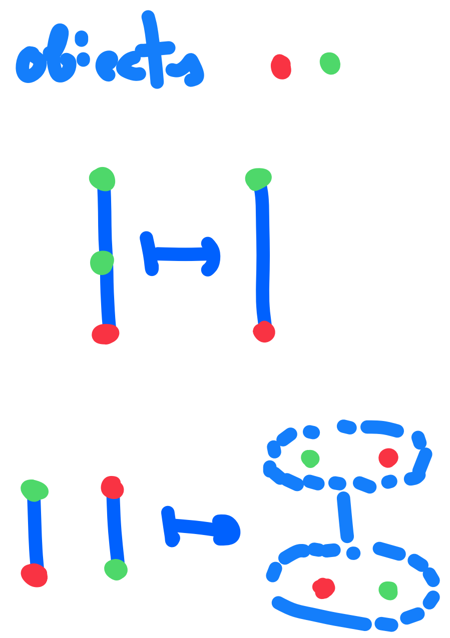
box operads

virtual double categories “=“
coloured box operads

double
categories







operads

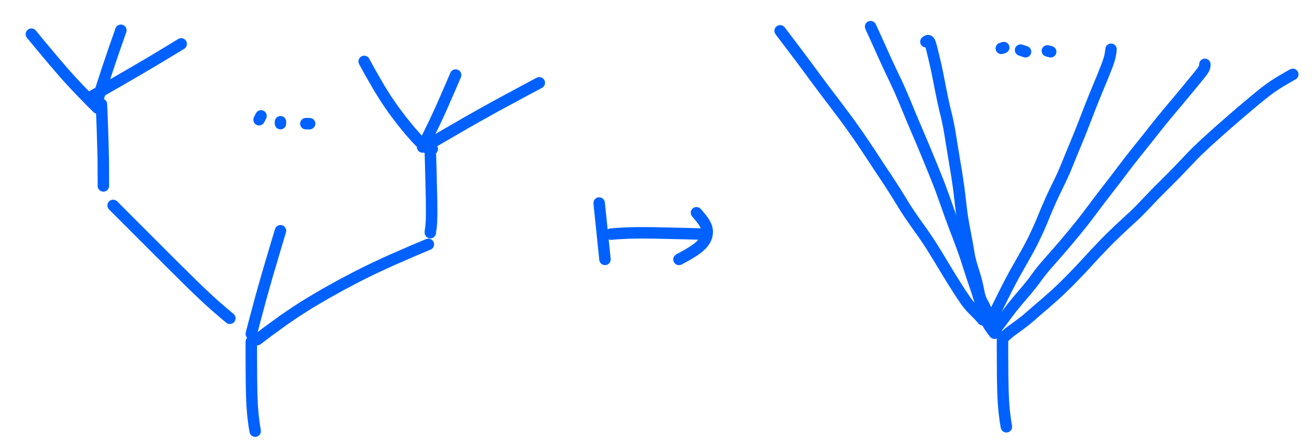
multicategories =
coloured operads

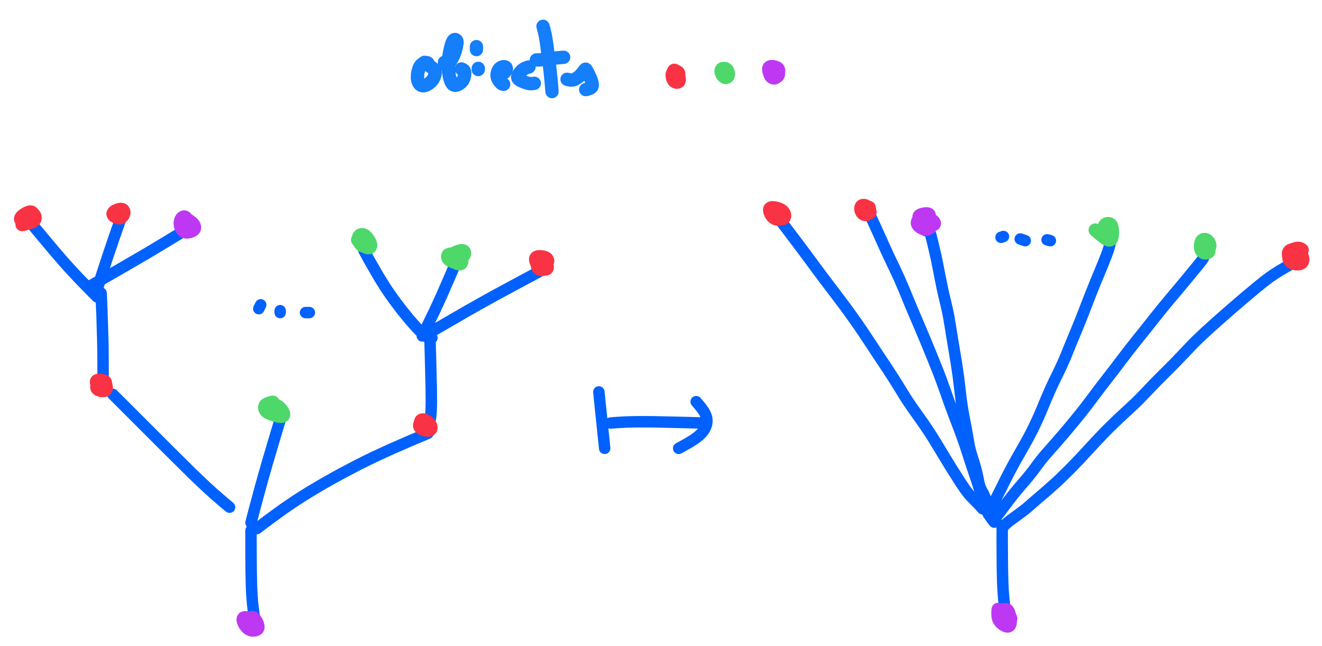
monoidal
categories

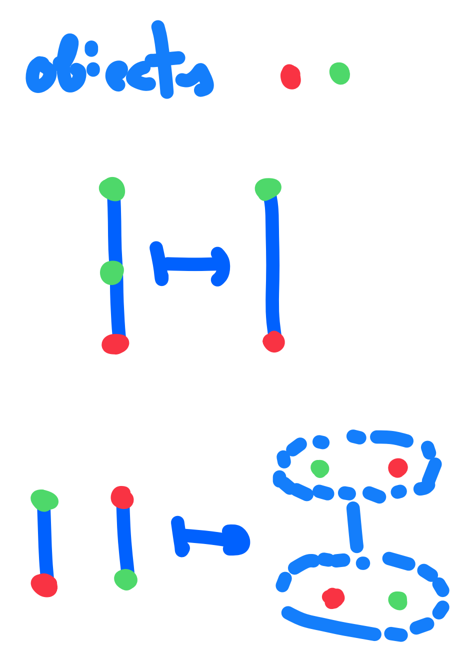
box operads

virtual double categories “=“
coloured box operads

double
categories







operads

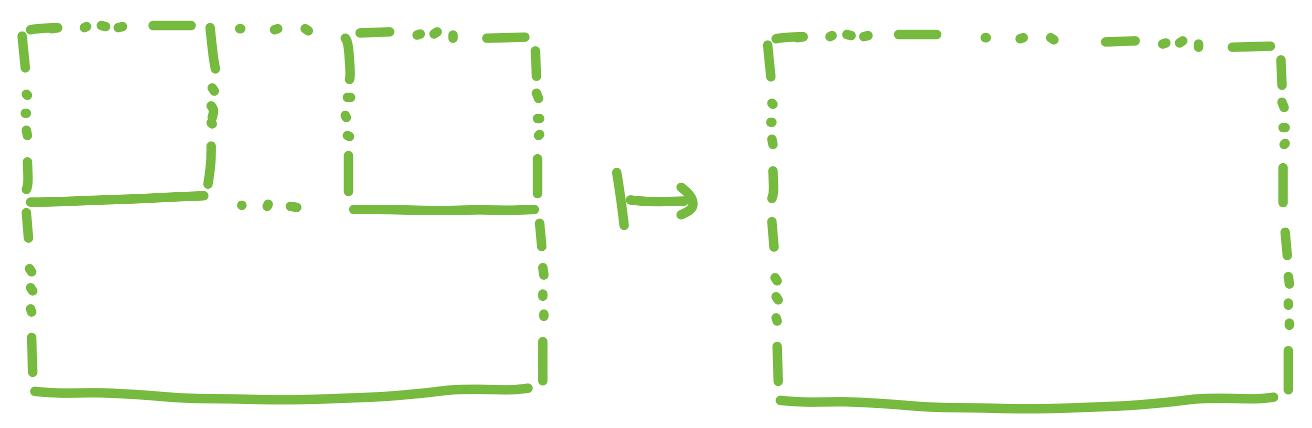
multicategories =
coloured operads

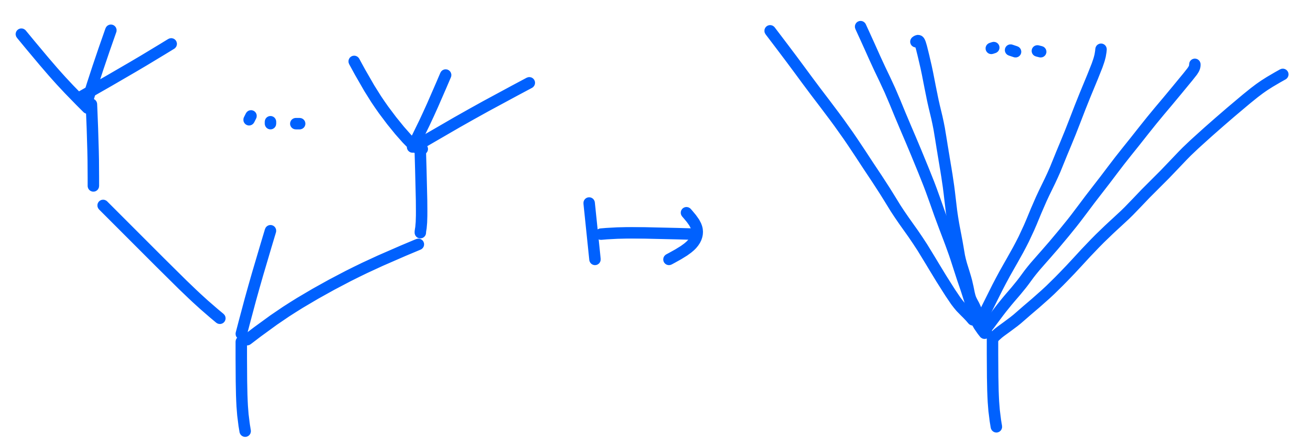
monoidal
categories

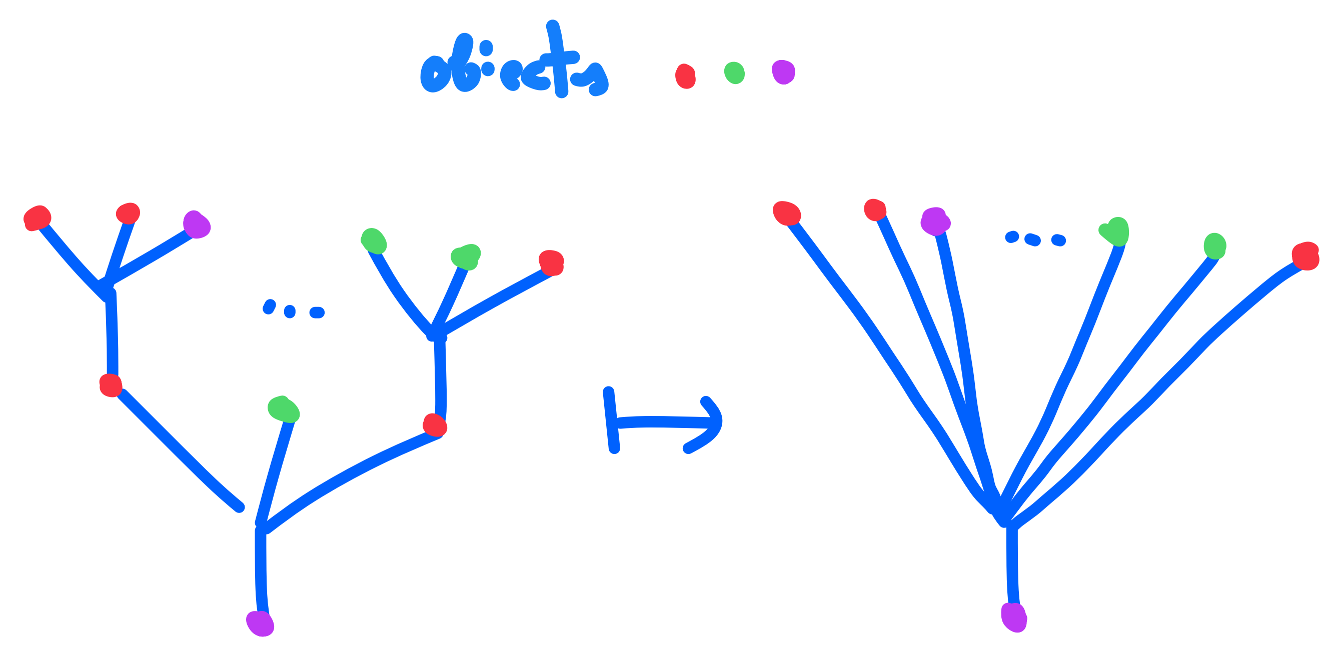
box operads

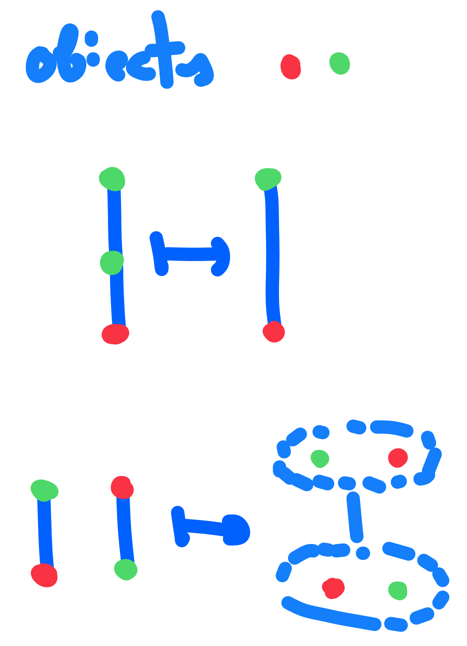
virtual double categories “=“
coloured box operads

double
categories









operads

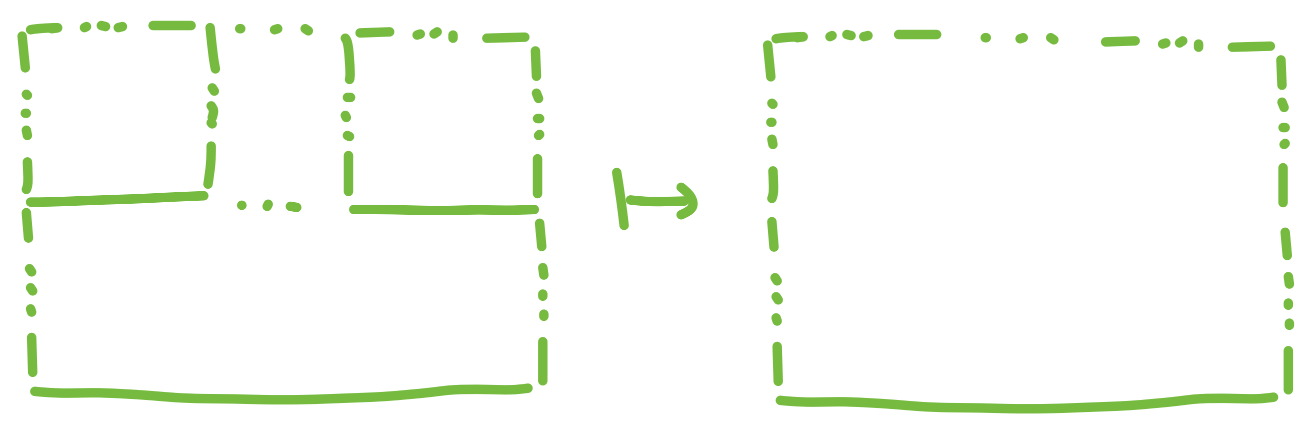
multicategories =
coloured operads

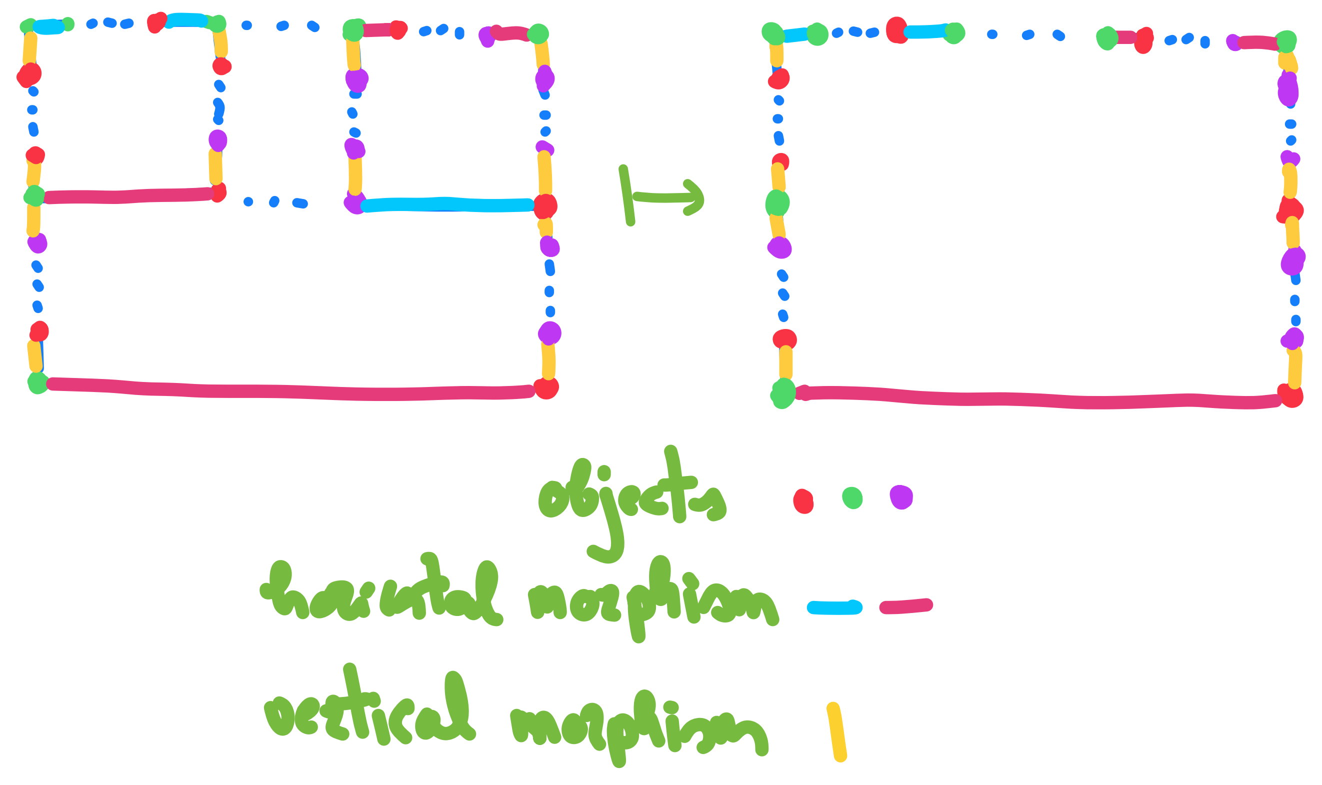
monoidal
categories

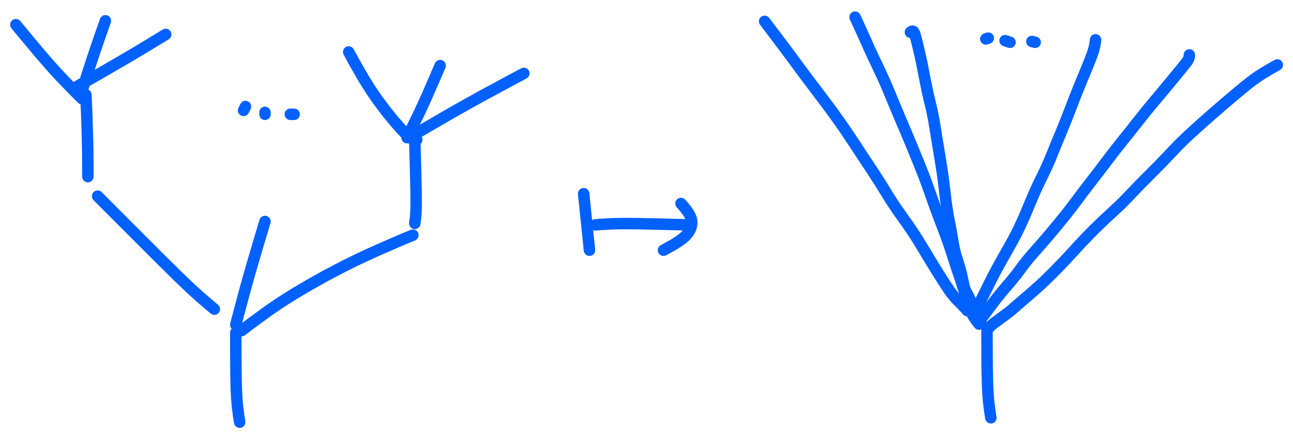
box operads

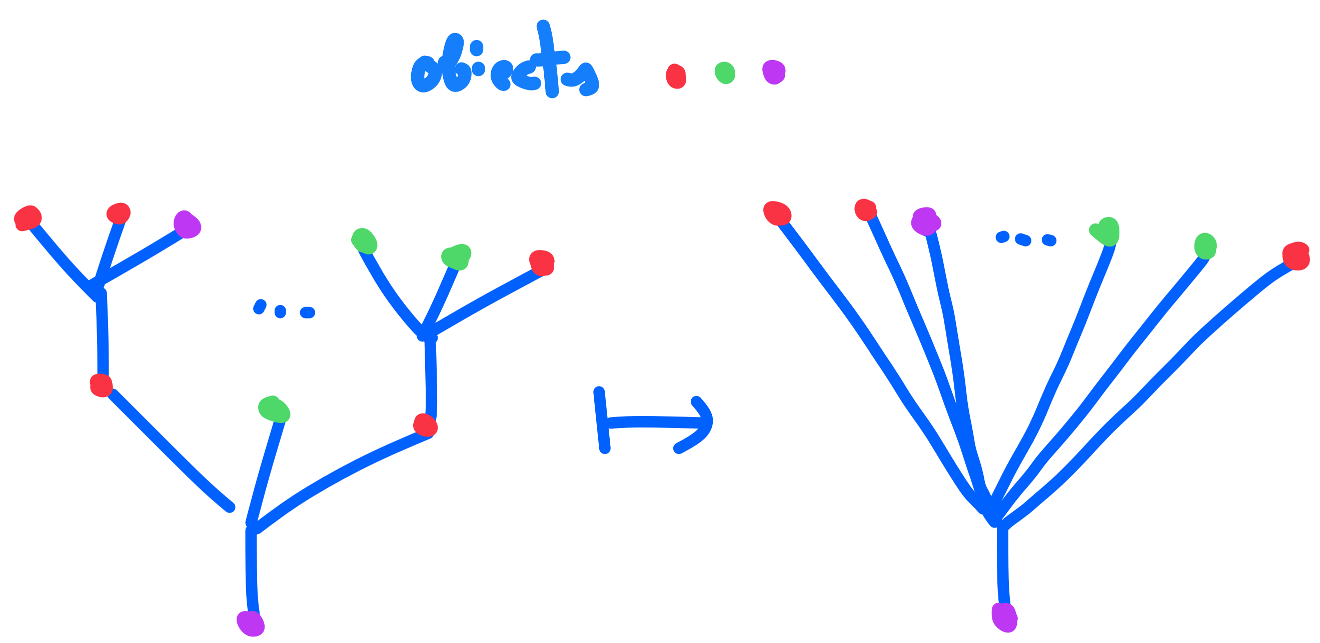
virtual double categories “=“
coloured box operads

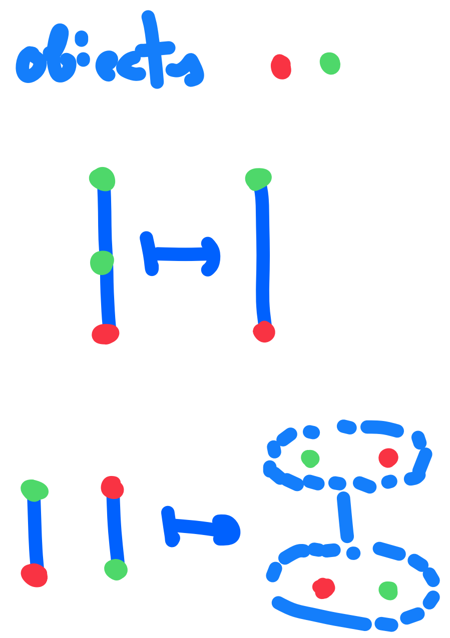
double
categories











operads

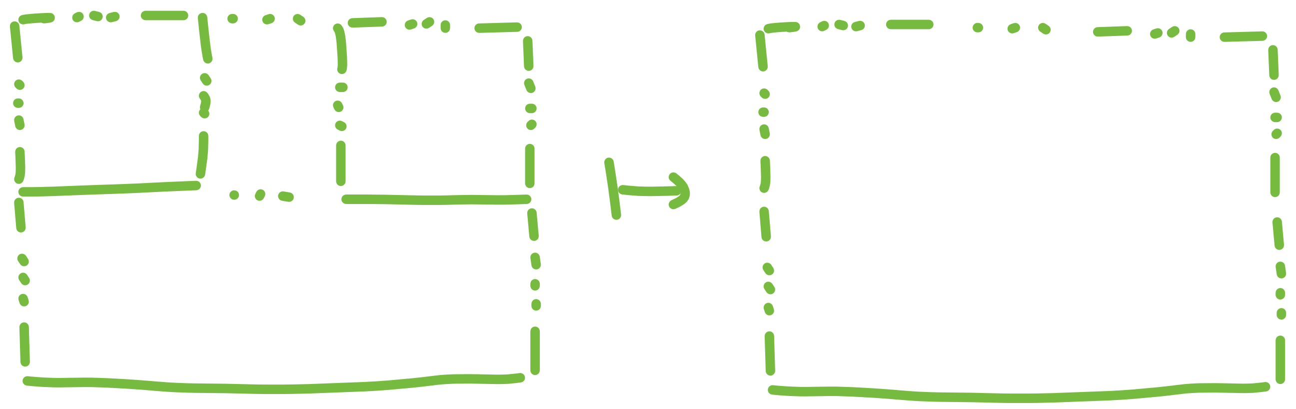
multicategories =
coloured operads

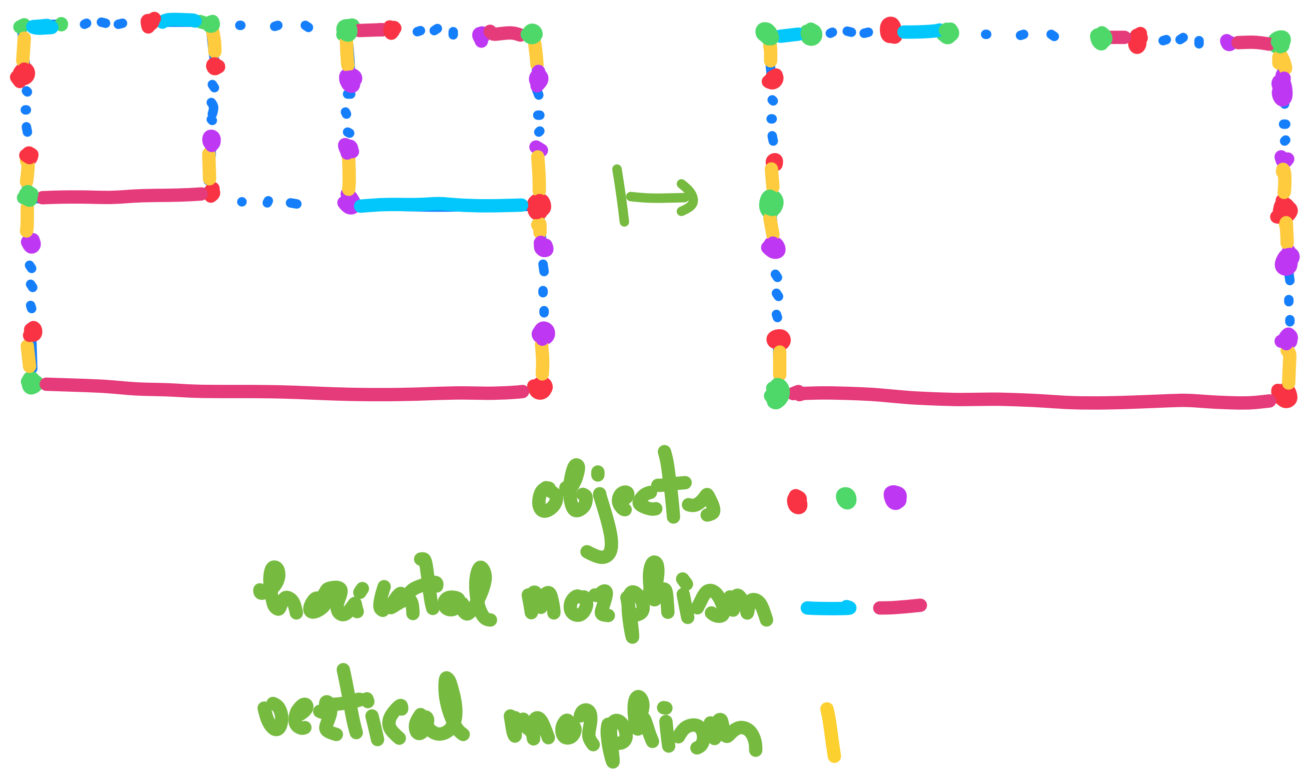
monoidal
categories

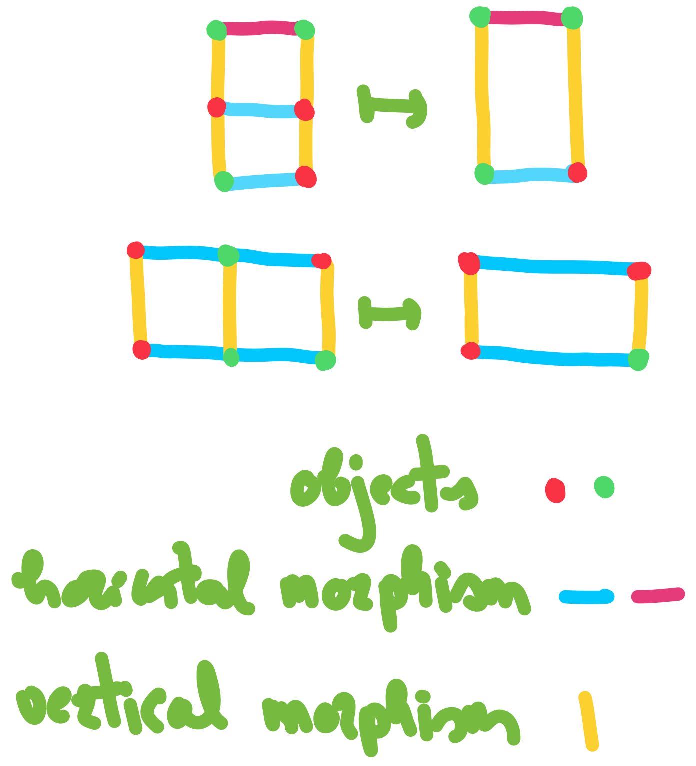
box operads

virtual double categories “=“
coloured box operads

double
categories







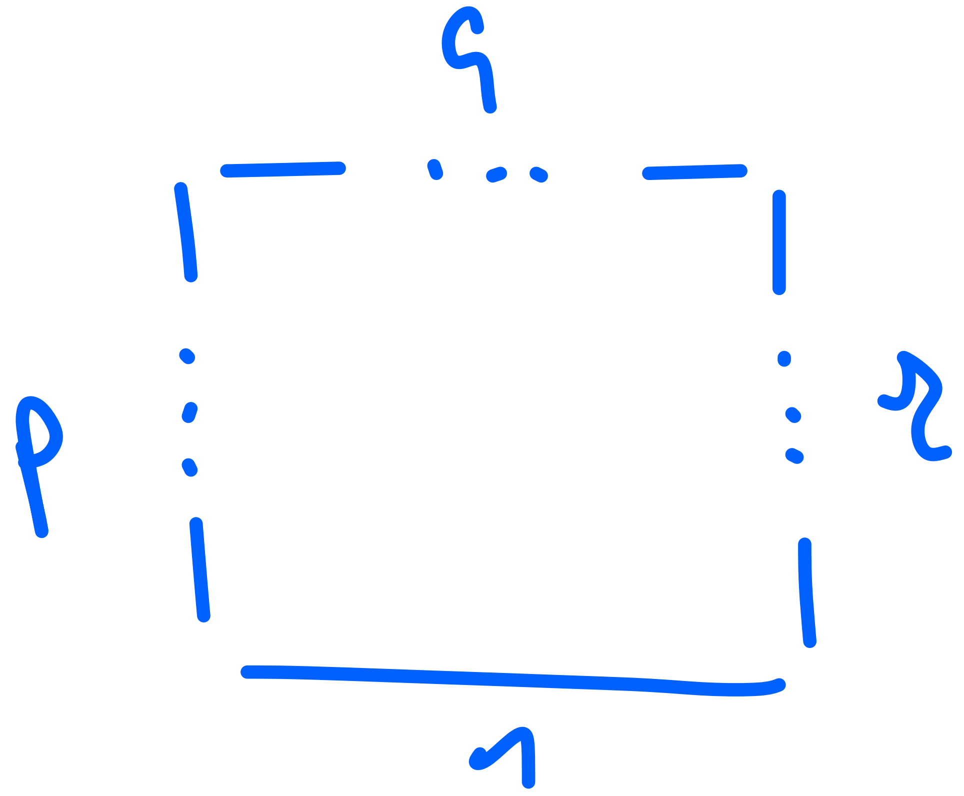
Box operad $\mathcal{B}$ enriched over $(\mathcal{V}, \otimes, I)$

Box operad $\mathcal{B}$ enriched over $(\mathcal{V}, \otimes, I)$ consists of

$\mathcal{V}$- objects $\big(\mathcal{B}(p,q,r)\big)_{p,q,r \geq 0}$

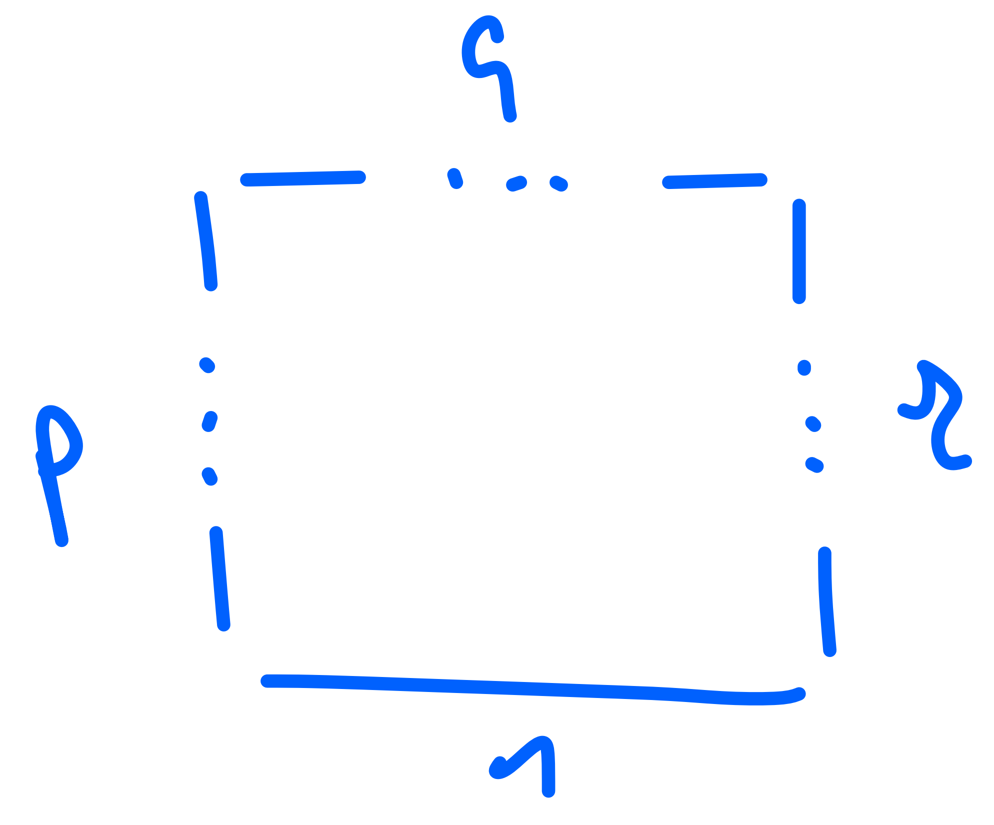
Box operad $\mathcal{B}$ enriched over $(\mathcal{V}, \otimes, I)$ consists of

$\mathcal{V}$- objects $\big(\mathcal{B}(p,q,r)\big)_{p,q,r \geq 0}$

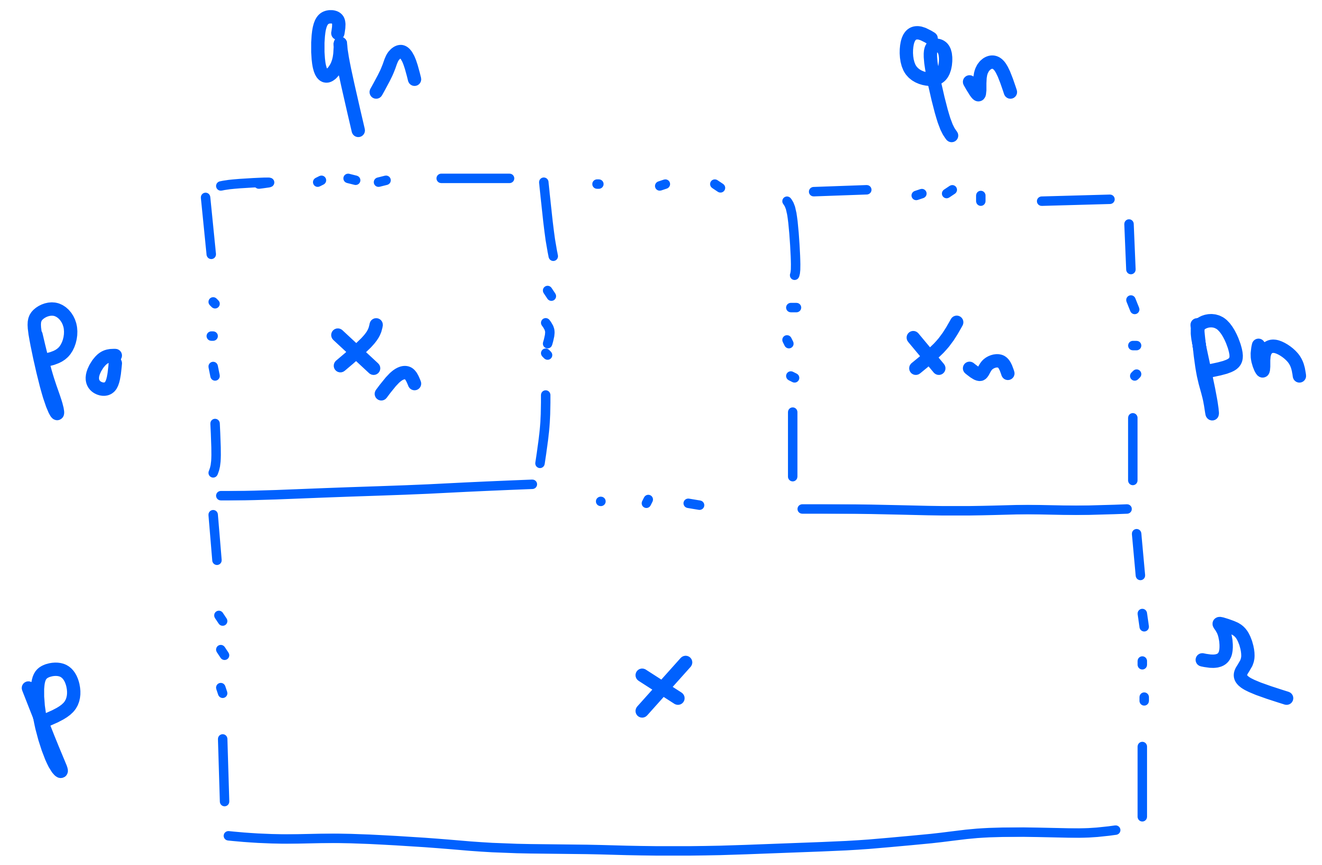


Box operad $\mathcal{B}$ enriched over $(\mathcal{V}, \otimes, I)$ consists of

$\mathcal{V}$-objects $\left(\mathcal{B}(p,q,r)\right)_{p,q,r \geq 0}$

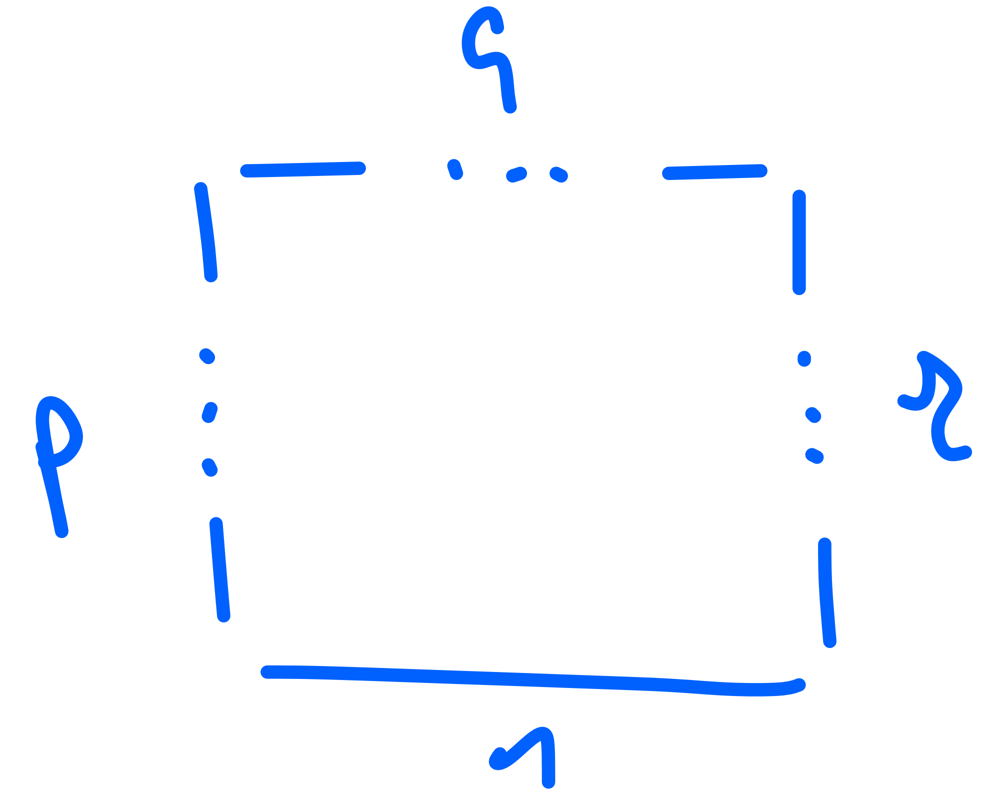


compositions

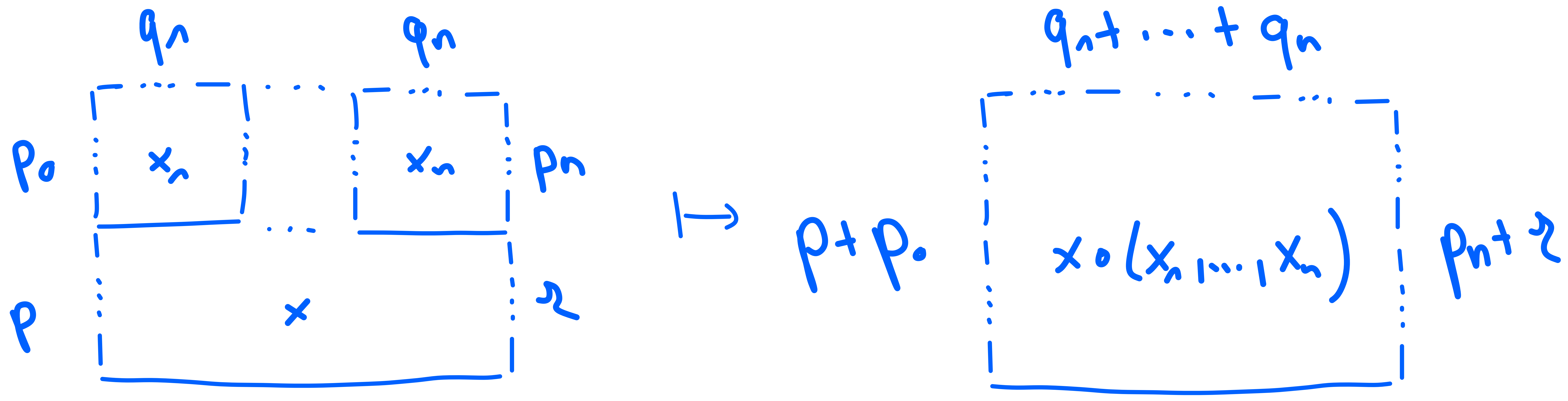


Box operad $\mathcal{B}$ enriched over $(\mathcal{V}, \otimes, I)$ consists of

$\mathcal{V}$- objects $\left(\mathcal{B}(p,q,r)\right)_{p,q,r \geq 0}$

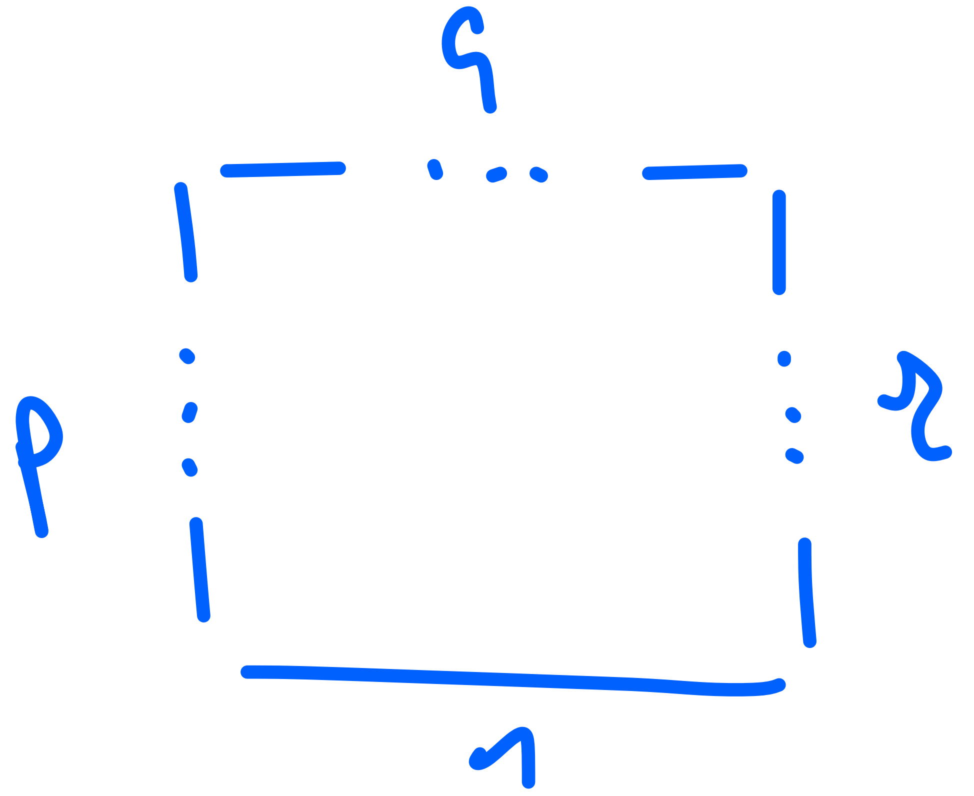


compositions

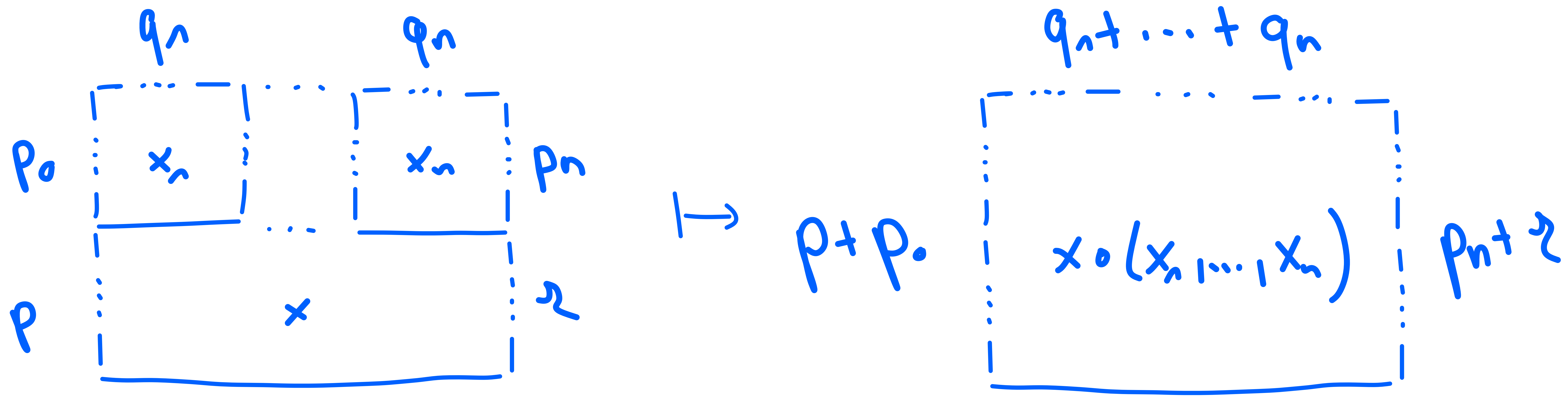


Box operad $\mathcal{B}$ enriched over $(\mathcal{V}, \otimes, I)$ consists of

$\mathcal{V}$- objects $\left(\mathcal{B}(p,q,r)\right)_{p,q,r \geq 0}$



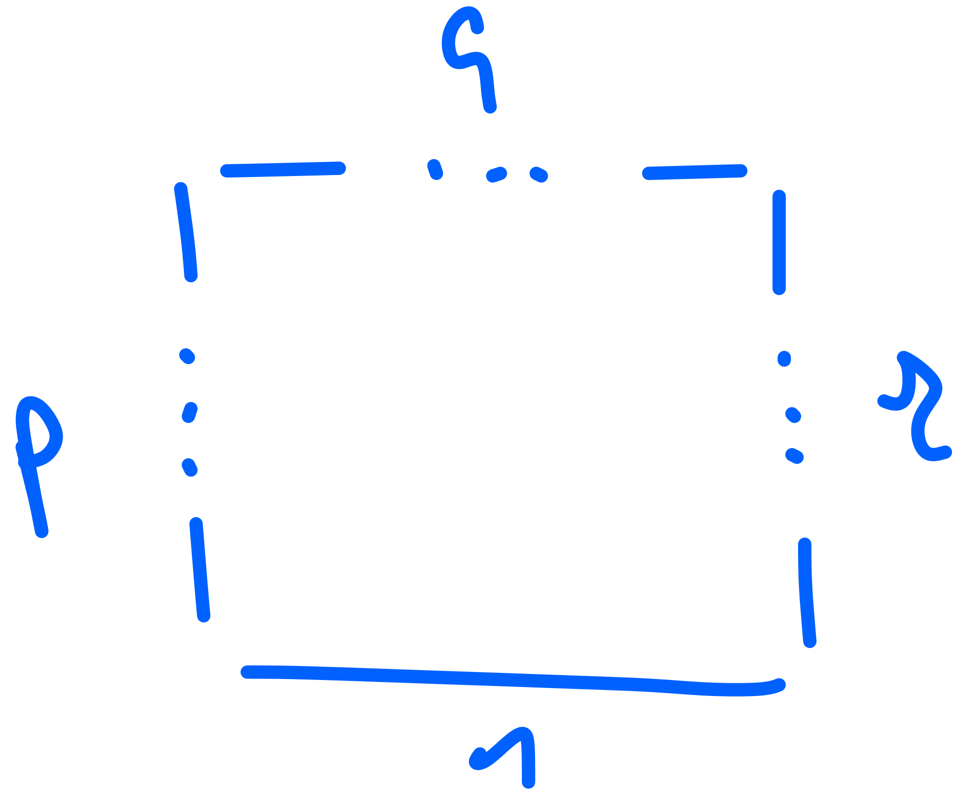
compositions



unit

Box operad $\mathcal{B}$ enriched over $(\mathcal{V},\otimes,I)$ consists of

$\mathcal{V}$-objects $\quad \big(\mathcal{B}(p,q,r)\big)_{p,q,r\geq 0}$



compositions $\quad \mathcal{B}(p,n,r)\otimes\bigotimes_{i=1}^{n}\mathcal{B}(p_{i-1},q_i,p_i)\xrightarrow{\mu}\mathcal{B}\big(p+p_0,\textstyle\sum q_i, r+p_n\big)$

unit $\quad I\xrightarrow{\eta}\mathcal{B}(0,1,0)$

Box operad $\mathcal{B}$ enriched over $(\mathcal{V}, \otimes, I)$

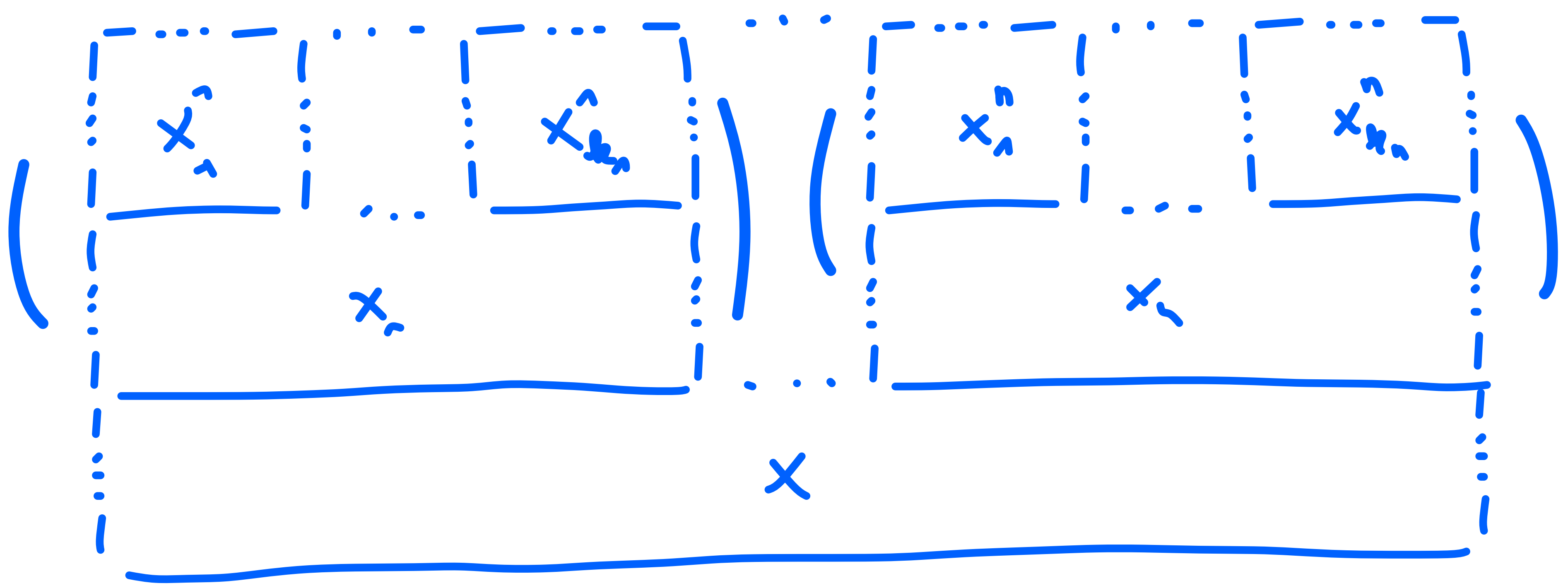
associativity

-

unit

Box operad $\mathcal{B}$ enriched over $(\mathcal{V}, \otimes, \mathcal{I})$

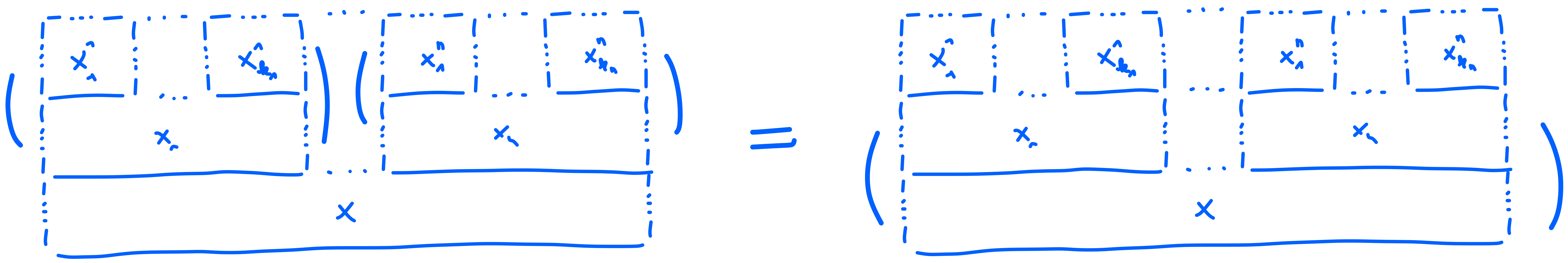
associativity



unit

Box operad $\mathcal{B}$ enriched over $(\mathcal{V}, \otimes, \mathcal{I})$

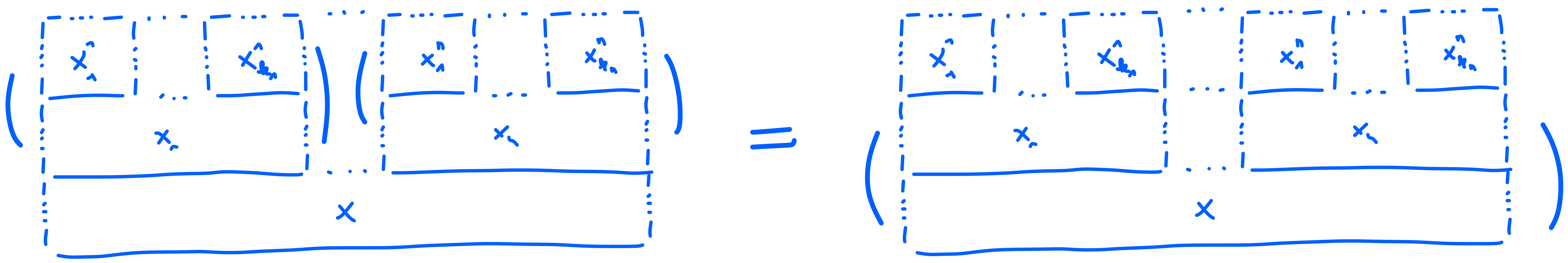
associativity



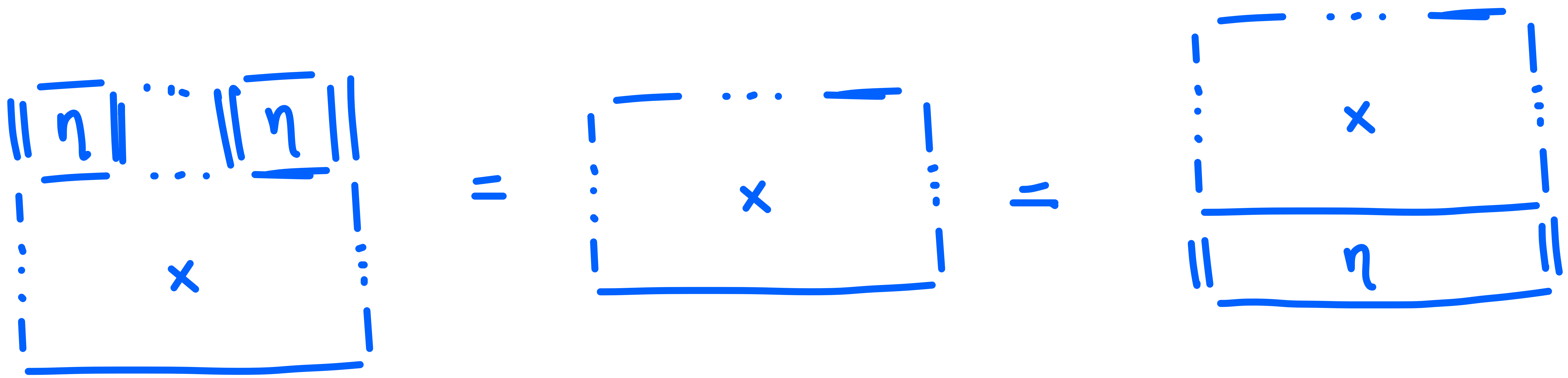
unit

Box operad $\mathcal{B}$ enriched over $(\mathcal{V}, \otimes, I)$

associativity



unit



Peculiar Degeneracies

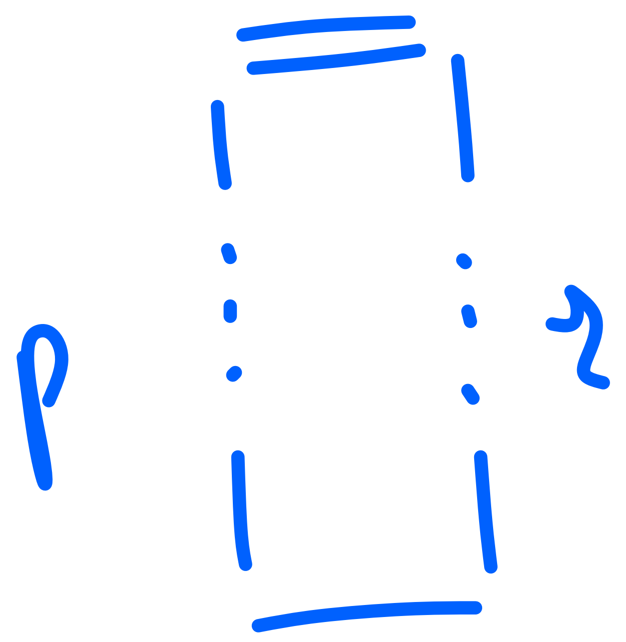
For $n = 0$

Peculiar Degeneracies

For n = 0 $\quad B(p, 0, z)$

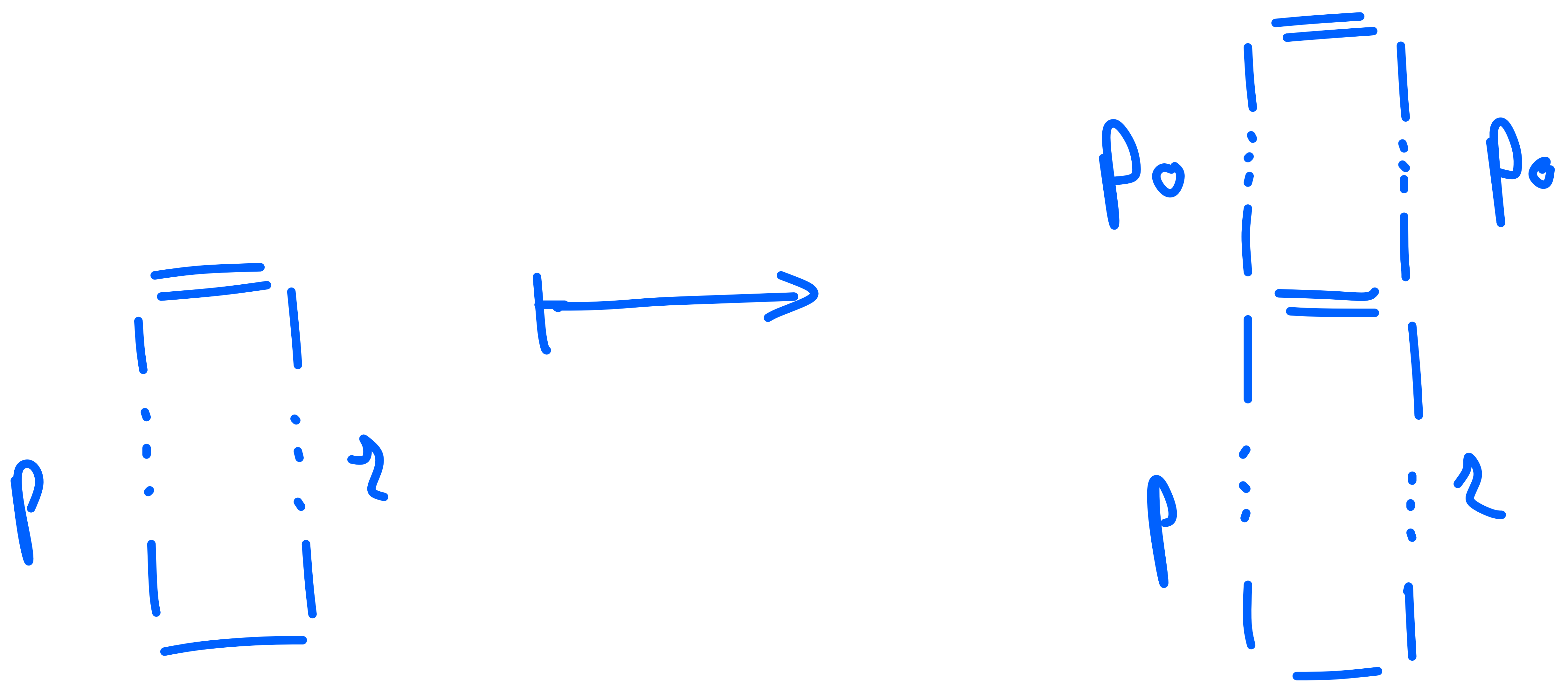
Peculiar Degeneracies

For n = 0 $\quad B(p, 0, q)$



Peculiar Degeneracies

For n = 0 $\quad B(p,0,q) \xrightarrow{\mu} B(p+p_0, 0, q+p_0)$



Operads = *thin* box operads

Operads = *thin* box operads

$$\text{BoxOperads} \hookleftarrow \text{Operads}$$

Operads = *thin* box operads

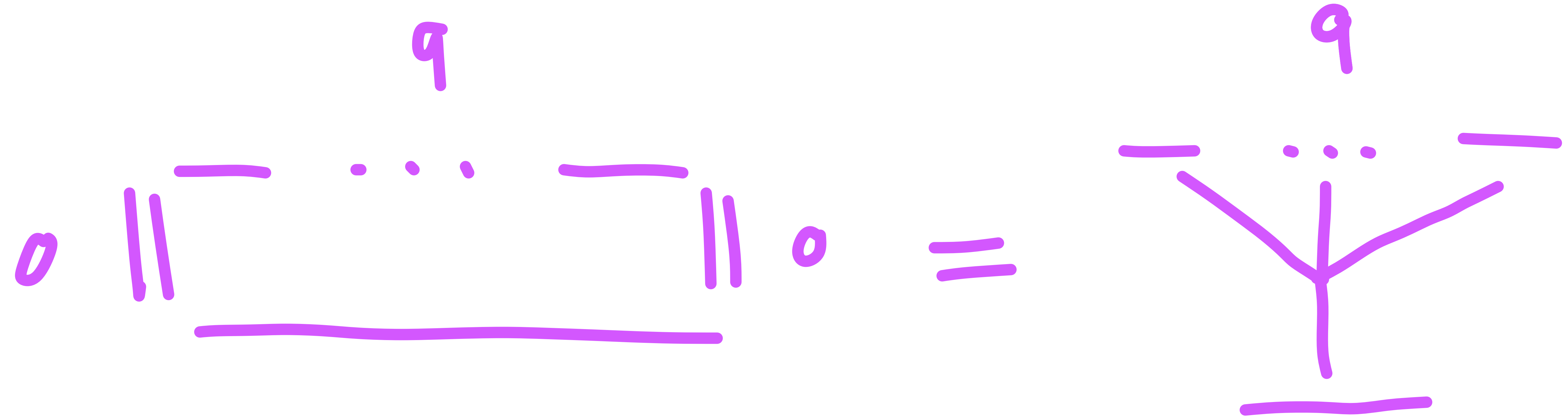
$$\mathrm{BoxOperads} \hookleftarrow \mathrm{Operads}$$

thin box

Operads = *thin* box operads

$$\mathrm{BoxOperads} \hookleftarrow \mathrm{Operads}$$

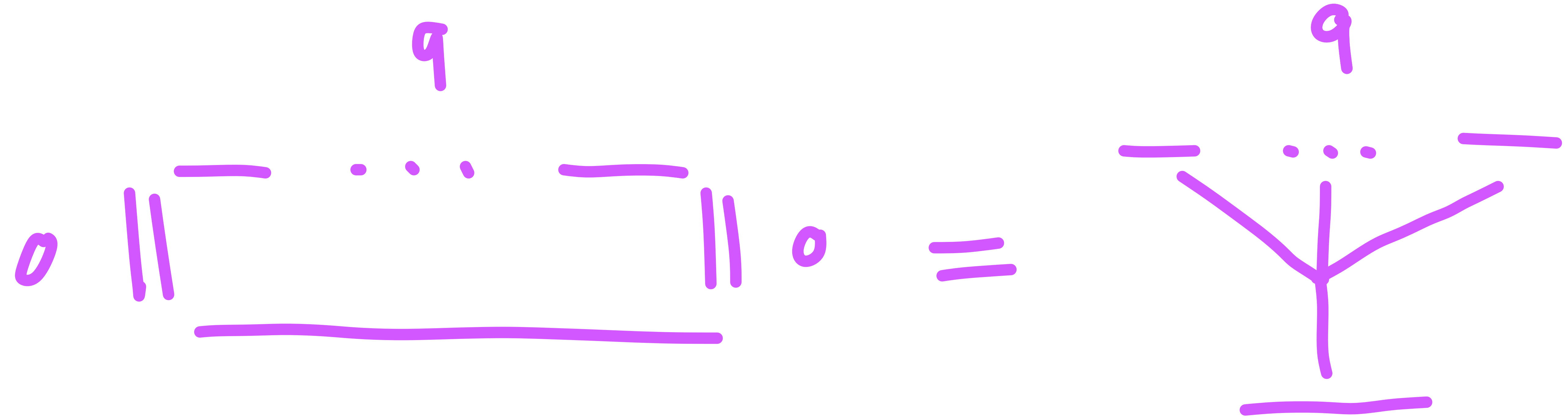
thin box



Operads = *thin* box operads

$\mathrm{BoxOperads} \hookleftarrow \mathrm{Operads}$

thin box

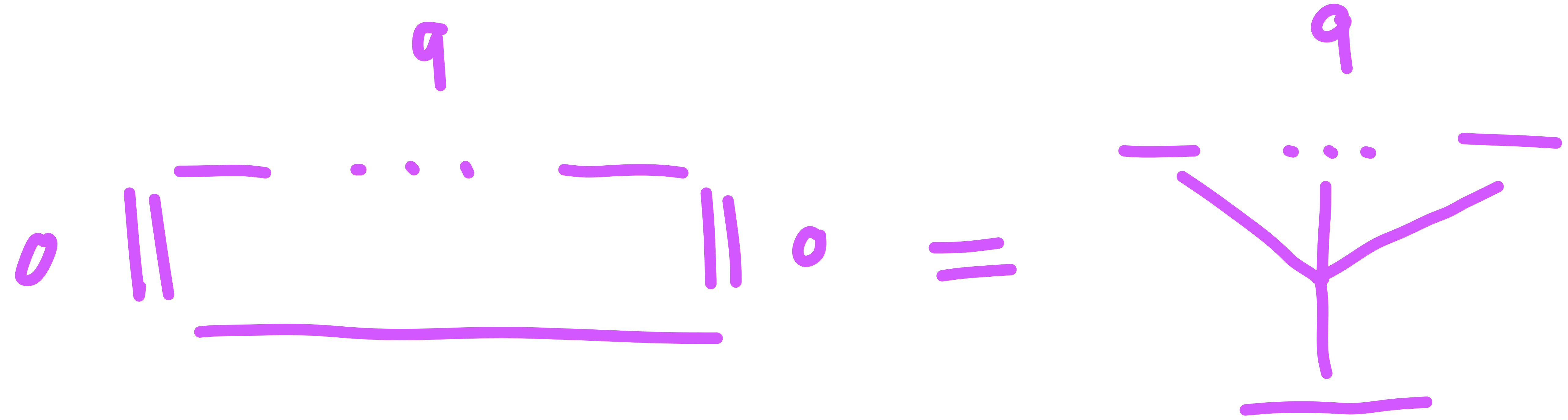


partial
composition

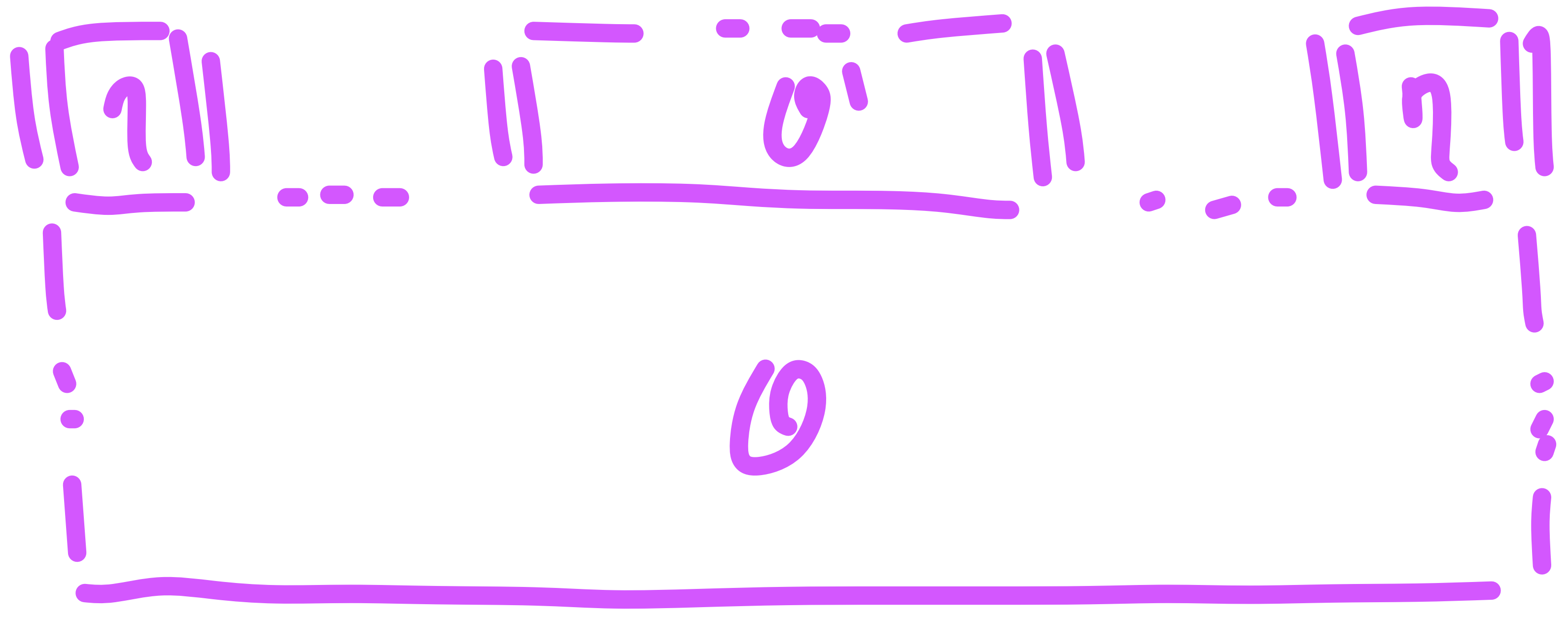
Operads = *thin* box operads

BoxOperads $\hookleftarrow$ Operads

thin box



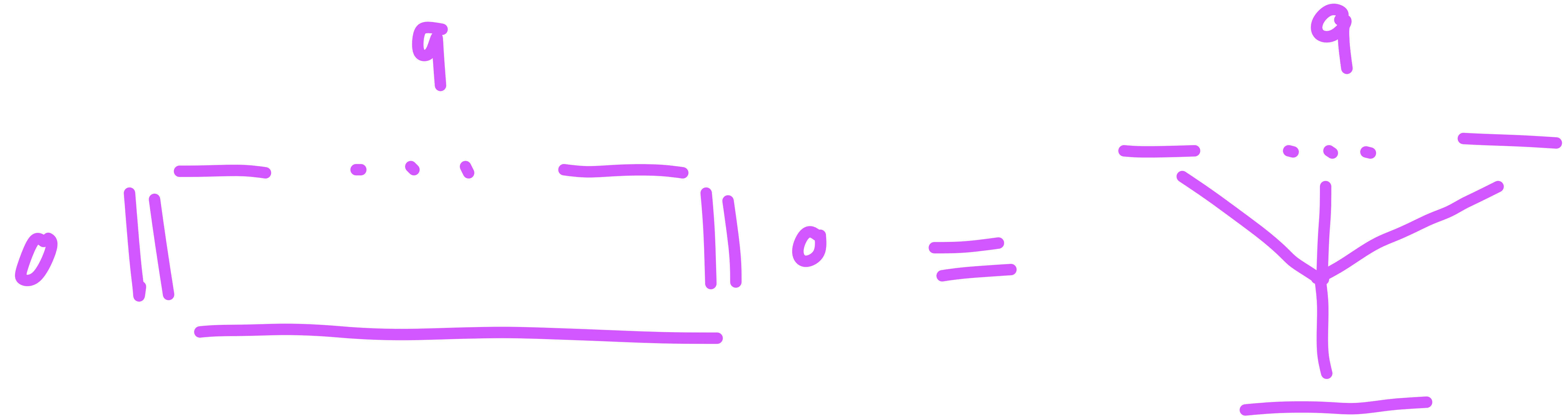
partial composition



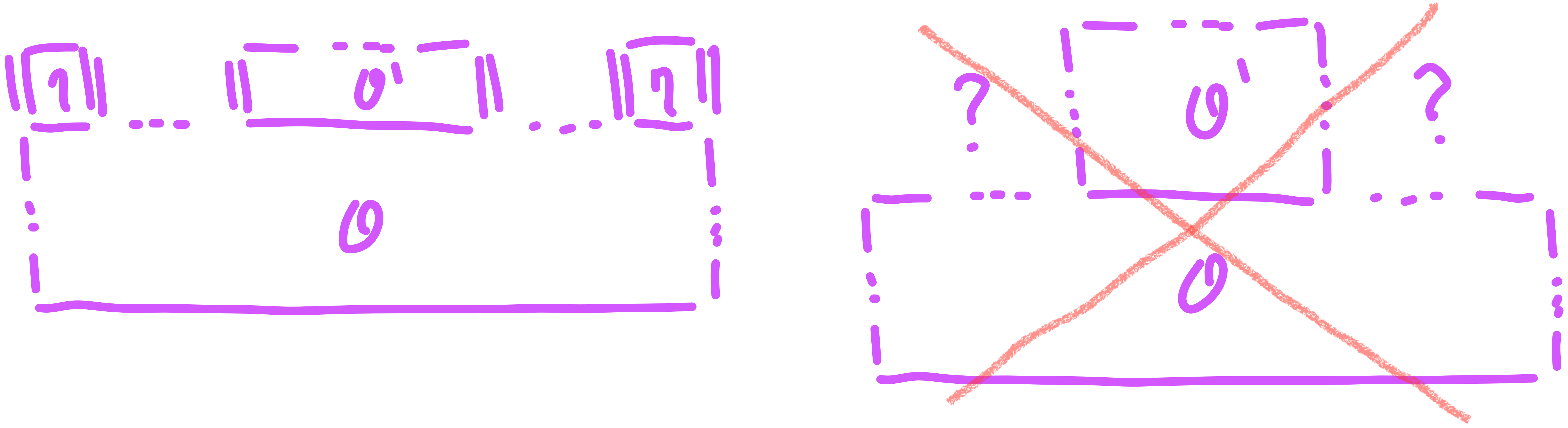
Operads = *thin* box operads

BoxOperads $\longleftrightarrow$ Operads

thin box

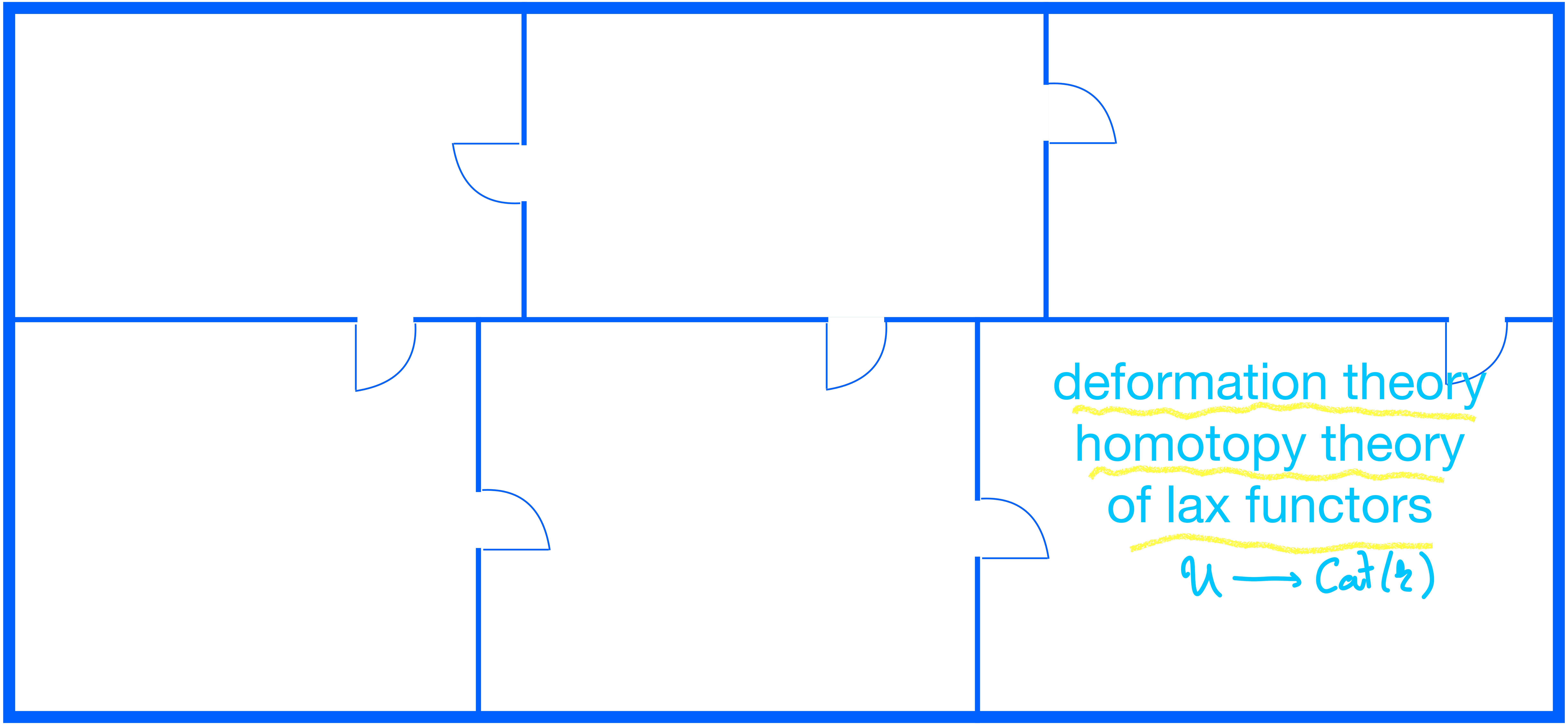


partial composition

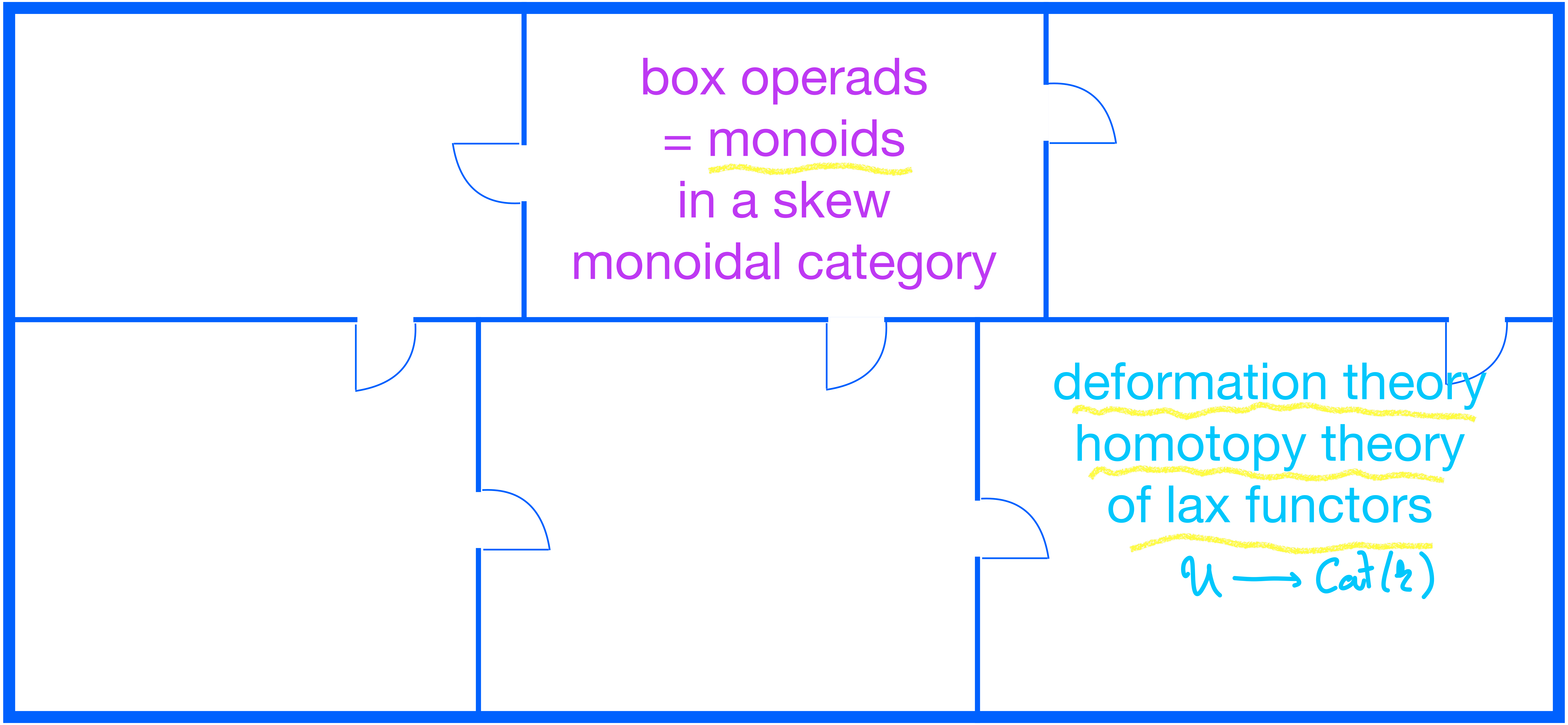


Part II

The Floorplan



The Floorplan

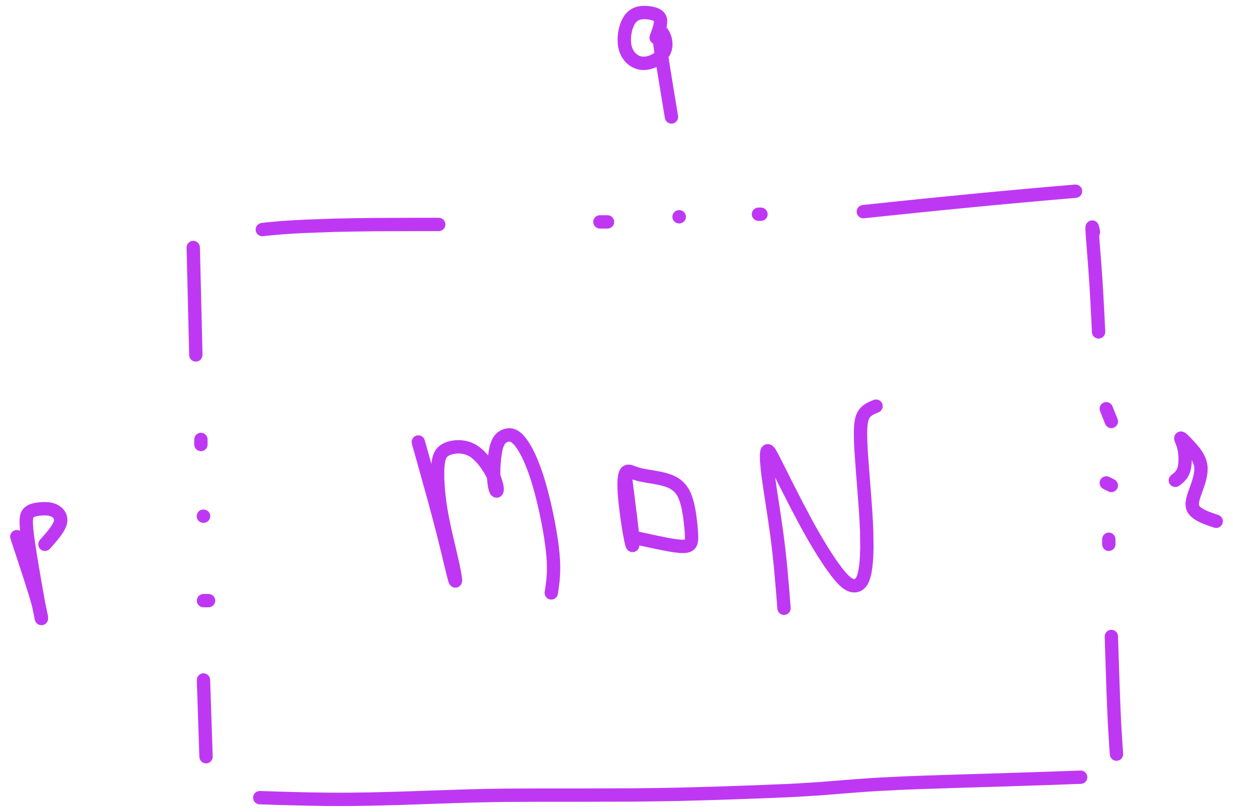


Box Composite

$$\Box : \mathcal{V}^{\mathbb{N}^3} \times \mathcal{V}^{\mathbb{N}^3} \longrightarrow \mathcal{V}^{\mathbb{N}^3}$$

Box Composite

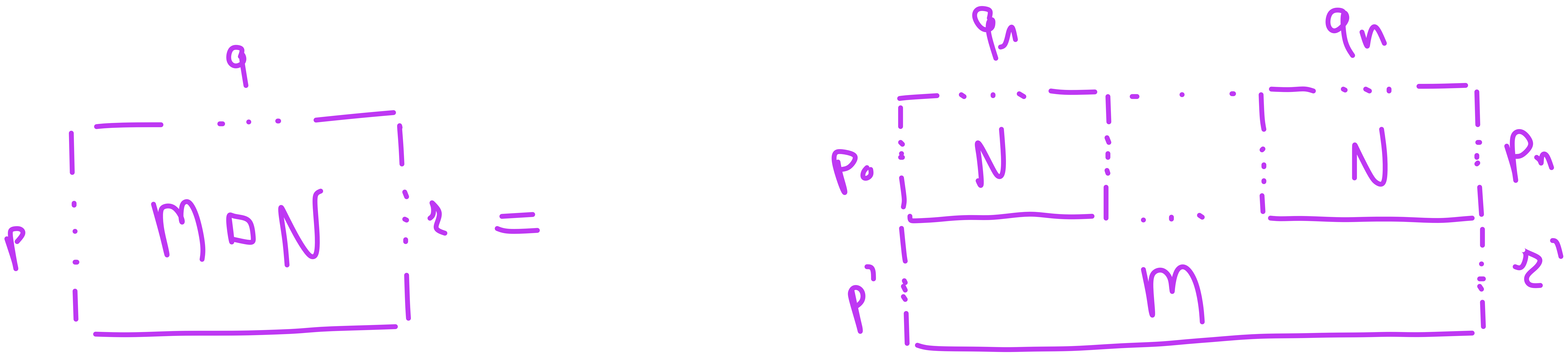
$$\Box : \mathcal{V}^{\mathbb{N}^3} \times \mathcal{V}^{\mathbb{N}^3} \longrightarrow \mathcal{V}^{\mathbb{N}^3}$$



$=$

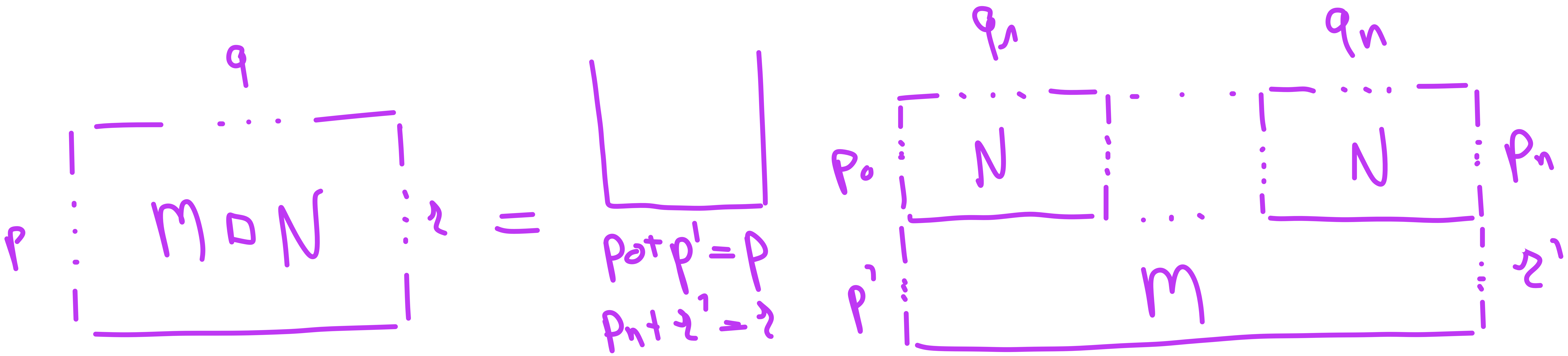
Box Composite

$$\Box : \mathcal{V}^{\mathbb{N}^3} \times \mathcal{V}^{\mathbb{N}^3} \longrightarrow \mathcal{V}^{\mathbb{N}^3}$$



Box Composite

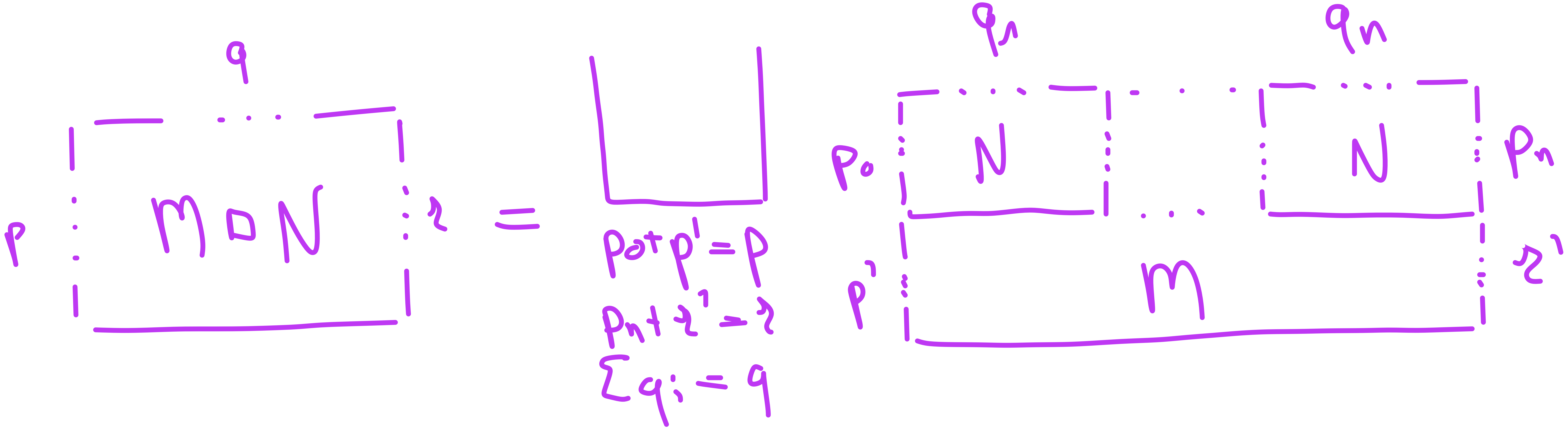
$$\square : \mathcal{V}^{\mathbb{N}^3} \times \mathcal{V}^{\mathbb{N}^3} \longrightarrow \mathcal{V}^{\mathbb{N}^3}$$



$$p_0 + p' = p$$
$$p_n + r' = r$$
$$\sum q_i = q$$

Box Composite

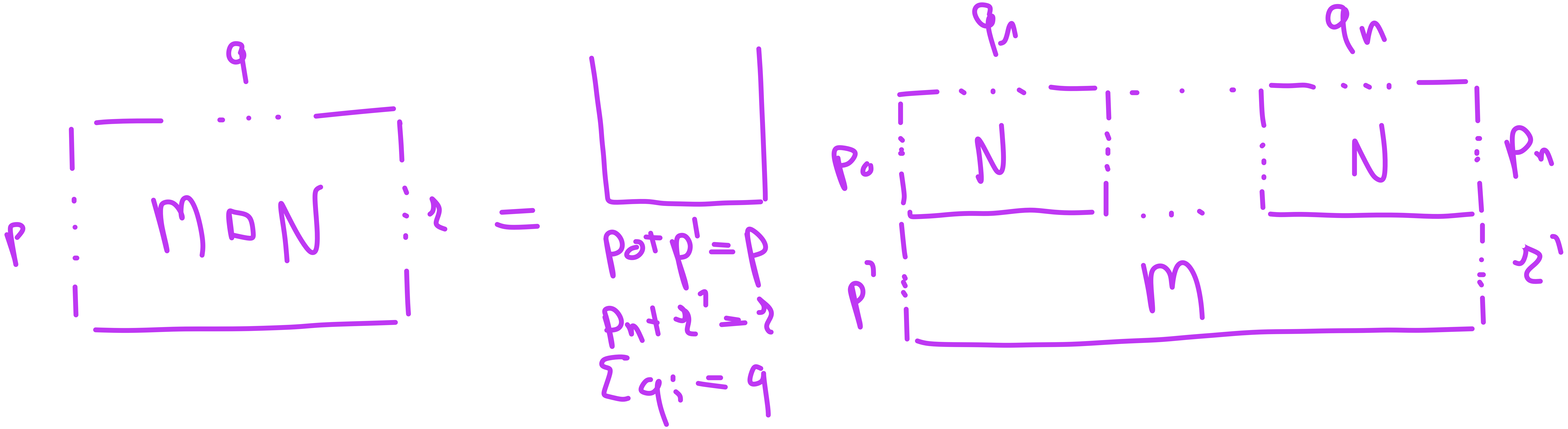
$$\Box : \mathcal{V}^{\mathbb{N}^3} \times \mathcal{V}^{\mathbb{N}^3} \longrightarrow \mathcal{V}^{\mathbb{N}^3}$$



Proposition $(\mathcal{V}^{\mathbb{N}^3}, \Box, I)$ is a skew monoidal category

Box Composite

$$\square : \mathcal{V}^{\mathbb{N}^3} \times \mathcal{V}^{\mathbb{N}^3} \longrightarrow \mathcal{V}^{\mathbb{N}^3}$$



Proposition $(\mathcal{V}^{\mathbb{N}^3}, \square, I)$ is a skew monoidal category

concentrated in $(0,1,0)$.

(left normal)

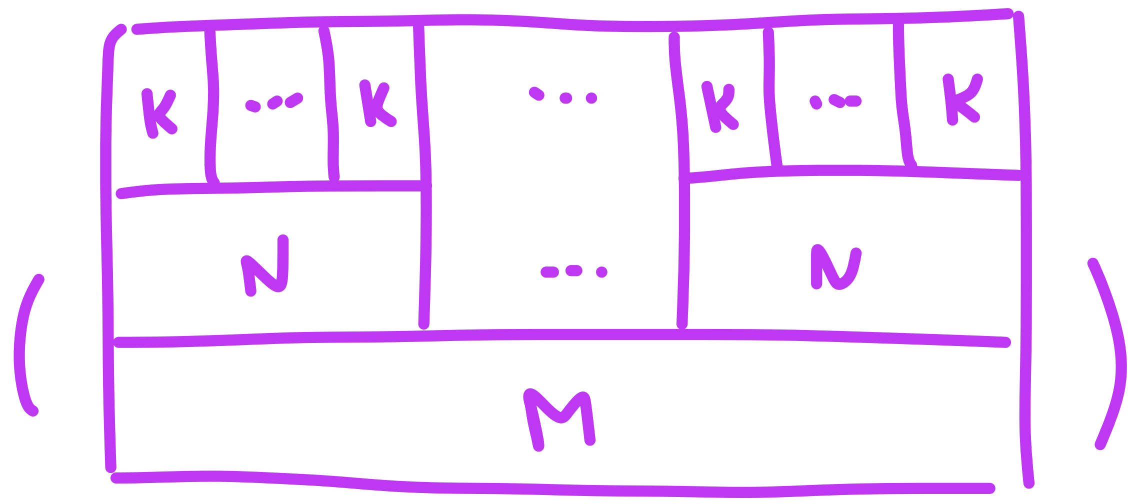
Proposition $\left(\mathcal{V}^{\mathbb{N}^3}, \square, \mathbb{I}\right)$ is a skew monoidal category

Proposition $(\mathcal{V}^{\mathbb{N}^3}, \square, I)$ is a skew monoidal category (left normal)

$$\begin{array}{c} (M \square N) \square K \\ \big\downarrow \\ M \square (N \square K) \end{array}$$

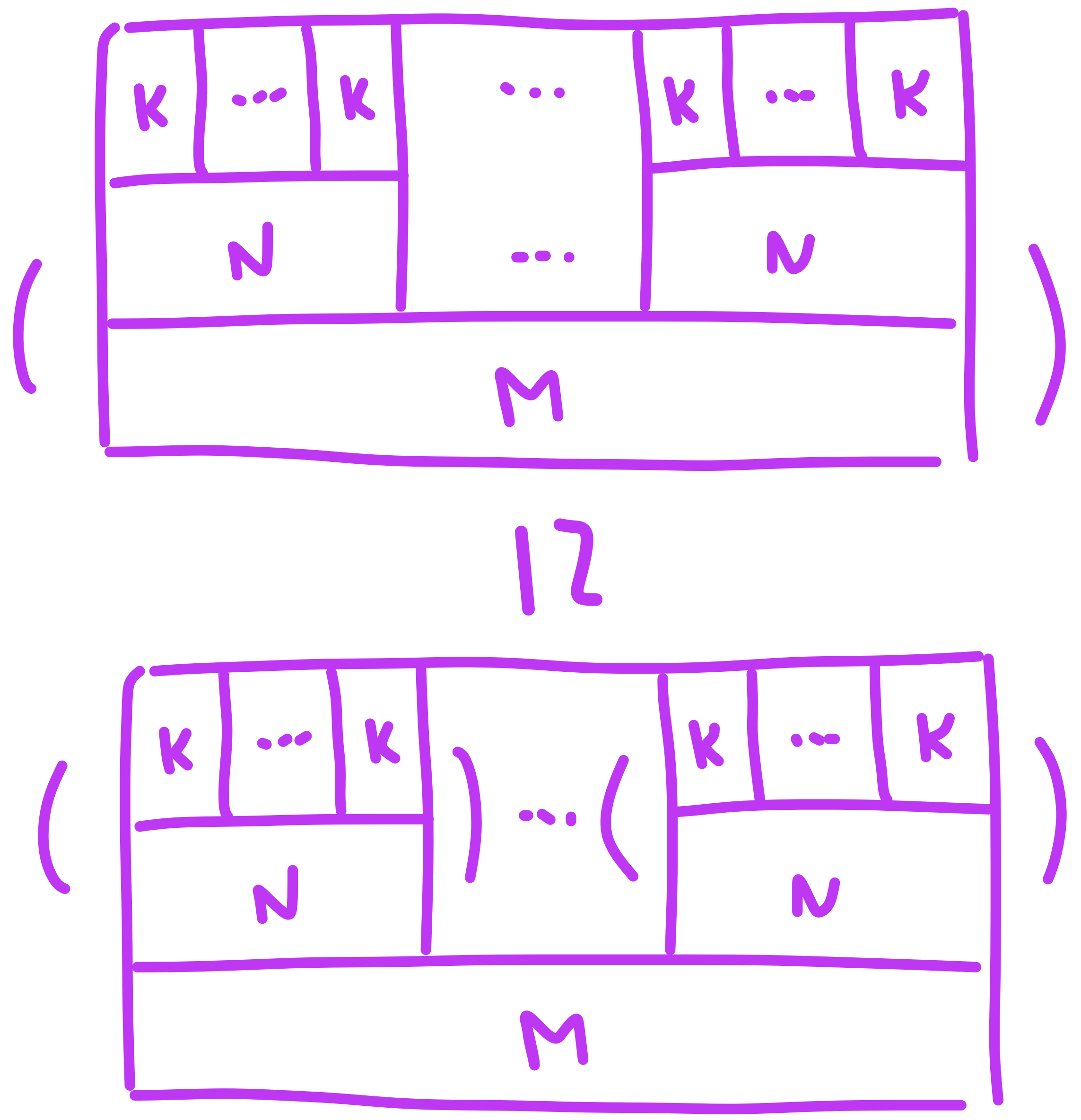
Proposition $\left(\mathcal{V}^{\mathbb{N}^3}, \square, I\right)$ (left normal) is a skew monoidal category

$$
\begin{array}{c}
(M \square N) \square K \\
\downarrow \\
M \square (N \square K)
\end{array}
$$



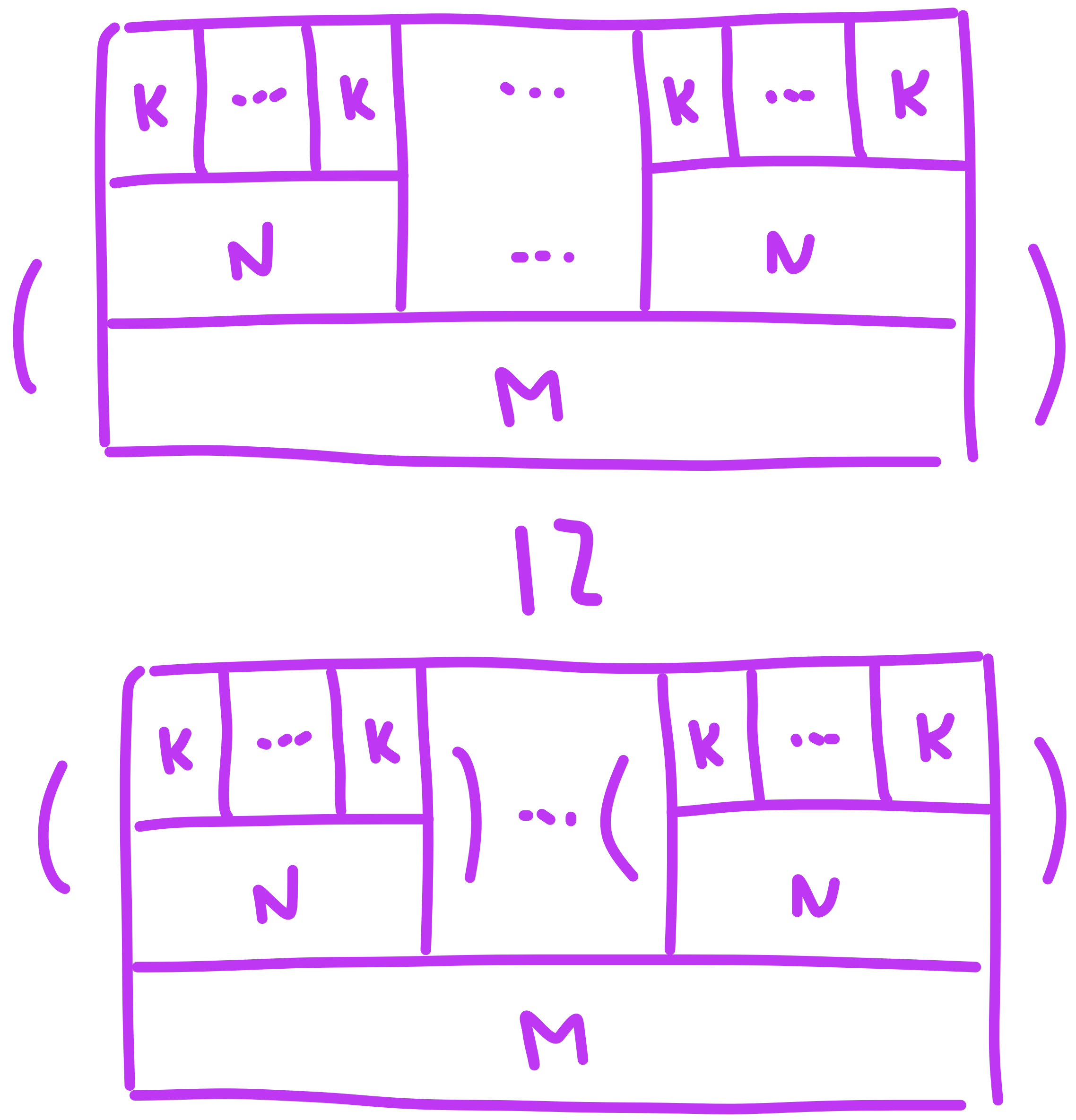
Proposition $(\mathcal{V}^{\mathbb{N}^3}, \square, I)$ (left normal) is a skew monoidal category

$$\begin{array}{c} (M \square N) \square K \\ \downarrow \\ M \square (N \square K) \end{array}$$



Proposition $(\mathcal{V}^{\mathbb{N}^3}, \square, I)$ is a skew monoidal category (left normal)

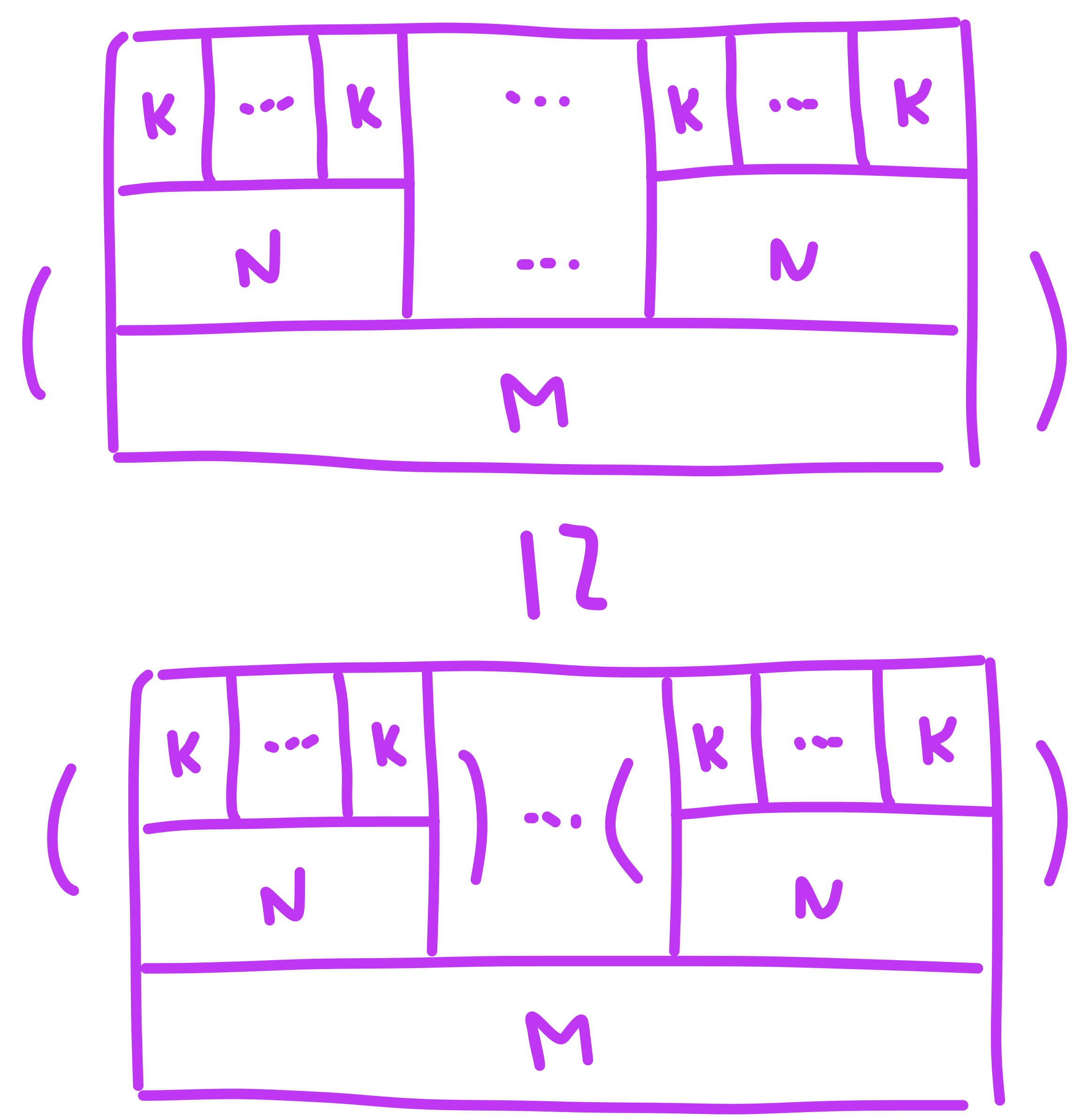
$$\begin{tikzcd}
(M \square N) \square K \arrow[d, shift left] \\
M \square (N \square K) \arrow[u, shift left, "\times" marking]
\end{tikzcd}$$



(left normal)

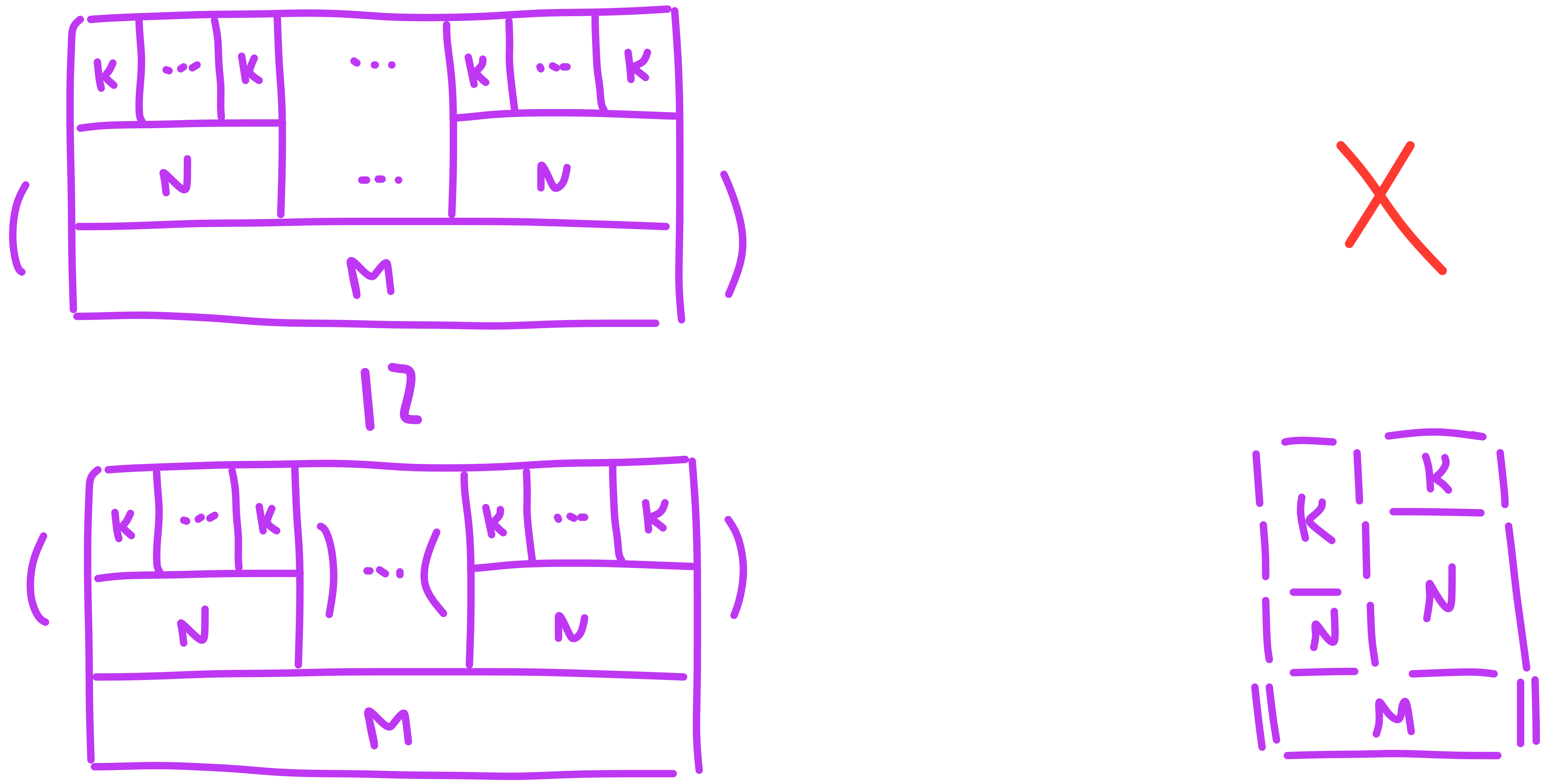
Proposition $(\mathcal{V}^{\mathbb{N}^3}, \square, I)$ is a skew monoidal category

$$\begin{array}{c}(M \square N) \square K \\ \downarrow \quad \not\uparrow \\ M \square (N \square K)\end{array}$$



Proposition $(\mathcal{V}^{\mathbb{N}^3}, \square, I)$ (left normal) is a skew monoidal category

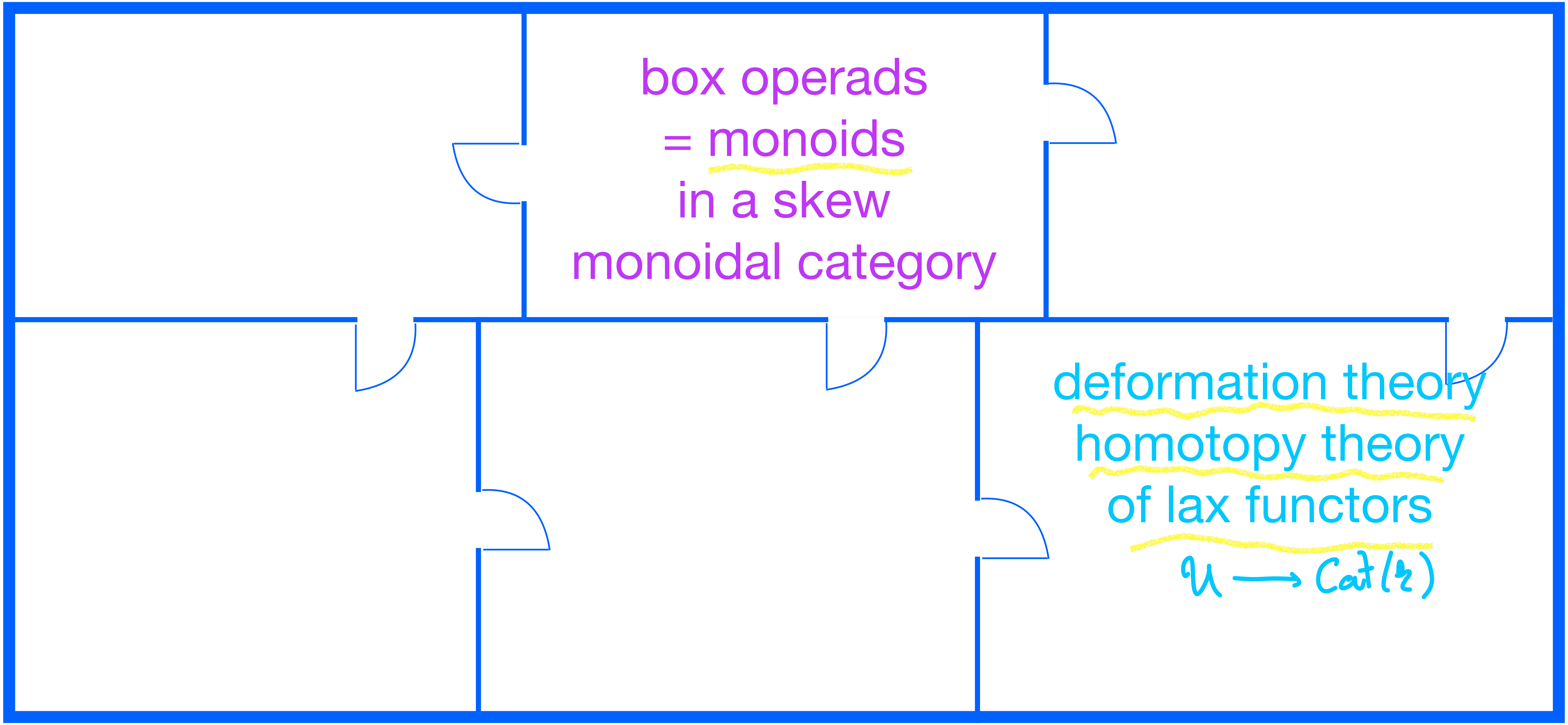
$$(M \square N) \square K \longrightarrow M \square (N \square K)$$



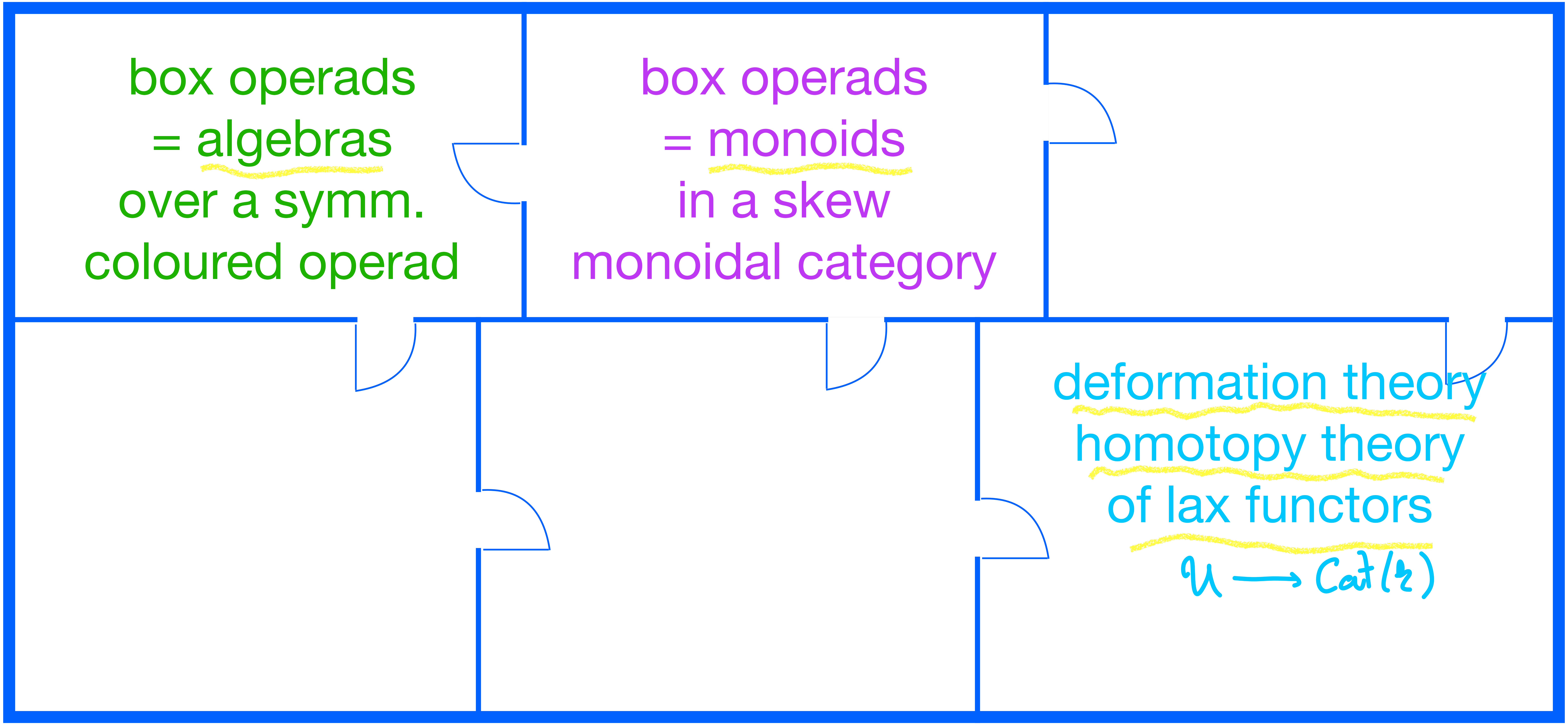
Proposition box operads are monoids in $(\mathcal{V}^{\mathbb{N}^3}, \square, I)$

$$B \square B \xrightarrow{m} B \qquad I \xrightarrow{\eta} B$$

The Floorplan



The Floorplan



The Floorplan

box operads
= algebras
over a symm.
coloured operad

box operads
= monoids
in a skew
monoidal category

totalisation
of a box operad
carries a
L-infinity structure

deformation theory
homotopy theory
of lax functors
$\mathcal{U} \longrightarrow \mathrm{Cat}(\Bbbk)$

Stackings

Stackings

a stacking S takes a sequence of boxes

Stackings

a stacking S takes a sequence of boxes

Stackings

a stacking S takes a sequence of boxes

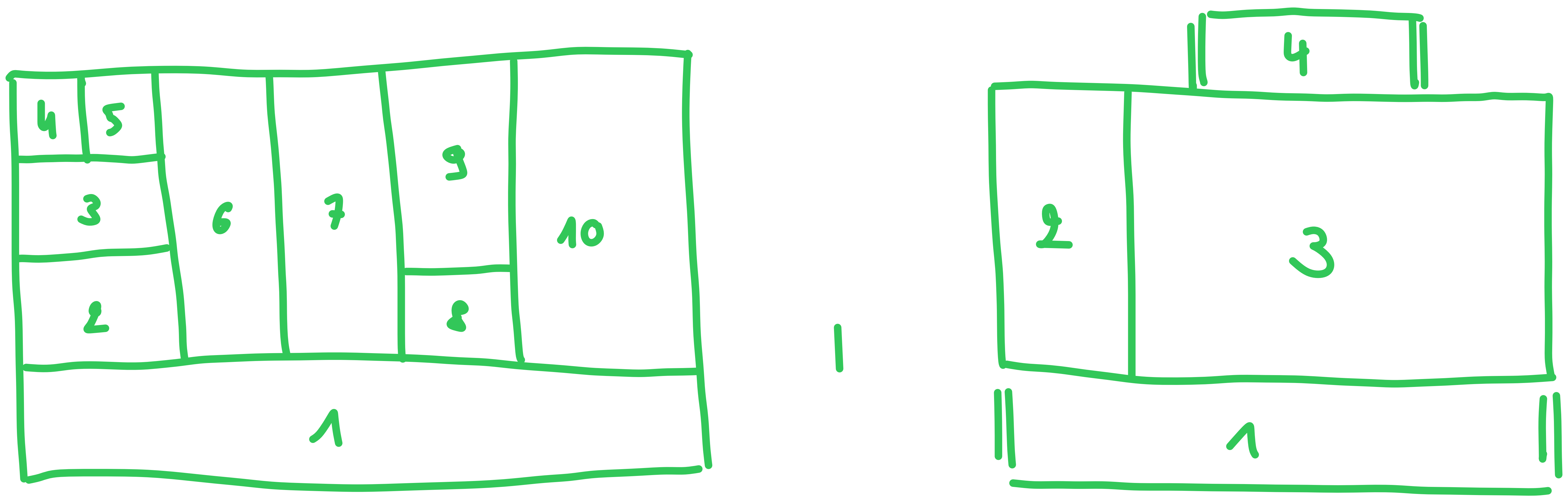
and stacks them to form a new box

Stackings

a stacking S takes a sequence of boxes

and stacks them to form a new box

e.g.

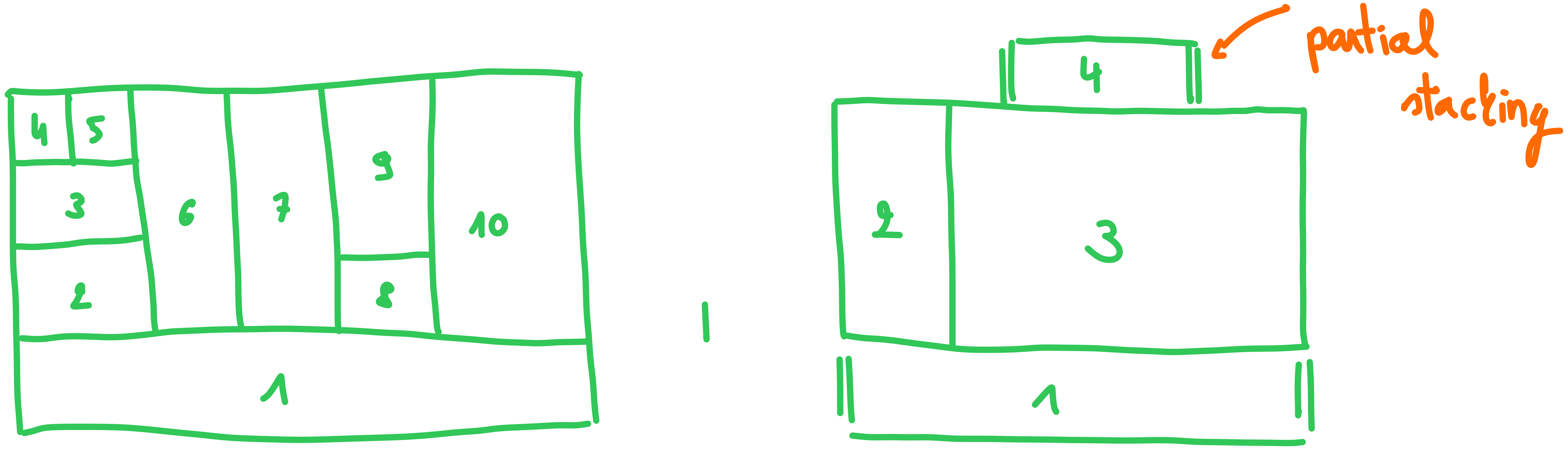


Stackings

a stacking S takes a sequence of boxes

and stacks them to form a new box

e.g.



Coloured symmetric operad $\square_p$

Coloured symmetric operad $\square p$

objects $\mathbb{N}^3$

Coloured symmetric operad $\square p$

objects $\mathbb{N}^3$

$\square p((p_1, q_1, r_1), \ldots, (p_n, q_n, r_n); (p, q, r)) =$

Coloured symmetric operad $\square p$

objects $\mathbb{N}^3$

$$\square p\big((p_1,q_1,r_1),\dots,(p_n,q_n,r_n);(p,q,r)\big) = \left\{ \begin{array}{l} \text{stacking of boxes} \\ {}_{p_1}\overset{q_1}{\square}_{r_1}, \dots, {}_{p_n}\overset{q_n}{\square}_{r_n} \\ \text{into a box } {}_{p}\overset{q}{\square}_{r} \end{array} \right\}$$

Coloured symmetric operad $\square_p$

objects $\mathbb{N}^3$

$$\square_p\big((p_1,q_1,r_1),\dots,(p_n,q_n,r_n);(p,q,r)\big) = \left\{ \begin{array}{l} \text{stacking of boxes} \\ p_1\,\overset{q_1}{\square}\,r_1\ ,\ \dots\ ,\ p_n\,\overset{q_n}{\square}\,r_n \\ \text{into a box} \quad p\,\overset{q}{\square}\,r \end{array} \right\}$$

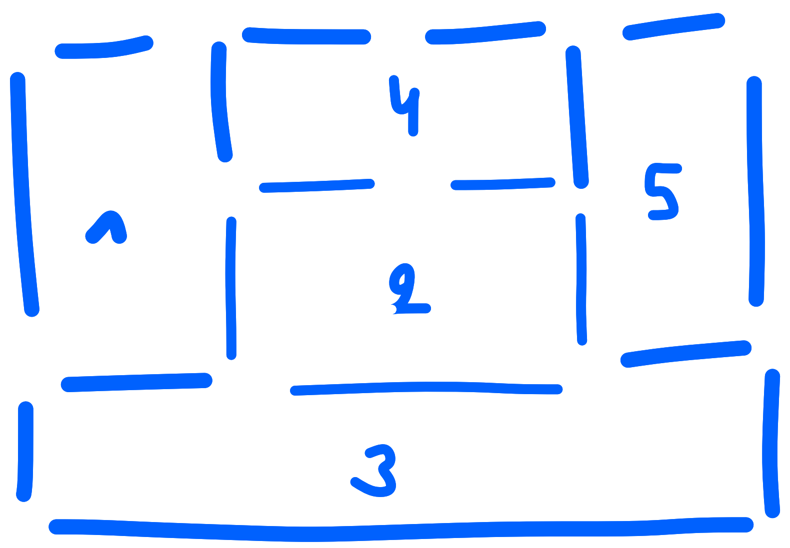
composition = substitution of a box by a stacking

Coloured symmetric operad $\square_p$

objects $\mathbb{N}^3$

$$\square_p\big((p_1,q_1,r_1),\dots,(p_n,q_n,r_n);(p,q,r)\big) = \left\{ \begin{array}{c} \text{stacking of boxes} \\ {}_{p_1}\overset{q_1}{\square}_{r_1}, \dots, {}_{p_n}\overset{q_n}{\square}_{r_n} \\ \text{into a box } {}_{p}\overset{q}{\square}_{r} \end{array} \right\}$$

composition = substitution of a box by a stacking

e.g. 

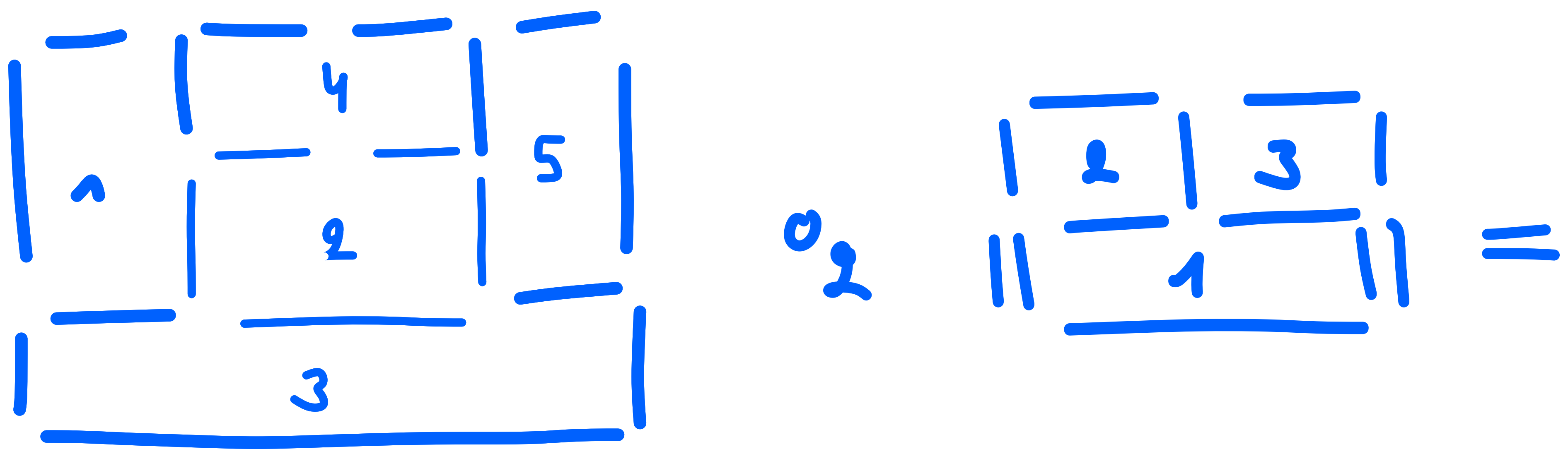
Coloured symmetric operad $\square_p$

objects $\mathbb{N}^3$

$$\square_p\big((p_1,q_1,r_1),\dots,(p_n,q_n,r_n);(p,q,r)\big) = \left\{ \begin{array}{l} \text{stacking of boxes} \\ {}_{p_1}\overset{q_1}{\square}{}_{r_1}, \dots, {}_{p_n}\overset{q_n}{\square}{}_{r_n} \\ \text{into a box } {}_{p}\overset{q}{\square}{}_{r} \end{array} \right\}$$

composition = substitution of a box by a stacking

e.g.

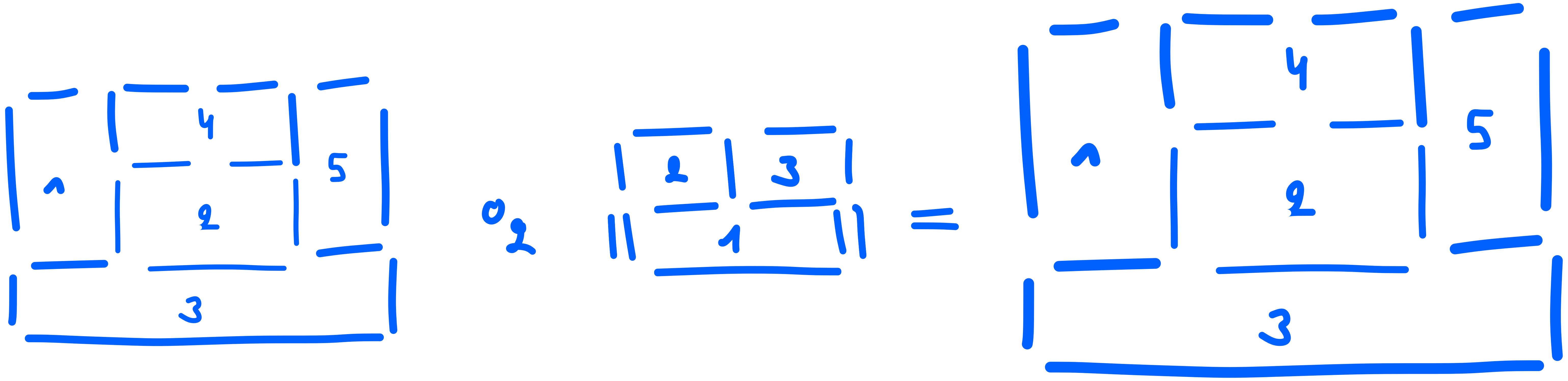


Coloured symmetric operad $\square p$

objects $\mathbb{N}^3$

$$\square p\big((p_1,q_1,r_1),\dots,(p_n,q_n,r_n);(p,q,r)\big) = \left\{ \begin{array}{l} \text{stacking of boxes } {}_{p_1}\overset{q_1}{\square}{}_{r_1}, \dots, {}_{p_n}\overset{q_n}{\square}{}_{r_n} \\ \text{into a box } {}_{p}\overset{q}{\square}{}_{r} \end{array} \right\}$$

composition = substitution of a box by a stacking

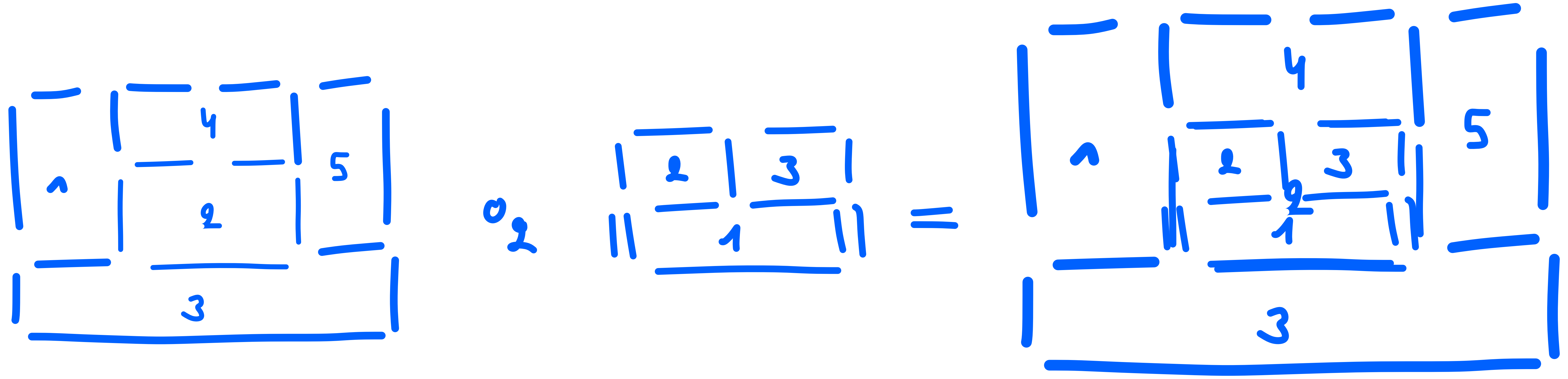
e.g. 

Coloured symmetric operad $\Box_p$

objects $\mathbb{N}^3$

$$\Box_p\big((p_1,q_1,r_1),\dots,(p_n,q_n,r_n);(p,q,r)\big) = \left\{ \begin{array}{c} \text{stacking of boxes} \\ {}_{p_1}\overset{q_1}{\boxed{}}{}_{r_1}\ ,\ \dots\ ,\ {}_{p_n}\overset{q_n}{\boxed{}}{}_{r_n} \\ \text{into a box}\quad {}_{p}\overset{q}{\boxed{}}{}_{r} \end{array} \right\}$$

composition = substitution of a box by a stacking

e.g. 

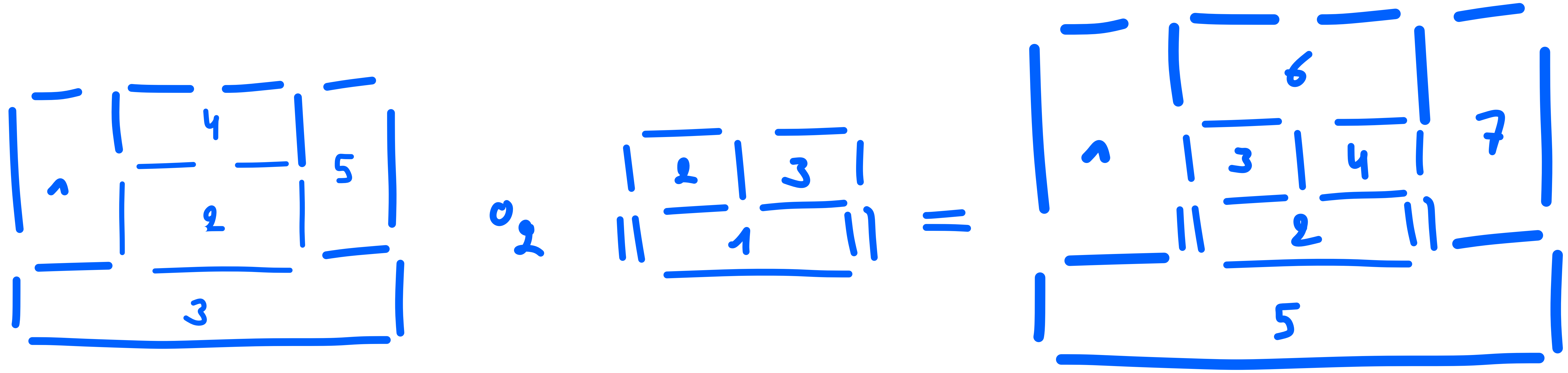
Coloured symmetric operad $\square_p$

objects $\mathbb{N}^3$

$$\square_p\Big((p_1,q_1,r_1),\dots,(p_n,q_n,r_n);(p,q,r)\Big) = \left\{ \begin{array}{l} \text{stacking of boxes} \\ {}_{p_1}\overset{q_1}{\square}_{r_1}\ ,\ \dots\ ,\ {}_{p_n}\overset{q_n}{\square}_{r_n} \\ \text{into a box}\quad {}_{p}\overset{q}{\square}_{r} \end{array} \right\}$$

composition = substitution of a box by a stacking

e.g.



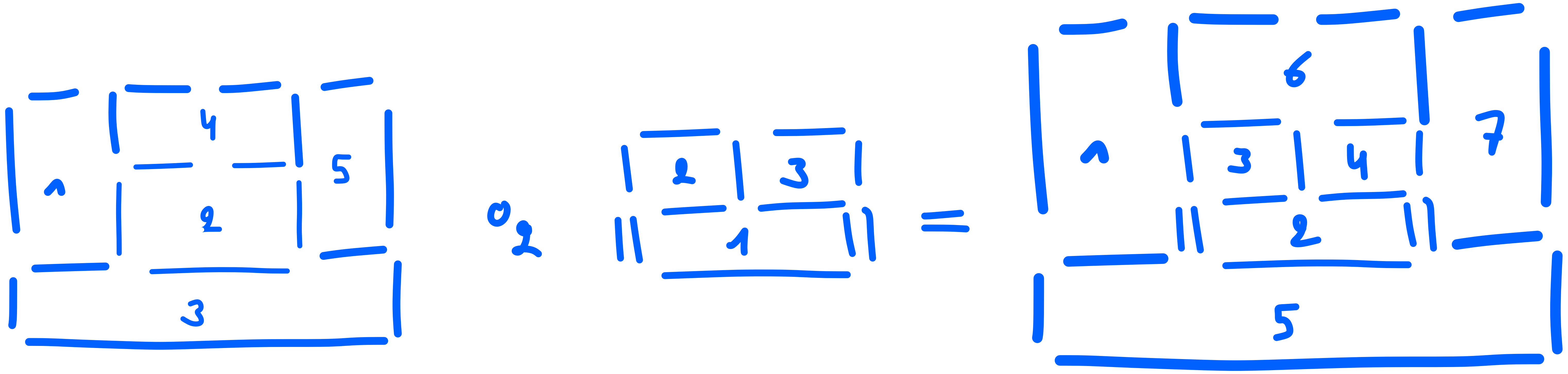
Coloured symmetric operad $\square_p$

objects $\mathbb{N}^3$

$$\square_p\big((p_1,q_1,r_1),\dots,(p_n,q_n,r_n);(p,q,r)\big) = \left\{ \begin{array}{l} \text{stacking of boxes} \\ {}_{p_1}\overset{q_1}{\square}{}_{r_1}, \dots, {}_{p_n}\overset{q_n}{\square}{}_{r_n} \\ \text{into a box} \quad {}_{p}\overset{q}{\square}{}_{r} \end{array} \right\}$$

composition = substitution of a box by a stacking

e.g.



Proposition box operads are algebras over $\square_p$

Higher Gerstenhaber brackets

Theorem

Higher Gerstenhaber brackets

Theorem we have a morphism

Higher Gerstenhaber brackets

Theorem   we have a morphism

$$L_\infty \longrightarrow \mathrm{Tot}(\square_p)$$

Higher Gerstenhaber brackets

Theorem we have a morphism

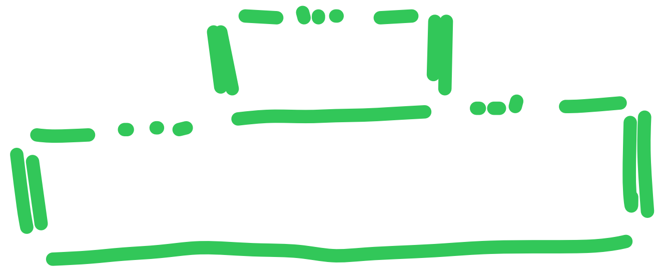
$$L_\infty \longrightarrow \mathrm{Tot}(\square_p)$$

$$\ell_n \longmapsto$$

Higher Gerstenhaber brackets

Theorem   we have a morphism

$$L_\infty \longrightarrow \mathrm{Tot}(\square_p)$$

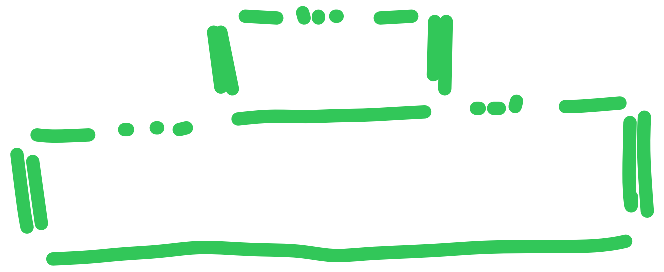
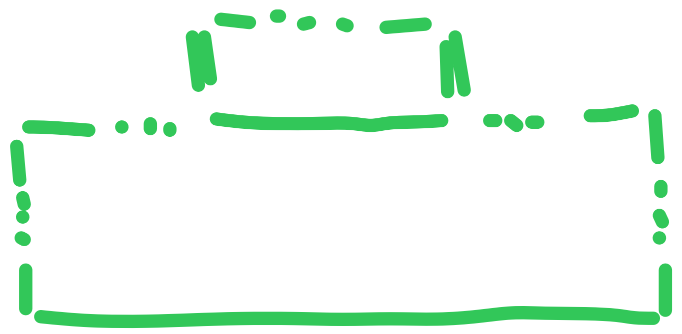
$$\ell_n \longmapsto \Sigma$$ 

2 thin boxes

Higher Gerstenhaber brackets

Theorem we have a morphism

$$L_\infty \longrightarrow \mathrm{Tot}(\square_p)$$

$$\ell_n \longmapsto \sum$$  $$+ \sum$$ 

2 thin boxes

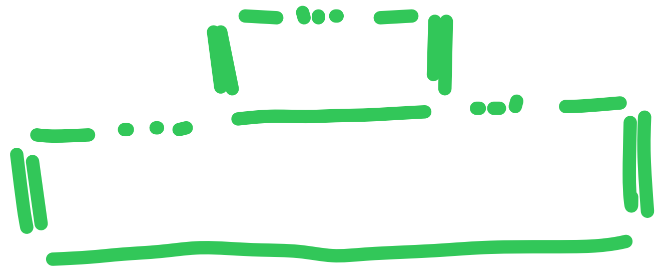
1 thin box
1 nonthin box

Higher Gerstenhaber brackets

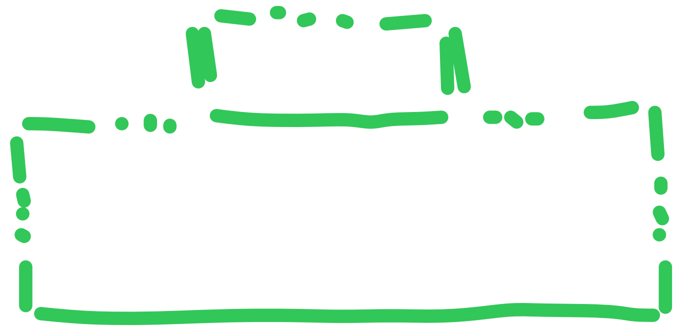
Theorem we have a morphism

$$L_\infty \longrightarrow \mathrm{Tot}(\square_p)$$

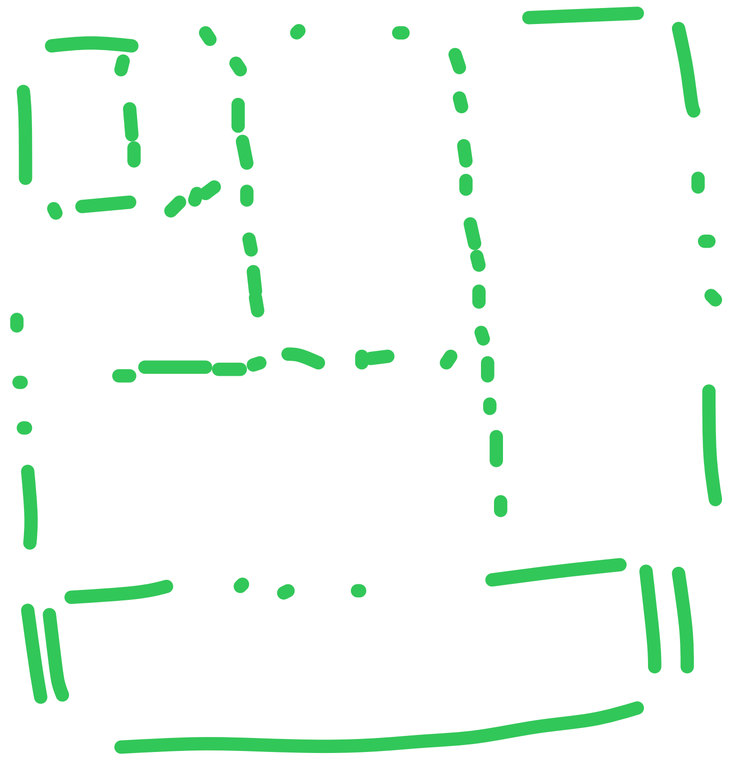
$$\ell_n \longmapsto \sum$$



$+ \sum$



$+ \sum$



2 thin boxes

1 thin box
1 nonthin box

1 thin box
n-1 nonthin box
topological condition

The Floorplan

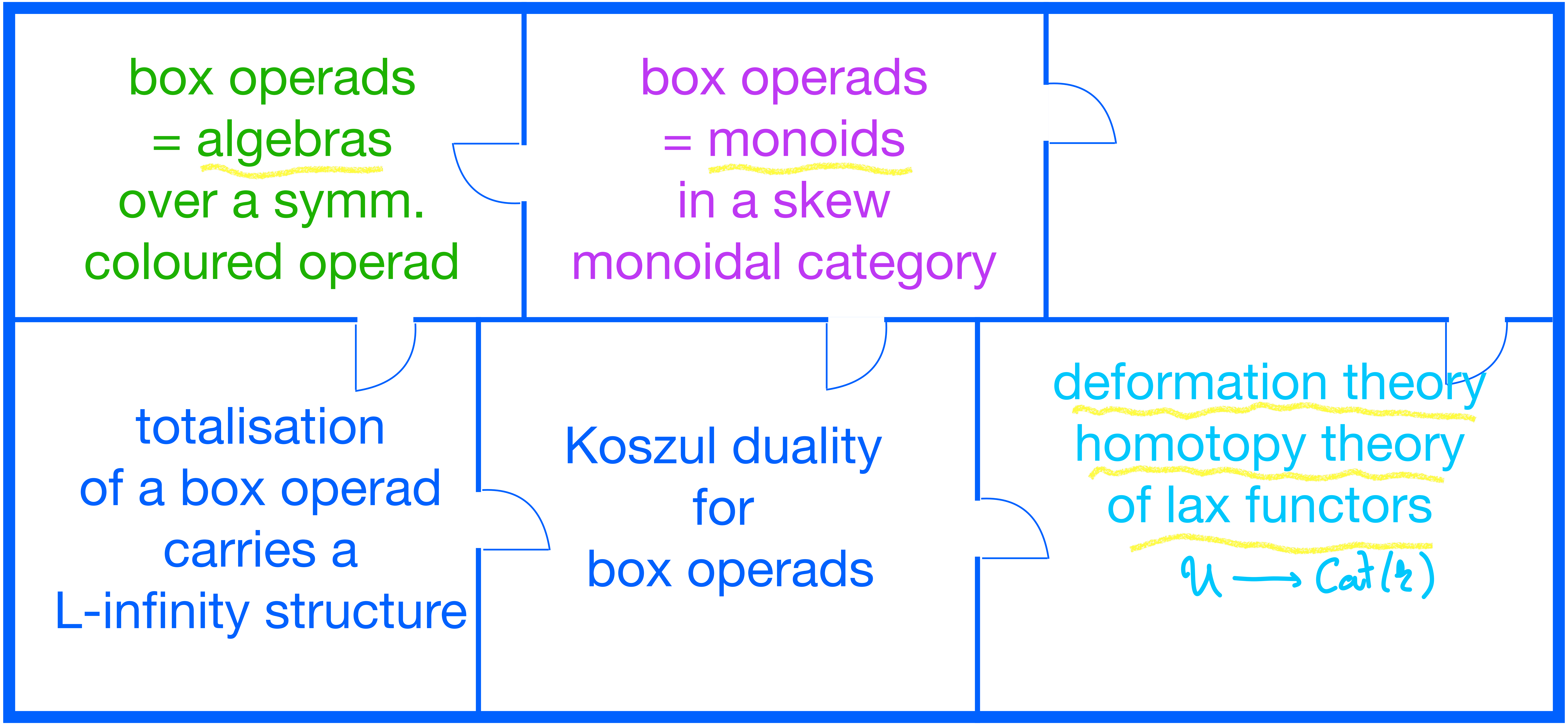
box operads
= algebras
over a symm.
coloured operad

box operads
= monoids
in a skew
monoidal category

totalisation
of a box operad
carries a
L-infinity structure

deformation theory
homotopy theory
of lax functors
$\mathcal{U} \longrightarrow \mathrm{Cat}(\Bbbk)$

The Floorplan



The Floorplan

box operads
= algebras
over a symm.
coloured operad

box operads
= monoids
in a skew
monoidal category

a coloured box
operad $\mathrm{Lax}_{\mathcal{U}}$
encoding lax functors
$\mathcal{U} \longrightarrow \mathrm{Cat}(\mathcal{V})$

totalisation
of a box operad
carries a
L-infinity structure

Koszul duality
for
box operads

deformation theory
homotopy theory
of lax functors
$\mathcal{U} \longrightarrow \mathrm{Cat}(k)$

The Floorplan

box operads
= algebras
over a symm.
coloured operad

box operads
= monoids
in a skew
monoidal category

a coloured box
operad $\mathrm{Lax}_{\mathcal{U}}$
encoding lax functors
$\mathcal{U} \longrightarrow \mathrm{Cat}(\mathcal{V})$

totalisation
of a box operad
carries a
L-infinity structure

Koszul duality
for
box operads

deformation theory
homotopy theory
of lax functors
$\mathcal{U} \longrightarrow \mathrm{Cat}(k)$
via minimal model
Lax_{∞}

The box operad Lax

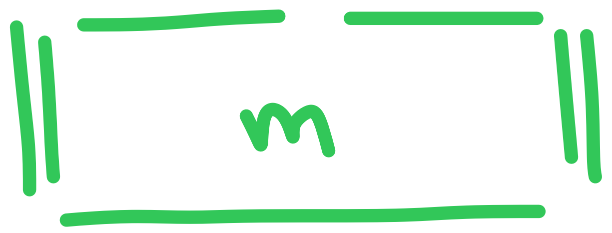
The box operad Lax

generators

relations

The box operad Lax

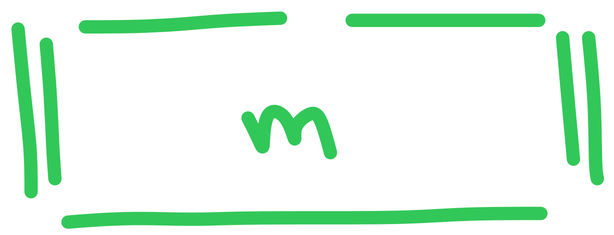
generators

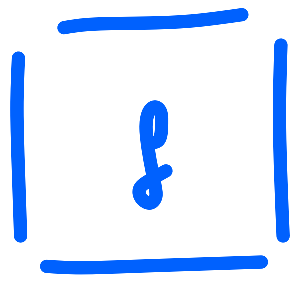


relations

The box operad Lax

generators

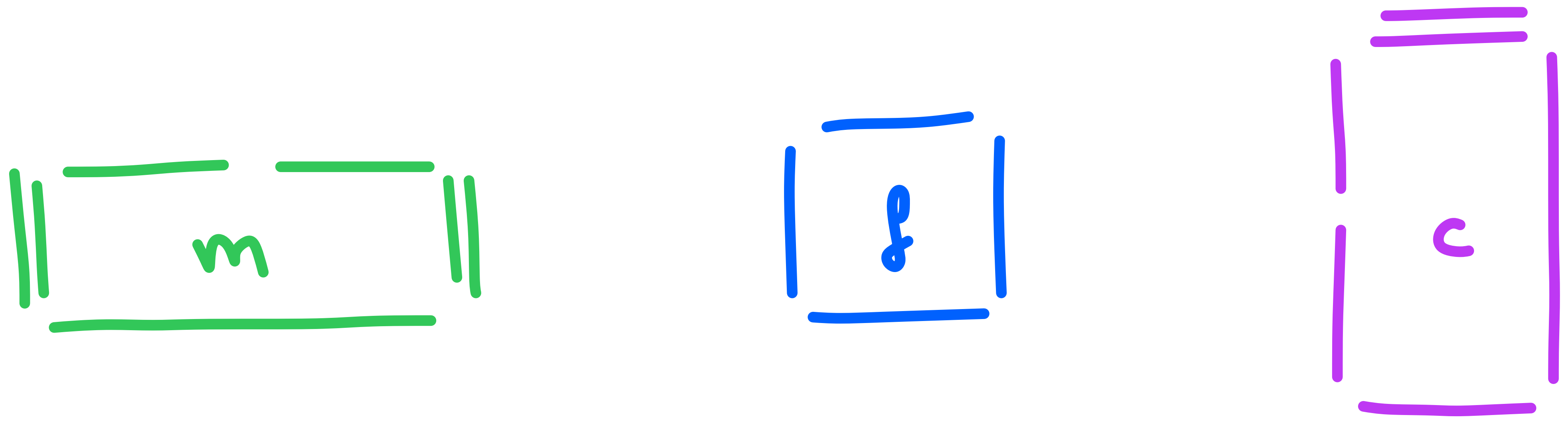




relations

The box operad Lax

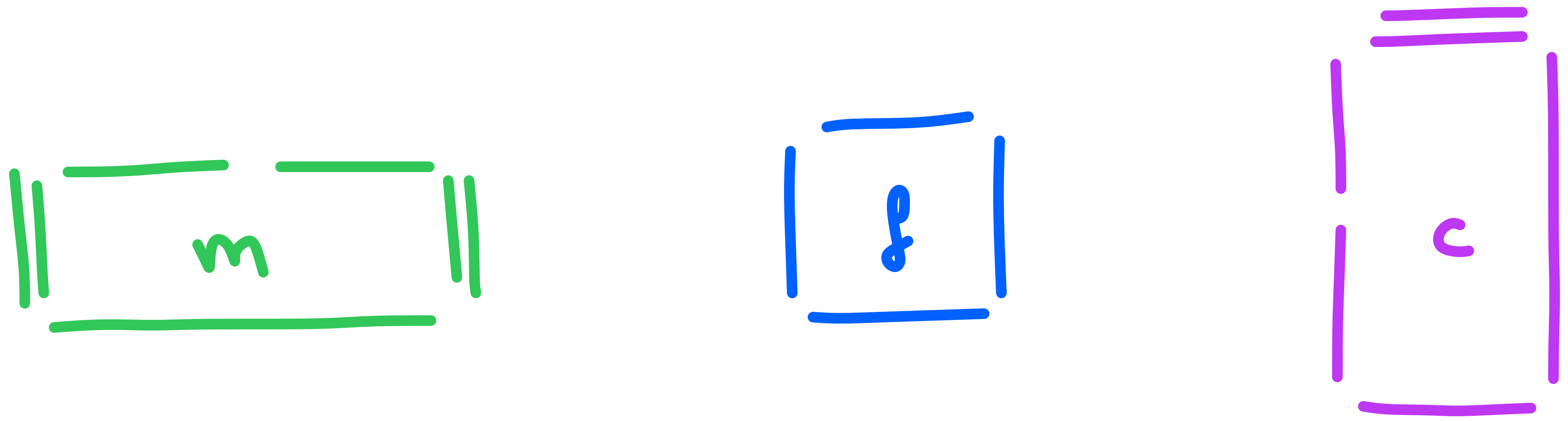
generators



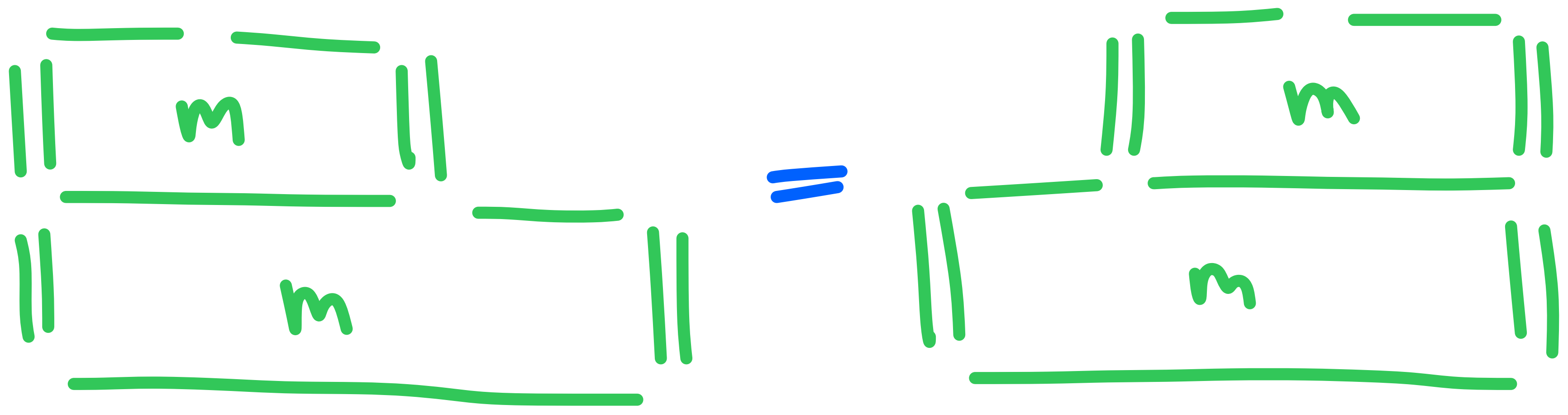
relations

The box operad Lax

generators

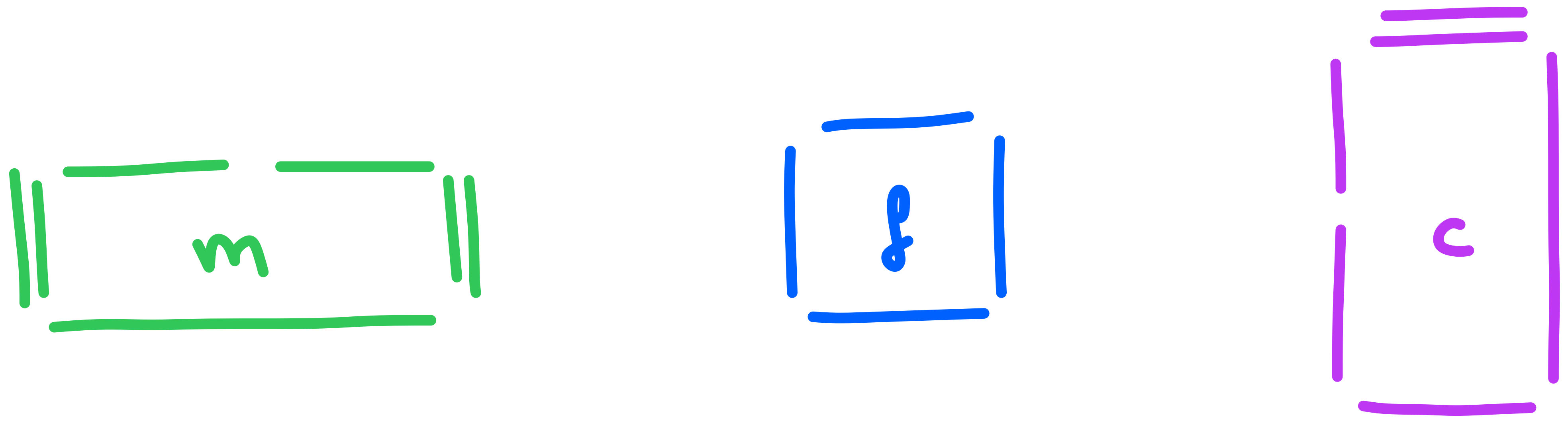


relations

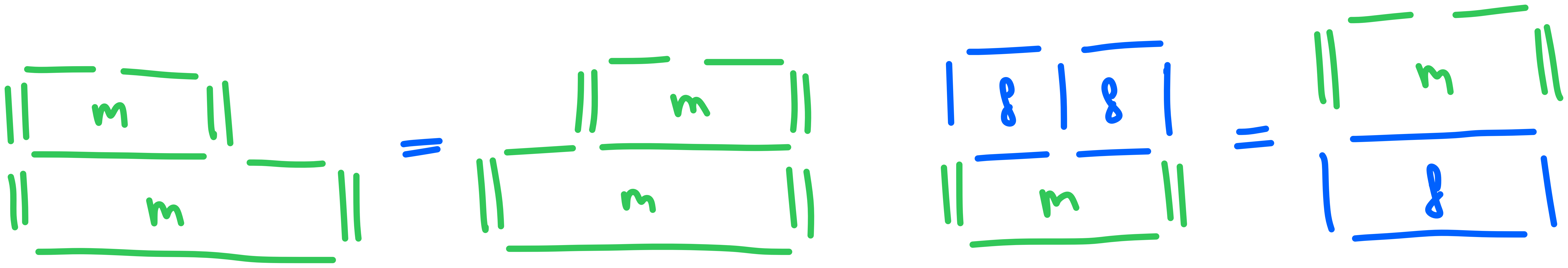


The box operad Lax

generators



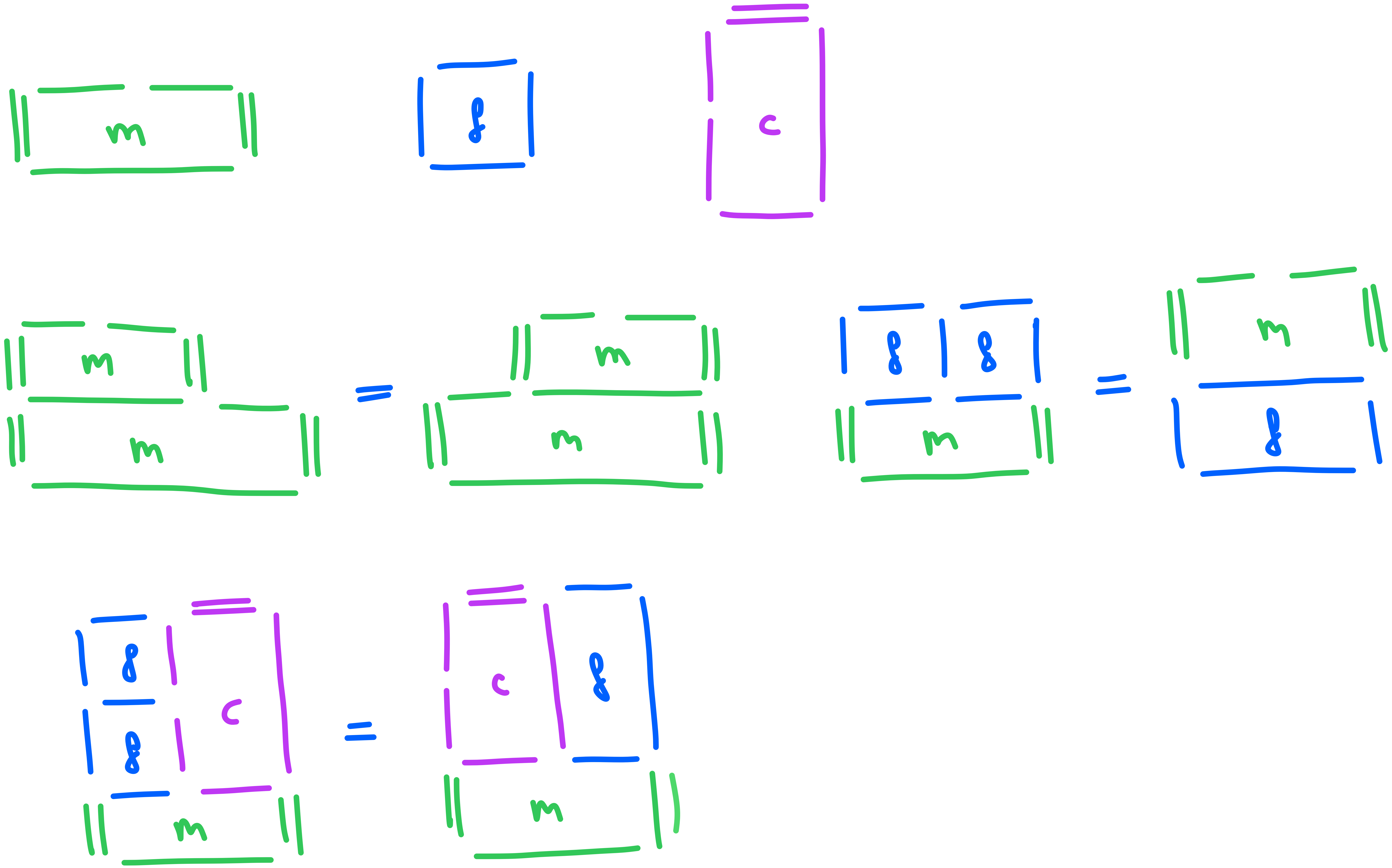
relations



The box operad Lax

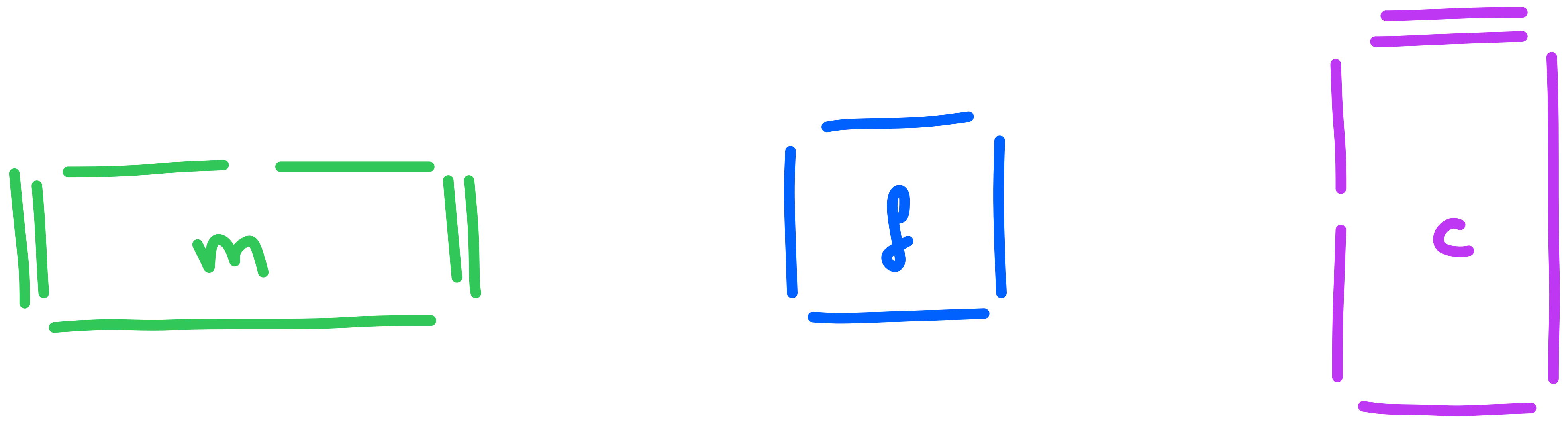
generators

relations

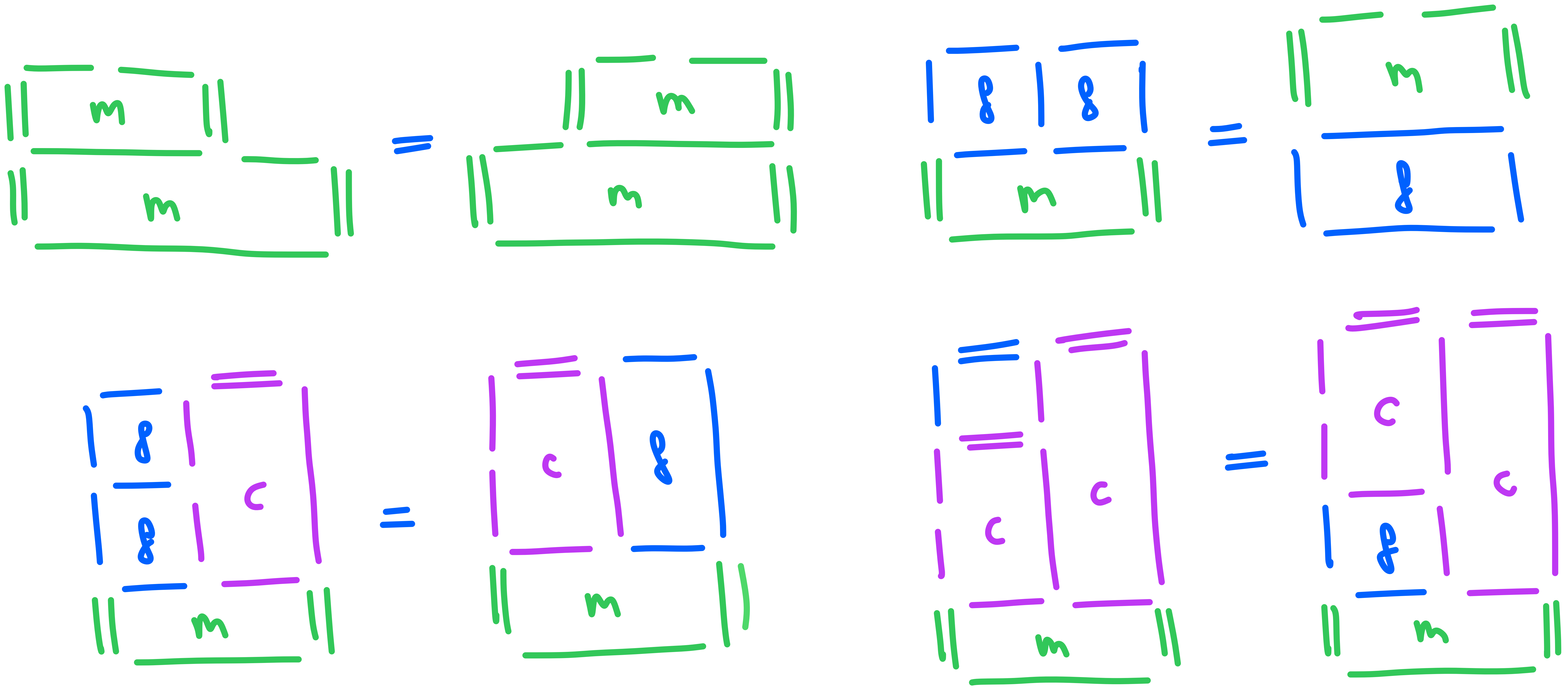


The box operad Lax

generators



relations



Coherence

Theorem

Coherence

Theorem $\quad \mathrm{Lax}(p, q, r) =$

Coherence

Theorem $$\mathrm{Lax}(p,q,r) = \begin{cases} \bigsqcup_{\mathrm{Part}(p,r)} I & p+q-r \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Coherence

Theorem

$$\mathrm{Lax}(p,q,r) = \begin{cases} \displaystyle\coprod_{\mathrm{Pat}(p,r)} I & p+q-r \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

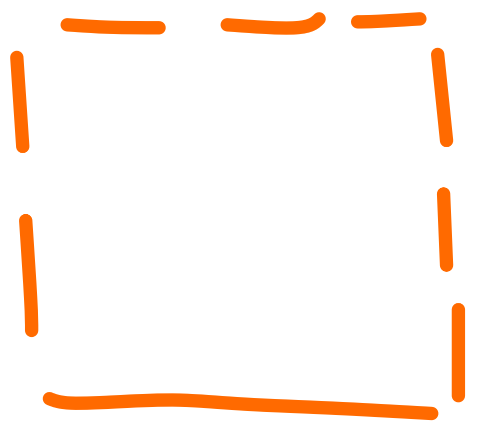
(1)

$$\boxed{\begin{array}{c} \cdots \\ p<r \\ \mathrm{Lax}(p,q,r) \end{array}} = 0$$

Coherence

Theorem $$\mathrm{Lax}(p,q,r) = \begin{cases} \displaystyle\bigsqcup_{\mathrm{Part}(p,r)} I & p+q-r \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

① $$\boxed{\overset{p<r}{\mathrm{Lax}(p,q,r)}} = 0$$ e.g. there is no element



Coherence

Theorem $\quad \mathrm{Lax}(p,q,r) = \begin{cases} \coprod_{\mathrm{Pat}(p,r)} I & p+q-r \geq 1 \\ 0 & \text{otherwise} \end{cases}$

① $\left\| \boxed{\overset{q\geq 1}{\mathrm{Lax}(0,q,0)}} \right\| = I \Rightarrow \mathrm{Lax}(0,q,0) = \mathrm{Assoc}(q) = m_q I$

Coherence

Theorem

$$\mathrm{Lax}(p,q,r) = \begin{cases} \bigsqcup\limits_{\mathrm{Pat}(p,r)} I & p+q-r \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

③ $\boxed{\mathrm{Lax}(p,q,1)}^{\,p+q\geq 2,\ p\geq 1} = I$

Coherence

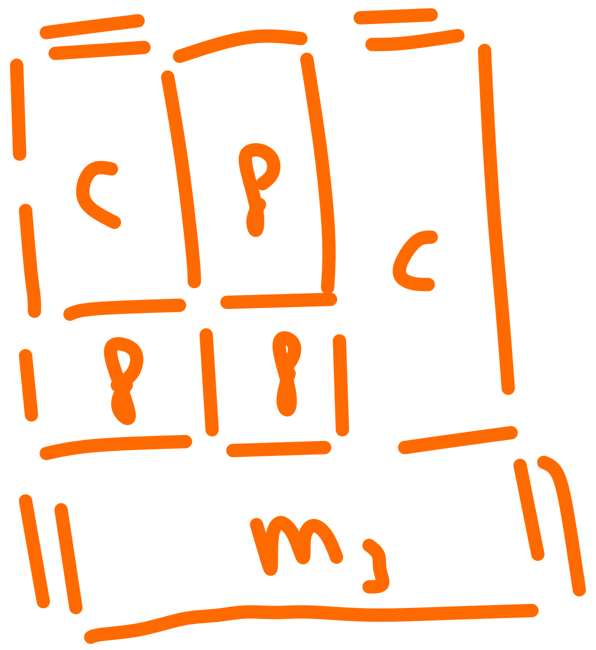
Theorem

$$\mathrm{Lax}(p,q,r) = \begin{cases} \bigsqcup_{\mathrm{Part}(p,r)} I & p+q-r \geqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

3

$$\boxed{\mathrm{Lax}(p,q,1)}^{\,p+q\geqslant 2,\ p\geqslant 1} = I$$

e.g.



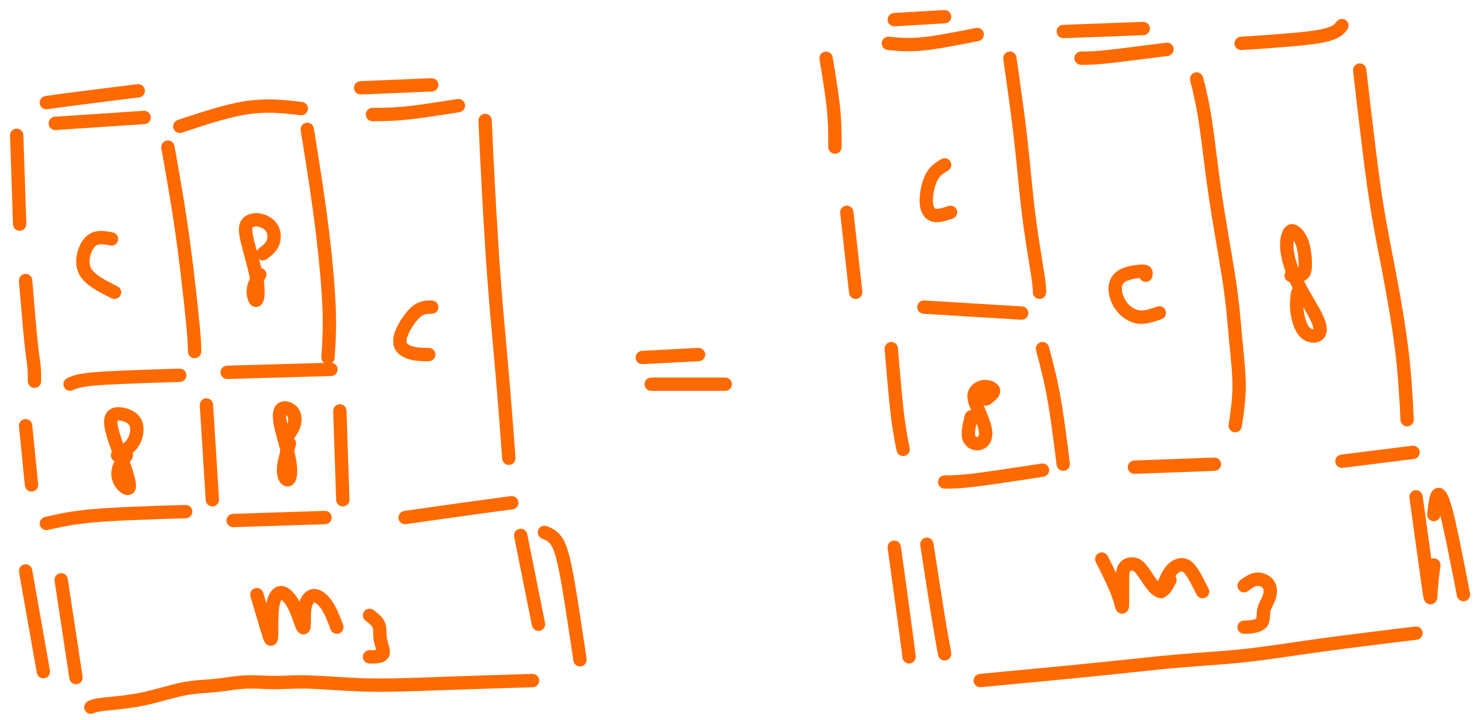
Coherence

Theorem

$$\mathrm{Lax}(p,q,r) = \begin{cases} \displaystyle\bigsqcup_{\mathrm{Pat}(p,r)} I & p+q-r \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

(3) $$\boxed{\overset{p+q\geq 2,\ p\geq 1}{\mathrm{Lax}(p,q,1)}} = I$$

e.g.



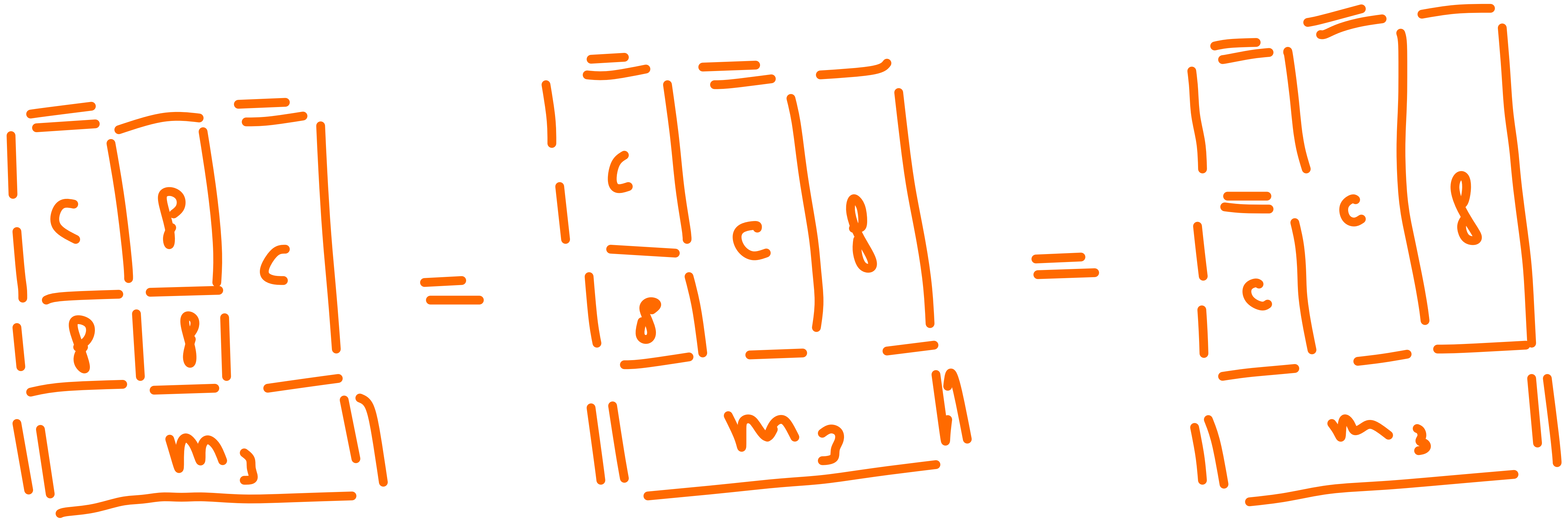
Coherence

Theorem

$$\mathrm{Lax}(p,q,r) = \begin{cases} \bigsqcup_{\mathrm{Pat}(p,r)} I & p+q-r \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

③ $\boxed{\overset{p+q\geq 2,\ p\geq 1}{\mathrm{Lax}(p,q,1)}} = I$

e.g.



Coherence

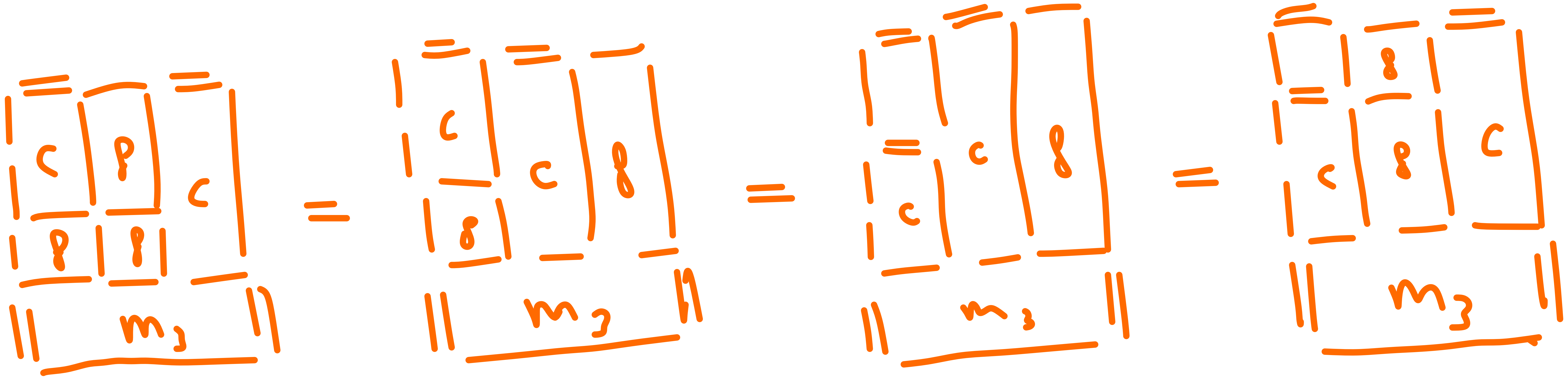
Theorem

$$\mathrm{Lax}(p,q,r) = \begin{cases} \displaystyle\bigsqcup_{\mathrm{Part}(p,r)} I & p+q-r \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

3

$$\boxed{\overset{p+q\geq 2,\ p\geq 1}{\mathrm{Lax}(p,q,1)}} = I$$

e.g.



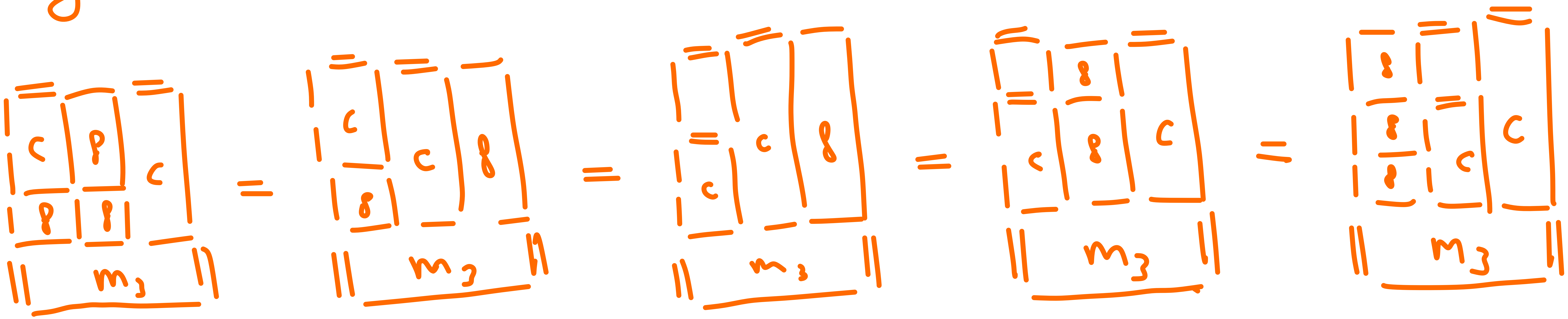
Coherence

Theorem

$$\mathrm{Lax}(p,q,r) = \begin{cases} \bigsqcup_{\mathrm{Pat}(p,r)} I & p+q-r \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

3 $\boxed{\begin{array}{c} p+q \geq 2,\ p \geq 1 \\ \mathrm{Lax}(p,q,1) \end{array}} = I$

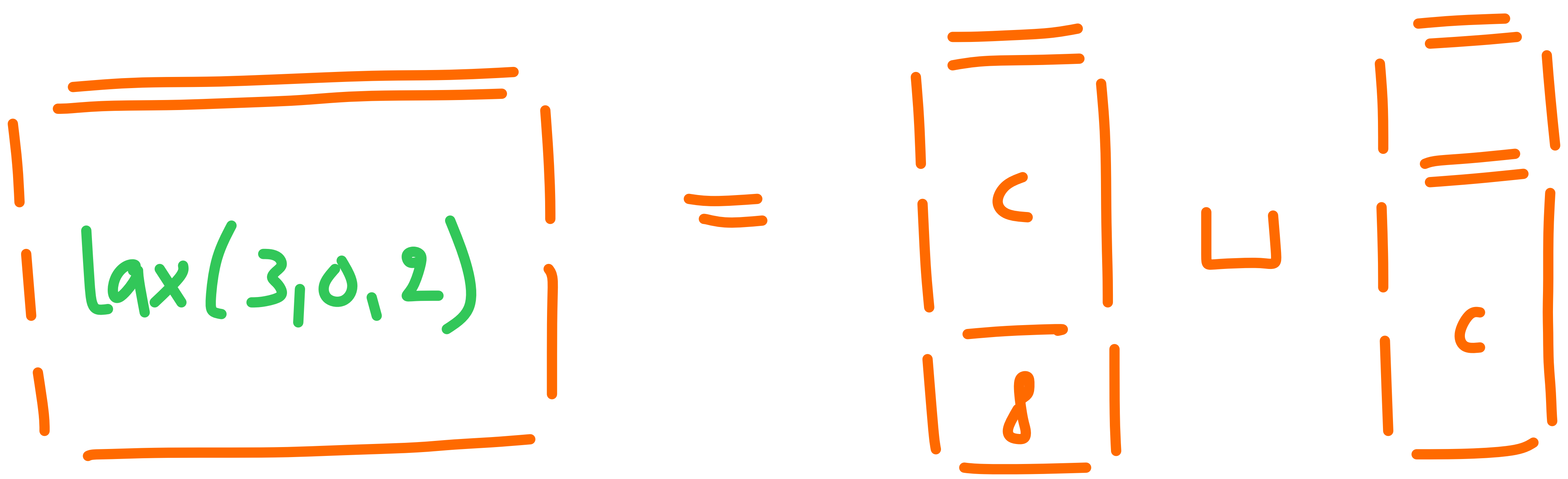
e.g.



Coherence

Theorem $$\mathrm{Lax}(p,q,r) = \begin{cases} \bigsqcup_{\mathrm{Pat}(p,r)} I & p+q-r \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

4



The coloured box operad $\mathrm{Lax}_{\mathcal{U}}$

FOR SMALL CATEGORY $\mathcal{U}$

generators

relations

The coloured box operad $\mathrm{Lax}_{\mathcal{U}}$

FOR SMALL CATEGORY $\mathcal{U}$

generators

$$
\begin{tikzcd}
U \arrow[r, dash] \arrow[d, equal, "id"'] & U \arrow[r, dash] \arrow[d, phantom, "m^U"] & U \arrow[d, equal, "id"] \\
U \arrow[rr, dash] & & U
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, dash] \arrow[d, dash, "u"'] \arrow[dr, phantom, "\rho^u"] & U \arrow[d, dash, "u"] \\
V \arrow[r, dash] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, equal] \arrow[d, dash, "u"'] \arrow[ddr, phantom, "c^{u,v}"] & U \arrow[dd, dash, "vu"] \\
V \arrow[d, dash, "v"'] & \\
W \arrow[r, dash] & W
\end{tikzcd}
$$

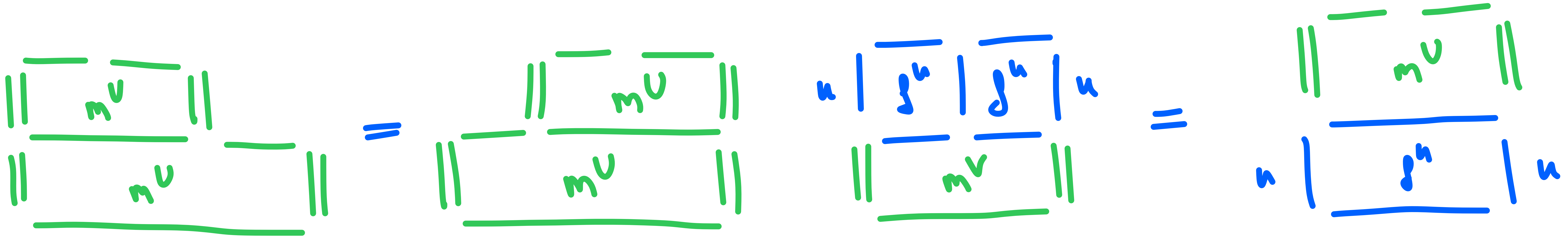
relations

The coloured box operad $\mathrm{Lax}_{\mathcal{U}}$

FOR SMALL CATEGORY $\mathcal{U}$

generators

relations

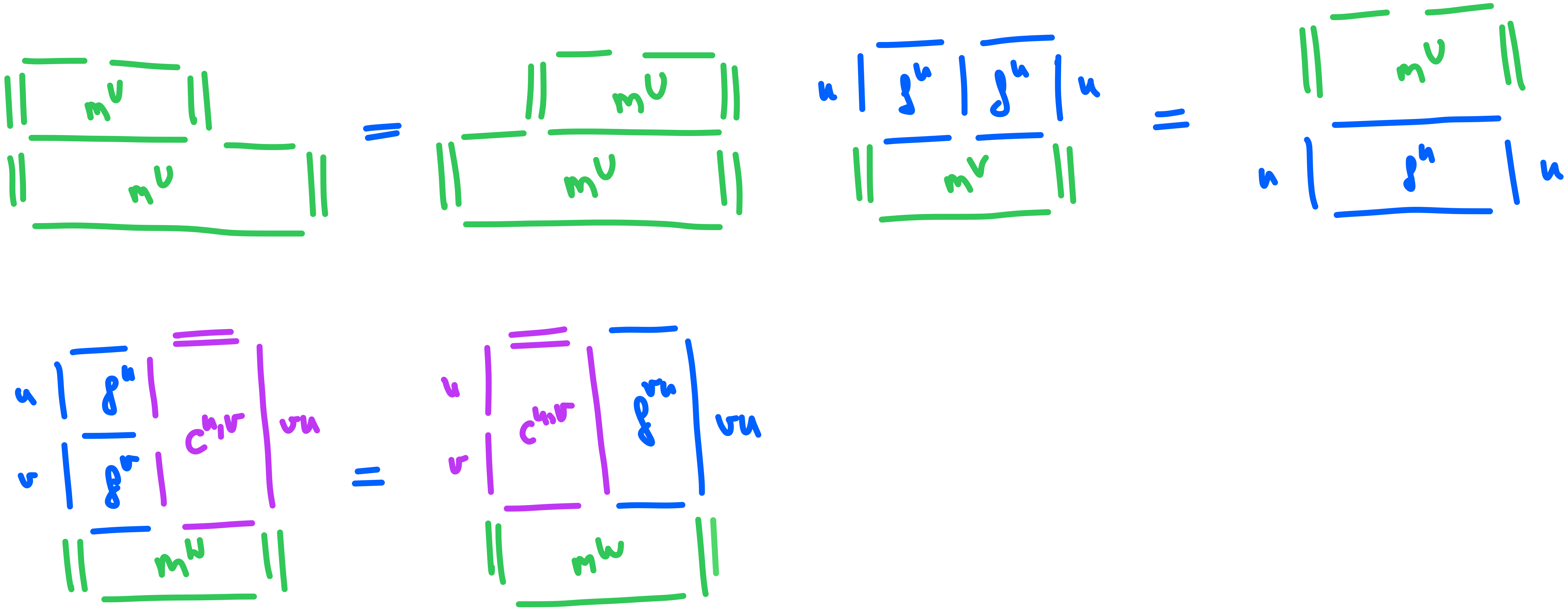


The coloured box operad Lax$_{\mathcal{U}}$

For small category $\mathcal{U}$

generators

relations

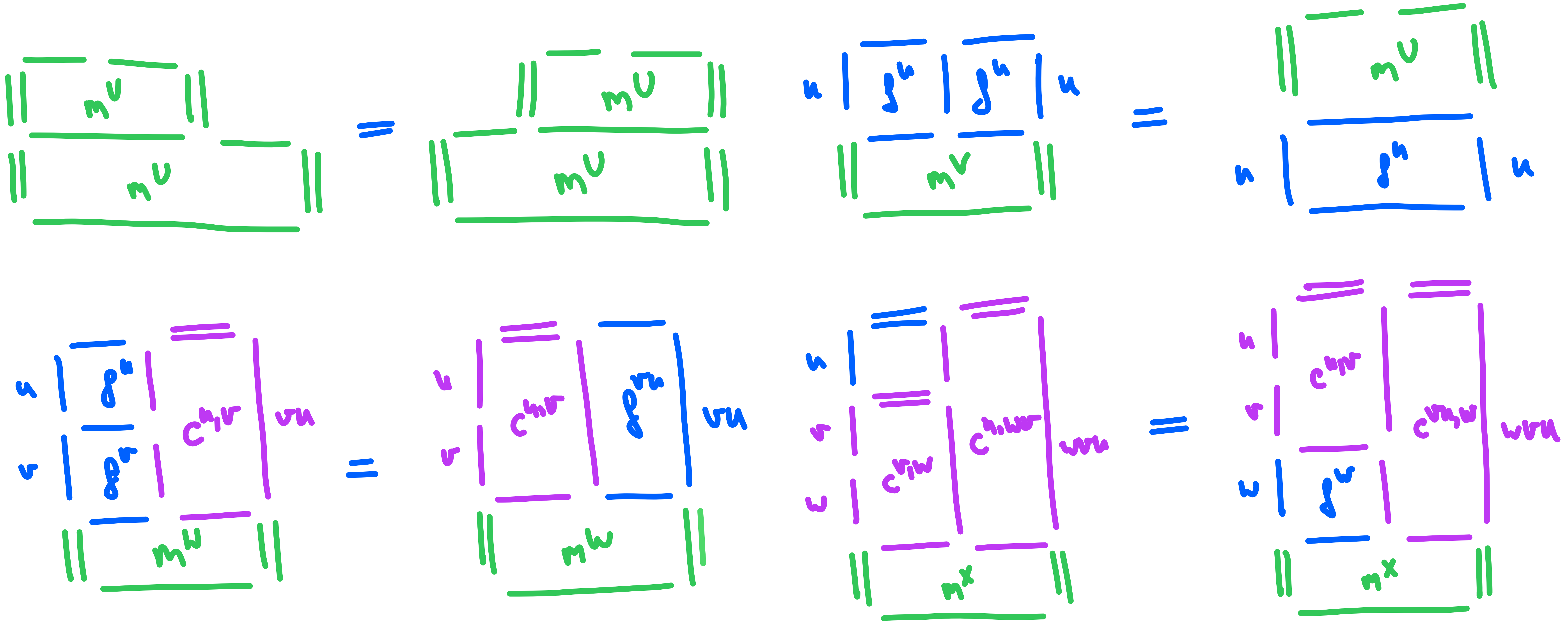


The coloured box operad $\mathrm{Lax}_{\mathcal{U}}$

FOR SMALL CATEGORY $\mathcal{U}$

generators

relations



Algebras over a coloured box operad $\mathcal{B}$

Algebras over a coloured box operad $\mathcal{B}$

definition $\mathrm{Alg}(\mathcal{B}) := \mathrm{Hom}(\mathcal{B}, \mathrm{Span}(\mathcal{V}))$

Algebras over a coloured box operad $\mathcal{B}$

virtual double category of

definition $\mathrm{Alg}(\mathcal{B}) := \mathrm{Hom}(\mathcal{B}, \mathrm{Span}(\mathcal{V}))$

Algebras over a coloured box operad $\mathcal{B}$

virtual double category of $\bullet$ = sets

definition $\mathrm{Alg}(\mathcal{B}) := \mathrm{Hom}(\mathcal{B}, \mathrm{Span}(\mathcal{V}))$

Algebras over a coloured box operad $\mathbb{B}$

definition $\mathrm{Alg}(\mathbb{B}) := \mathrm{Hom}(\mathbb{B}, \mathrm{Span}(\mathbb{V}))$

virtual double category of
$\bullet$ = sets
$\downarrow$ = functions

Algebras over a coloured box operad $\mathcal{B}$

virtual double category of

$\bullet$ = sets

$\downarrow$ = functions

$\rightarrow$ = spans enriched over $\mathcal{V}$

definition $\quad \mathrm{Alg}(\mathcal{B}) := \mathrm{Hom}(\mathcal{B}, \mathrm{Span}(\mathcal{V}))$

Algebras over a coloured box operad $\mathbb{B}$

virtual double category of

- $\bullet$ = sets
- $\downarrow$ = functions
- $\rightarrow$ = spans enriched over V
- $\Downarrow$ = morphisms of V-spans

definition $\quad \mathrm{Alg}(\mathbb{B}) := \mathrm{Hom}(\mathbb{B}, \mathrm{Span}(V))$

Algebras over a coloured box operad $\mathbb{B}$

definition $\mathrm{Alg}(\mathbb{B}) := \mathrm{Hom}(\mathbb{B}, \mathrm{Span}(\mathcal{V}))$

algebra over $\mathbb{B}$ = morphism $\mathbb{B} \xrightarrow{A} \mathrm{Span}(\mathcal{V})$

virtual double category of

• = sets

↓ = functions

→ = spans enriched over $\mathcal{V}$

⇓ = morphisms of $\mathcal{V}$-spans

Algebras over a coloured box operad $\mathbb{B}$

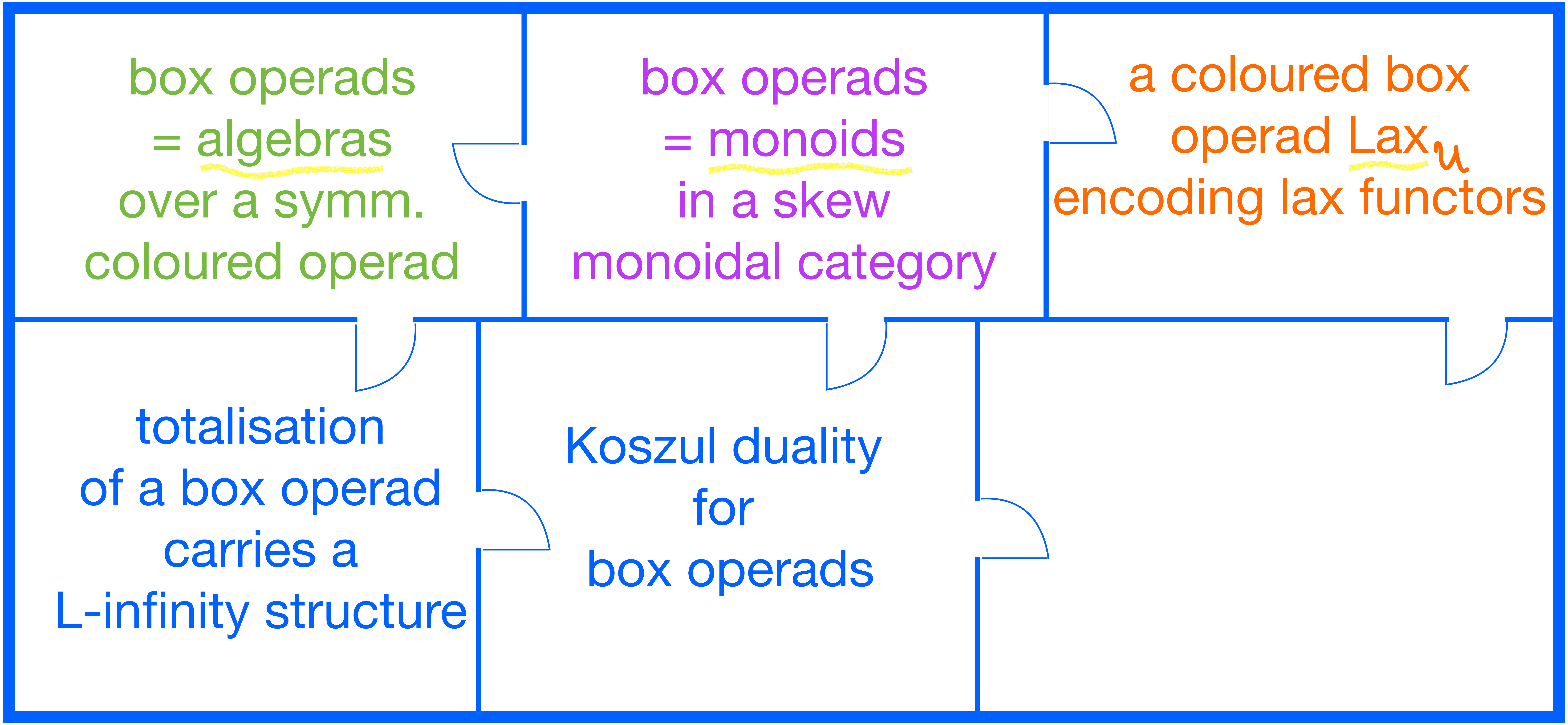
virtual double category of
• = sets
$\downarrow$ = functions
$\rightarrow$ = spans enriched over V
$\Downarrow$ = morphisms of V-spans

definition $\quad \mathrm{Alg}(\mathbb{B}) := \mathrm{Hom}(\mathbb{B}, \mathrm{Span}(V))$

algebra over $\mathbb{B}$ = morphism $\mathbb{B} \xrightarrow{A} \mathrm{Span}(V)$

Proposition $\quad \mathrm{Alg}(\mathrm{Lax}_{\mathcal{U}}) = \mathrm{Lax}(\mathcal{U}, \mathrm{Cat}(V))$

The Floorplan



The Floorplan

box operads
= algebras
over a symm.
coloured operad

box operads
= monoids
in a skew
monoidal category

a coloured box
operad $\mathrm{Lax}_{\mathcal{U}}$
encoding lax functors

totalisation
of a box operad
carries a
L-infinity structure

Koszul duality
for
box operads

deformation theory
+ homotopy theory
of lax functors
$\mathcal{U} \longrightarrow \mathrm{Cat}(\Bbbk)$
via minimal model
Lax_∞

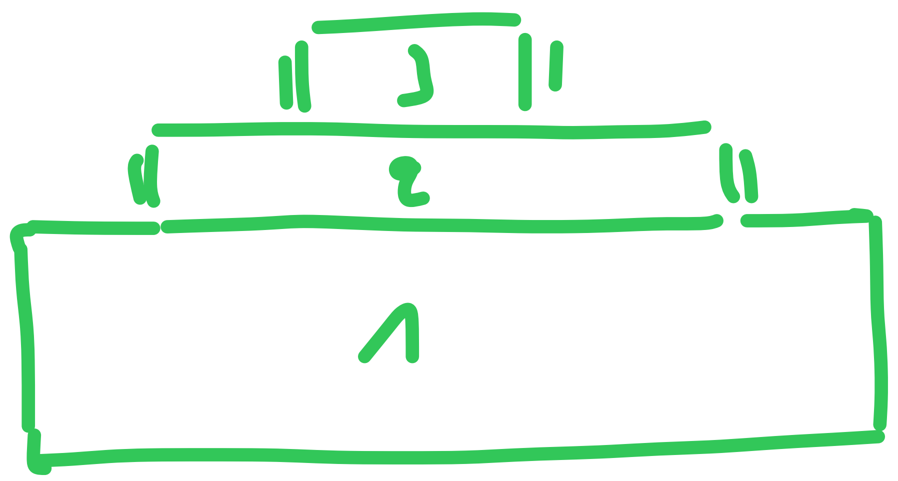
More details in

- *Box operads and higher Gerstenhaber braces* - Dinh Van, Hermans and Lowen, arXiv preprint (2023)

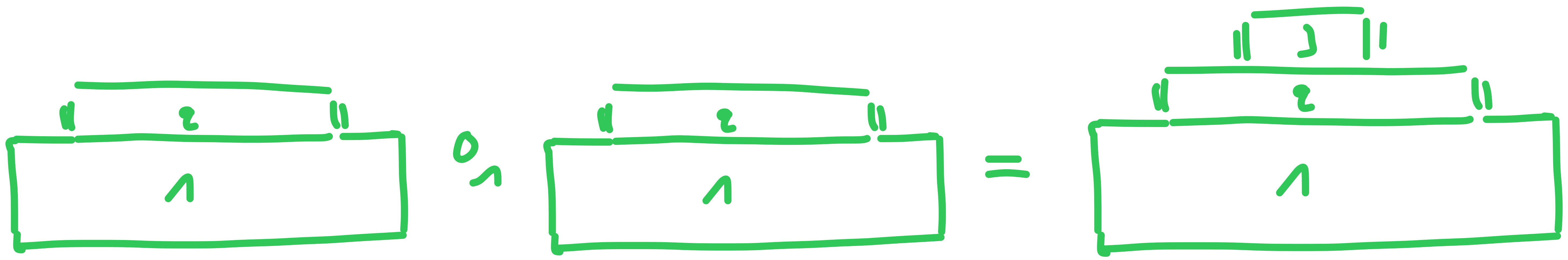
- *A minimal model for lax prestacks via Koszul duality for box operads* - Hermans, arXiv preprint (2023)

Proof

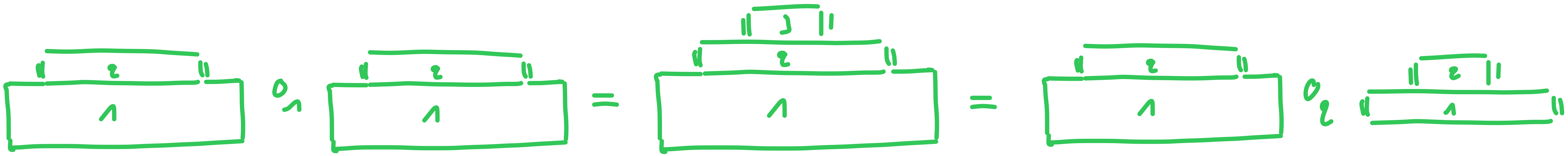
Proof



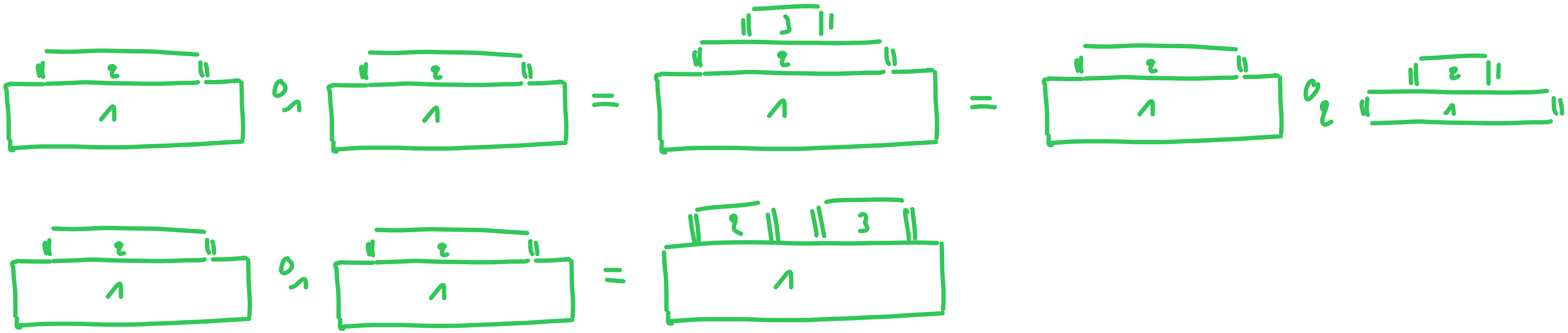
Proof



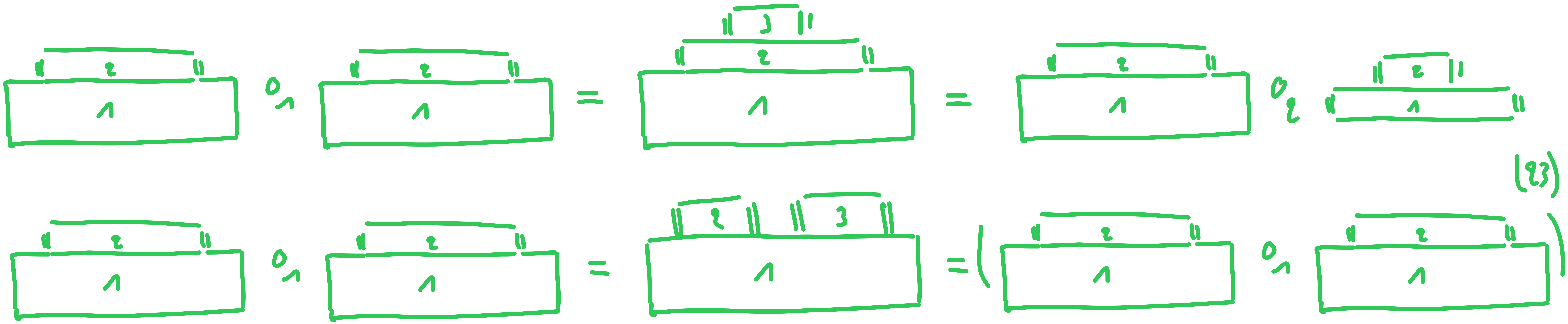
Proof



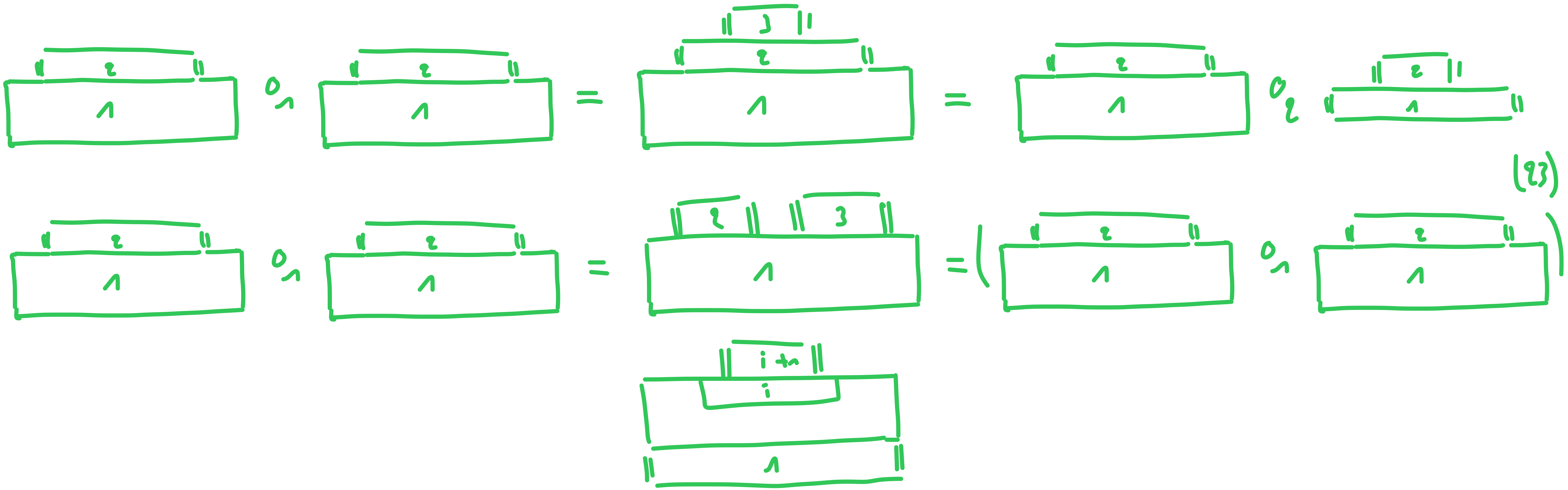
Proof



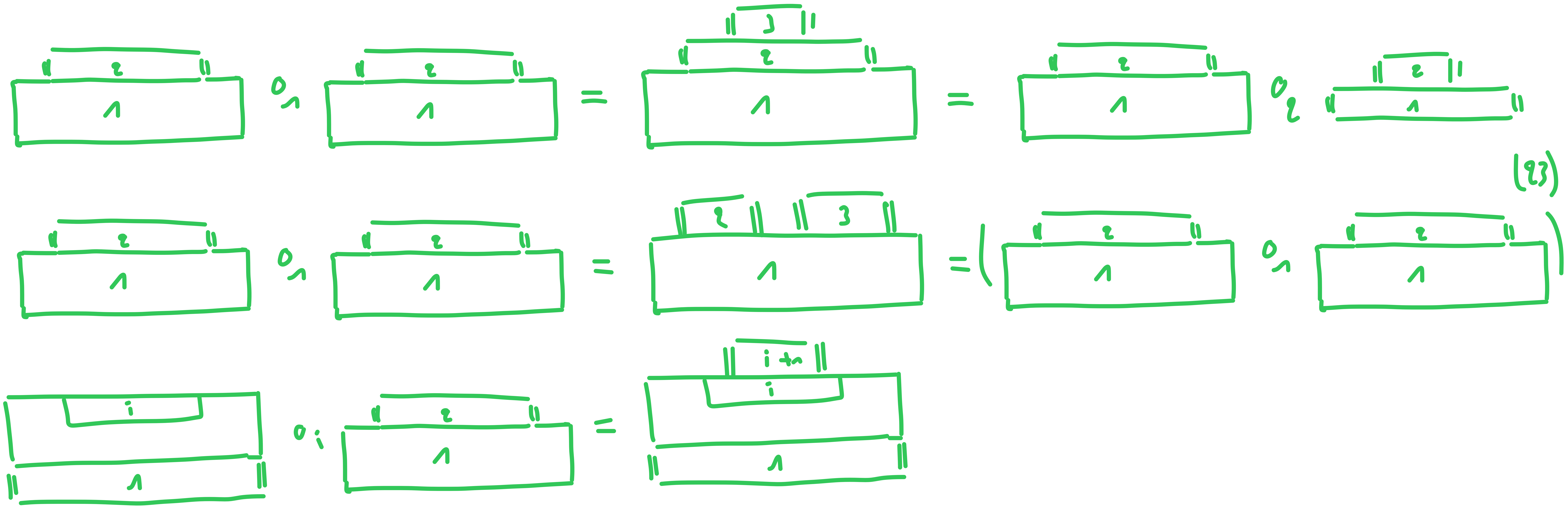
Proof



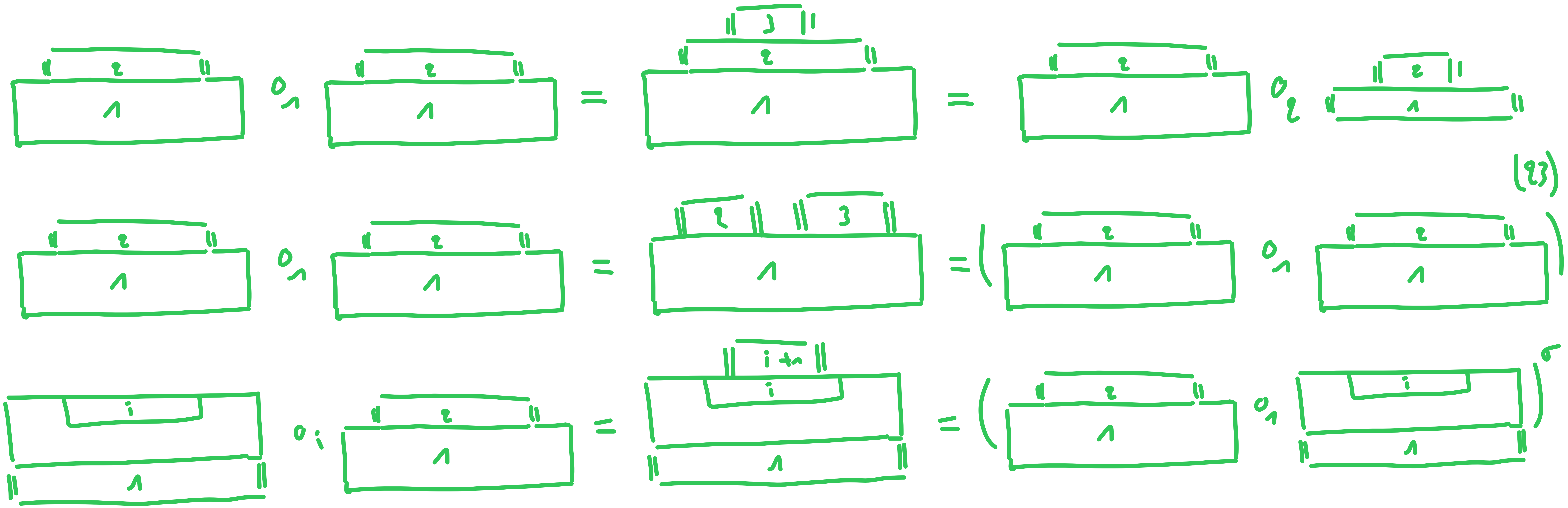
Proof



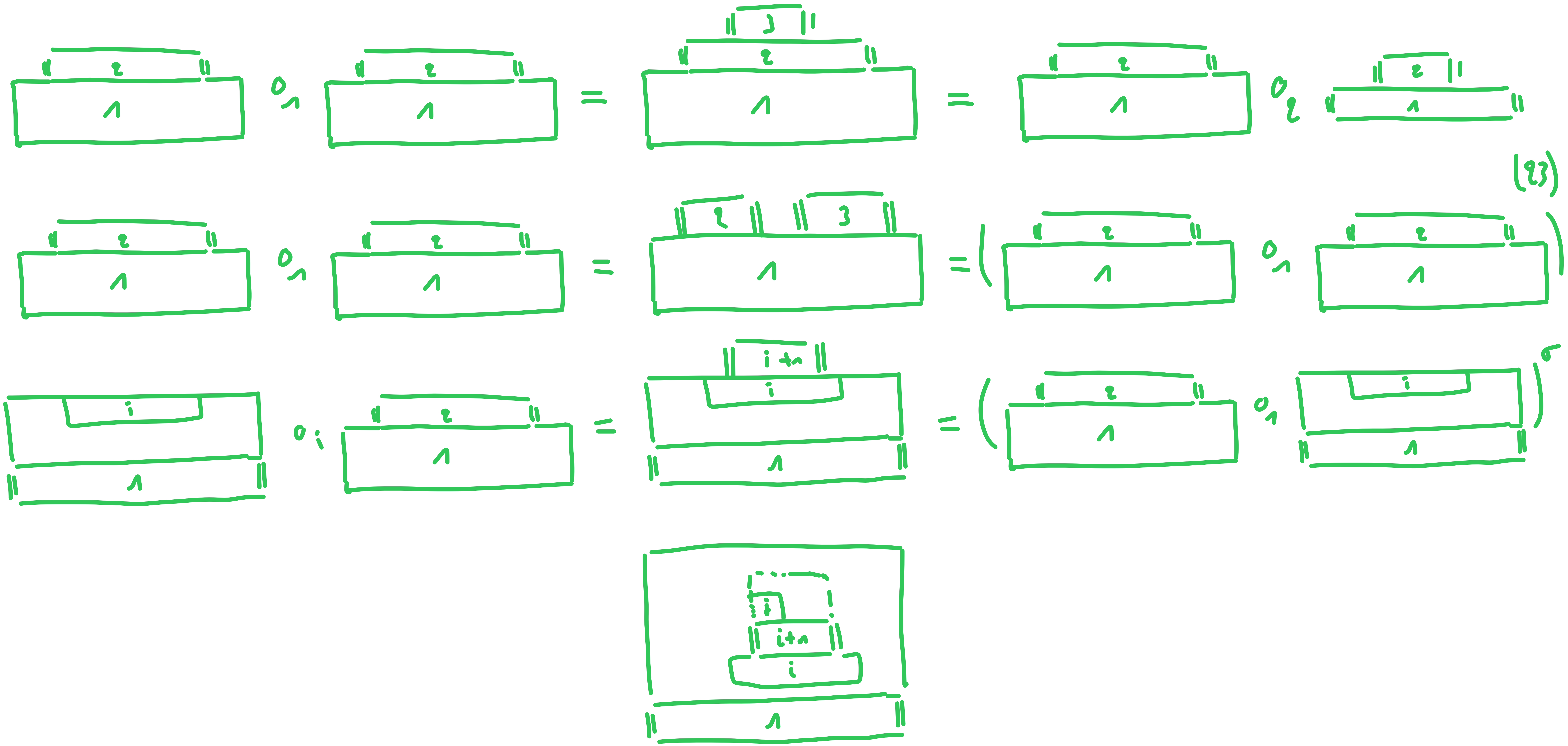
Proof



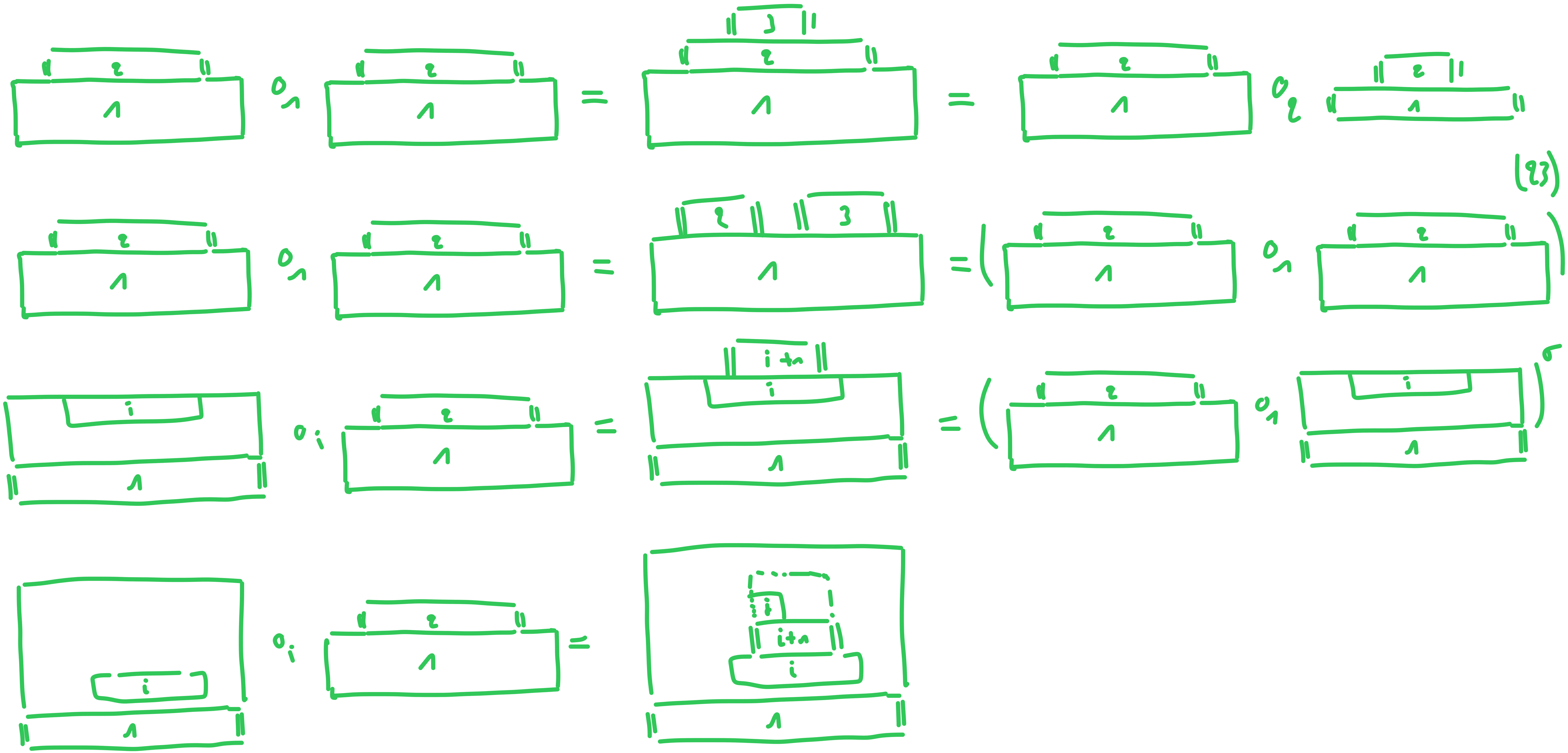
Proof



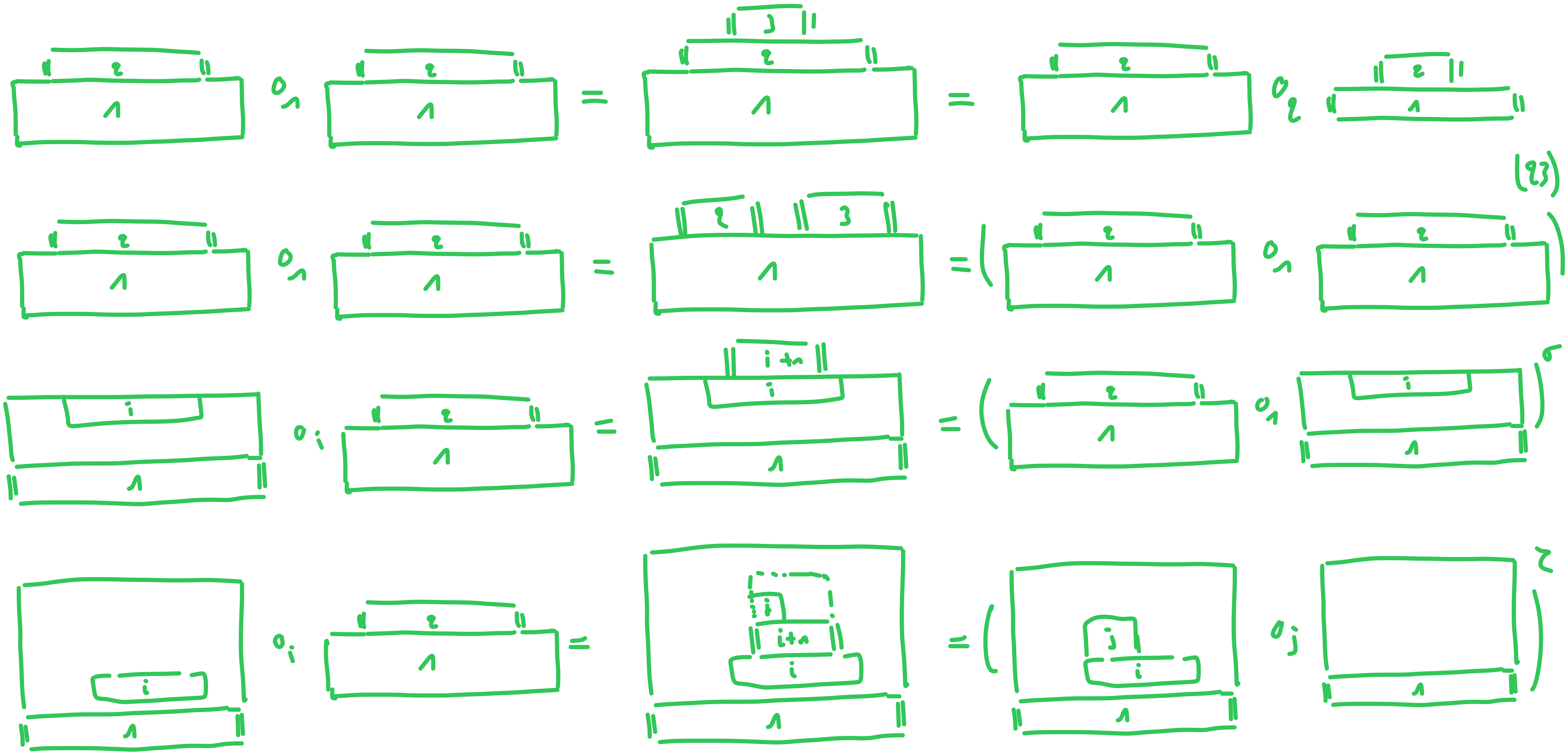
Proof



Proof



Proof



Virtual double category $\mathcal{B}$ enriched over $(\mathcal{V}, \otimes, I)$ consists of

- a set of objects $\mathcal{B}_0$
- a set of vertical arrows $\mathcal{B}_\downarrow \rightrightarrows \mathcal{B}_0$
- a set of horizontal arrows $\mathcal{B}_\rightarrow \rightrightarrows \mathcal{B}_0$
- a collection of $\mathcal{V}$-objects of rectangular 2-arrows

$$\begin{tikzcd}
\bullet \arrow[r, "m_1"] \arrow[d, "f"'] \arrow[drrr, phantom, "\mathcal{B}\left(f;\, {m_1, \dots, m_n \atop m};\, f'\right)"] & \bullet \arrow[r, phantom, "\cdots"] & \bullet \arrow[r, "m_n"] & \bullet \arrow[d, "f'"] \\
\bullet \arrow[rrr, "m"'] & & & \bullet
\end{tikzcd}$$

$$
\begin{tikzcd}
\bullet \arrow[d, "f"'] \\
\bullet \arrow[d, "g"'] \\
\bullet
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
\bullet \arrow[dd, "g \circ f"] \\
\\
\bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bullet \arrow[r, "m_1^1"] \arrow[d, "f_0"'] & \bullet \cdots \bullet \arrow[r, "m_{k_1}^1"] & \bullet \arrow[d, "f_1"] & \cdots & \bullet \arrow[r, "m_1^n"] \arrow[d, "f_{n-1}"'] & \bullet \cdots \bullet \arrow[r, "m_{k_n}^n"] & \bullet \arrow[d, "f_n"] \\
\bullet \arrow[rr, "m_1"'] \arrow[d, "f"'] & \alpha_1 & \bullet & \cdots & \bullet \arrow[rr, "m_n"'] & \alpha_n & \bullet \arrow[d, "f'"] \\
\bullet \arrow[rrrrrr, "m"'] & & & \alpha & & & \bullet
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
\bullet \arrow[r, "m_1^1"] \arrow[dd, "f \circ f_0"'] & \bullet \cdots \bullet \arrow[r, "m_{k_1}^1"] & \bullet & \cdots & \bullet \arrow[r, "m_1^n"] & \bullet \cdots \bullet \arrow[r, "m_{k_n}^n"] & \bullet \arrow[dd, "f' \circ f_n"] \\
& & & \alpha \circ (\alpha_1, \ldots, \alpha_n) & & & \\
\bullet \arrow[rrrrrr, "m"'] & & & & & & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
a \arrow[d, equal, "id_a"] \\
a
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
a \arrow[r, "m"] \arrow[d, equal, "id_a"'] & b \arrow[d, equal, "id_b"] \\
a \arrow[r, "m"'] & b
\end{tikzcd}
$$

compositions

$$
\begin{tikzcd}
\bullet \arrow[d, "f"'] \\
\bullet \arrow[d, "g"'] \\
\bullet
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\bullet \arrow[d, "g \circ f"] \\
\bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bullet \arrow[r, "m^1_1"] \arrow[d, "f_0"'] & \bullet \cdots \bullet \arrow[r, "m^1_{k_1}"] & \bullet \arrow[d, "f_1"] & \cdots & \bullet \arrow[r, "m^n_1"] \arrow[d, "f_{n-1}"'] & \bullet \cdots \bullet \arrow[r, "m^n_{k_n}"] & \bullet \arrow[d, "f_n"] \\
\bullet \arrow[rr, "m_1"'] \arrow[d, "f"'] \arrow[r, phantom, "\alpha_1"] & & \bullet & \cdots & \bullet \arrow[rr, "m_n"'] \arrow[r, phantom, "\alpha_n"] & & \bullet \arrow[d, "f'"] \\
\bullet \arrow[rrrrrr, "m"'] \arrow[rrrrrr, phantom, "\alpha", shift left=3ex] & & & & & & \bullet
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\bullet \arrow[r, "m^1_1"] \arrow[d, "f \circ f_0"'] & \bullet \cdots \bullet \arrow[r, "m^1_{k_1}"] & \bullet \cdots \bullet \arrow[r, "m^n_1"] & \bullet \cdots \bullet \arrow[r, "m^n_{k_n}"] & \bullet \arrow[d, "f' \circ f_n"] \\
\bullet \arrow[rrrr, "m"'] \arrow[rrrr, phantom, "\alpha \circ (\alpha_1, \ldots, \alpha_n)", shift left=4ex] & & & & \bullet
\end{tikzcd}
$$

units

$$
\begin{tikzcd}
a \arrow[d, equal, "id_a"] \\
a
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
a \arrow[r, "m"] \arrow[d, equal, "id_a"'] & b \arrow[d, equal, "id_b"] \\
a \arrow[r, "m"'] & b
\end{tikzcd}
$$