Double categories of relations relative to factorisation systems.

Virtual workshop on double categories 2024

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1. Double category of relations

2. Double categories vs factorisation systems

- $\mathcal{C}$: category with finite limits

· A relation $A \overset{R}{\nrightarrow} B$ is a monomorphism $|R| \rightarrowtail A \times B$.

· A span $A \overset{F}{\nrightarrow} B$ is a morphism $|F| \longrightarrow A \times B$

$\mathcal{M}$: class of mophisms in $\mathcal{C}$

- An $\mathcal{M}$-relation $A \xrightarrow{R} B$ is a morphism $|R| \longrightarrow A \times B$ in $\mathcal{M}$

Now define "cells" as follows

$$\begin{tikzcd}
A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] & {} \arrow[d, phantom, "\overset{\text{def}}{\iff}"] & {|R|} \arrow[r] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & A \times B \arrow[d, "f \times g"] \\
C \arrow[r, "S"', "\shortmid"{marking}] & D & {} & {|S|} \arrow[r] & C \times D
\end{tikzcd}$$

Fact

When $\mathcal{M}$ is the right class of a stable orthogonal factorisation system, then those data form a double caty $\mathbb{R}\mathrm{el}(\mathcal{C}, \mathcal{M})$

SOFS ⇝ dbl caty

Definition

A stable orthogonal factorisation system is a pair $(\mathcal{E}, \mathcal{M})$ of classes of morphisms s.t.

OFS

(i)
$$\begin{tikzcd} \cdot \arrow[rr, "\forall f"] \arrow[dr, "\mathcal{E} \ni \exists e"'] & & \cdot \\ & \cdot \arrow[ur, "\exists m \in \mathcal{M}"'] & \end{tikzcd}$$
(factorisation)

(ii)
$$\begin{tikzcd} \cdot \arrow[r, "\forall u"] \arrow[d, "\mathcal{E} \ni e"'] & \cdot \arrow[d, "m \in \mathcal{M}"] \\ \cdot \arrow[r, "\forall v"'] \arrow[ur, dotted, "\exists ! k"] & \cdot \end{tikzcd}$$
$(\mathcal{E} \perp \mathcal{M})$

(iii) $\mathcal{E}$ and $\mathcal{M}$ form replete subcaties of $\mathcal{E}^{\to}$

Stable

(iv)
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "e'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, "e \in \mathcal{E}"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$
$\Rightarrow e' \in \mathcal{E}$.

SOFS ⇝ dbl caty

Recall a double category $\mathbb{D}$ is a pseudo-category

- $$\begin{tikzcd} \mathbb{D}_0 \times_{\mathbb{D}_1} \mathbb{D}_0 \arrow[r, shift left=2] \arrow[r, "\text{comp.}"] \arrow[r, shift right=2] & \mathbb{D}_1 \arrow[r, shift left=2, "\text{dom.}"] \arrow[r, shift right=2, "\text{cod.}"'] & \underline{\mathbb{D}_0} \arrow[l, "\text{unit}" description] \end{tikzcd}$$

  $\mathbb{V}(\mathbb{D})$

  in Cat.

- $\mathbb{D}$ is an equipment

  iff $\mathbb{D}_1 \xrightarrow{\langle dom, cod \rangle} \mathbb{D}_0 \times \mathbb{D}_0$ is a bifibration

$$\begin{tikzcd} {} \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\text{opc}"] & {} \arrow[d] \\ {} \arrow[r, "|" marking] & {} \end{tikzcd} \quad , \quad \begin{tikzcd} {} \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & {} \arrow[d] \\ {} \arrow[r, "|" marking] & {} \end{tikzcd}$$

opcartesian arrow   cartesian arrow

with respect to $\mathbb{D}_1 \xrightarrow{\langle dom, cod \rangle} \mathbb{D}_0 \times \mathbb{D}_0$

SOFS ⇝ dbl caty

Recall a double category $\mathbb{D}$ is a pseudo-category

- $$\begin{tikzcd} \mathbb{D}_0 \times_{\mathbb{D}_1} \mathbb{D}_0 \arrow[r, shift left=2] \arrow[r, "\text{comp.}"] \arrow[r, shift right=2] & \mathbb{D}_1 \arrow[r, shift left=2, "\text{dom.}"] \arrow[r, shift right=2, "\text{cod.}"'] & \mathbb{D}_0 \arrow[l, "\text{unit}"'] \end{tikzcd}$$
  $\mathbb{D}_0$ — $V(\mathbb{D})$ in Cat.
- $\mathbb{D}$ is an equipment

  iff $\mathbb{D}_1 \xrightarrow{\langle \text{dom}, \text{cod} \rangle} \mathbb{D}_0 \times \mathbb{D}_0$ is a bifibration

companion of f — f_* conjoint of f — f^*

$$\begin{tikzcd} {} \arrow[rr, "f_*", "|" marking] \arrow[dr, "f"'] & {} & {} \arrow[dl, equal] \\ & {} \arrow[u, phantom, "\text{cart}"] & \end{tikzcd} , \quad \begin{tikzcd} {} \arrow[rr, "f^*", "|" marking] \arrow[dr, equal] & {} & {} \arrow[dl, "f"] \\ & {} \arrow[u, phantom, "\text{cart}"] & \end{tikzcd}$$

Fact $f_* \dashv f^*$

SOFS ⇝ dbl caty

Recall a double category $\mathbb{D}$ is a category

- $$\begin{tikzcd} \mathbb{D}_0 \times_{\mathbb{D}_1} \mathbb{D}_0 \arrow[r, shift left=2] \arrow[r, "\text{comp.}" description] \arrow[r, shift right=2] & \mathbb{D}_1 \arrow[r, shift left=2, "\text{dom.}"] \arrow[r, shift right=2, "\text{cod.}"'] & \mathbb{D}_0 \arrow[l, "\text{unit}" description] \end{tikzcd}$$

$V(\mathbb{D})$ in Cat.

- $\mathbb{D}$ is an equipment iff $\mathbb{D}_1 \xrightarrow{\langle dom, cod\rangle} \mathbb{D}_0 \times \mathbb{D}_0$ is a bifibration

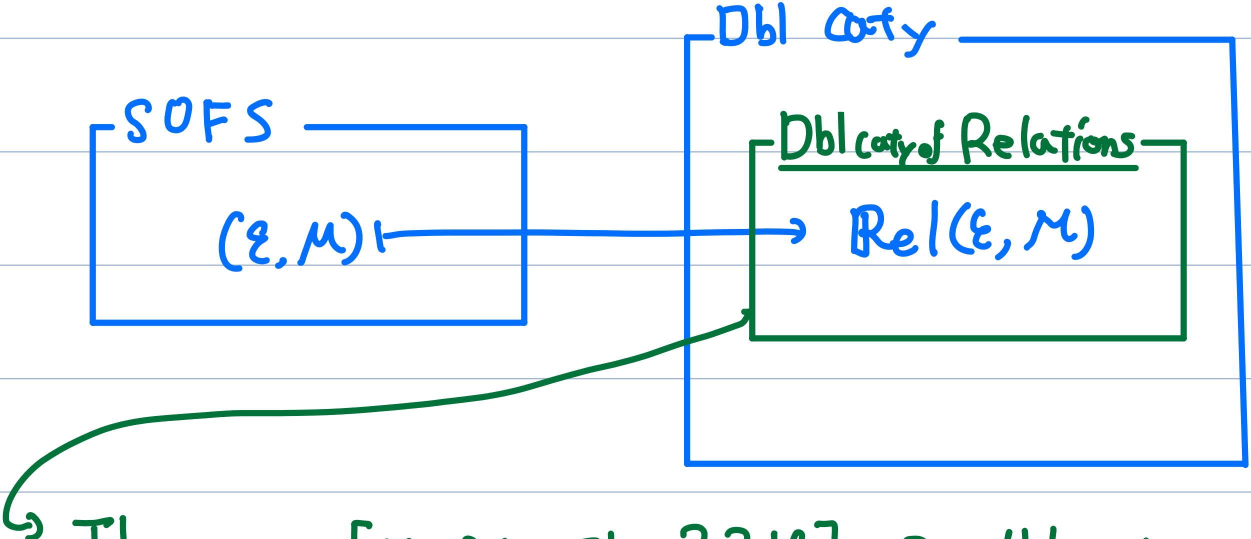
- $\mathrm{Rel}(\mathcal{E}, M)_1$ is defined by a pullback:

$$\begin{tikzcd} \mathrm{Rel}(\mathcal{E}, M)_1 \arrow[r] \arrow[d, "\langle dom, cod\rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "cod"] \\ \mathcal{E} \times \mathcal{E} \arrow[r, "\times"'] & \mathcal{E} \end{tikzcd}$$

M : full sub category of $\mathcal{E}^{\rightarrow}$

Fact $\mathrm{Rel}(\mathcal{E}, M)_1 \overset{dom}{\underset{cod}{\rightrightarrows}} \mathcal{E}$ extends to an equipment if $(\mathcal{E}, M)$ is an SOFS.

(Shulman 2008 : monoidal BC bifibration $\overset{\mathrm{Fr}}{\rightsquigarrow}$ equipment)



Theorem [HN24, Thm 3.3.16] $\mathbb{D}$: dbl caty.

$\mathbb{D} \cong \mathbb{R}\mathrm{el}(\mathcal{E},\mathcal{M})$ for some $(\mathcal{E},\mathcal{M})$: SOFS iff.

- $\mathbb{D}$ is a cartesian equipment
- $\mathbb{D}$ has Beck-Chevalley pullbacks and strong tabulators.
- Fibrations are closed under compositions

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$\mathbb{D}$ is a cartesian equipment

$\overset{\text{def}}{\Longleftrightarrow}$ $\mathbb{D} \xrightarrow[\Delta]{} \mathbb{D}\times\mathbb{D}$, $\mathbb{D} \xrightarrow{!} 1$ have right adjoints

[Verity]

[Aleiferi] in $\mathcal{E}\mathrm{quip}$: 2-caty of equipments + vertical naturals.

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A (pullback) square $\mathbf{V}\mathbb{D}\lrcorner$

$$\begin{tikzcd}
 & \bullet \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[dd, phantom, "\vee", very near start] & \\
\bullet \arrow[dr, "f"'] & & \bullet \arrow[dl, "g"] \\
 & \bullet &
\end{tikzcd}$$

is Beck-Chevalley

def $\Leftrightarrow$ The id cell factors as

$$\begin{tikzcd}
 & \bullet \arrow[dl, "h"'] \arrow[dr, "k"] & \\
\bullet \arrow[dr, "f"'] \arrow[rr, phantom, "="] & & \bullet \arrow[dl, "g"] \\
 & \bullet &
\end{tikzcd}
=
\begin{tikzcd}
 & \bullet \arrow[dl, "h"'] \arrow[dr, "k"] & \\
\bullet \arrow[dr, "f"'] \arrow[rr, "|" marking, "\mathrm{opc}", "\mathrm{cart}"'] & & \bullet \arrow[dl, "g"] \\
 & \bullet &
\end{tikzcd}$$

[Walters & Wood]

Theorem [HN24, Thm 3.3.16] $\mathbb{D}$: dbl cat'y.

$\mathbb{D} \simeq \mathbb{R}\mathrm{el}(\mathcal{E}, \mathcal{M})$ for some $(\mathcal{E}, \mathcal{M})$: OFS iff.

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A (pullback) square

$$\begin{tikzcd}
V\mathbb{D} & \cdot \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[dd, phantom, "\lrcorner" very near start] & \\
\cdot \arrow[dr, "f"'] & & \cdot \arrow[dl, "g"] \\
& \cdot &
\end{tikzcd}$$

is Beck-Chevalley $\overset{\text{def}}{\Leftrightarrow}$ The id cell factors as

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& \cdot \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[d, phantom, "\mathrm{opc}"] & \\
\cdot \arrow[rr, "|" marking] \arrow[dr, "f"'] & \mathrm{cart} & \cdot \arrow[dl, "g"] \\
& \cdot &
\end{tikzcd}$$

$\mathcal{C}$: category

$\mathcal{C}$: discrete $\Leftrightarrow$

$$\left(\begin{tikzcd}
\mathbb{P}\mathrm{rof} & \cdot \arrow[dl, "\exists h"'] \arrow[dr, "\exists k"] \arrow[dd, phantom, "\lrcorner" very near start] & \\
\cdot \arrow[dr, "\forall f"'] & & \cdot \arrow[dl, "g \forall"] \\
& \mathcal{C} &
\end{tikzcd} : \mathrm{B.C.} \right)$$

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"Every object is discrete"

A (pullback) square

$$\begin{tikzcd}[column sep=small] & \cdot \arrow[dl, "h"'] \arrow[dr, dash, "k"] \arrow[d, phantom, "\vee"] & \\ \cdot \arrow[dr, "f"'] & {} & \cdot \arrow[dl, "g"] \\ & \cdot & \end{tikzcd} \quad \text{in } \mathbb{V}\mathbb{D}$$

is Beck-Chevalley

$\overset{\text{def}}{\Longleftrightarrow}$ The id cell factors as

$$\begin{tikzcd}[column sep=small] & \cdot \arrow[dl, "h"'] \arrow[dr, dash, "k"] & \\ \cdot \arrow[dr, dash, "f"'] & = & \cdot \arrow[dl, "g"] \\ & \cdot & \end{tikzcd} = \begin{tikzcd}[column sep=small] & \cdot \arrow[dl, "h"'] \arrow[dr, dash, "k"] \arrow[d, phantom, "\mathrm{opc}"] & \\ \cdot \arrow[rr, "|"] \arrow[dr, dash, "f"'] & \mathrm{cart} & \cdot \arrow[dl, "g"] \\ & \cdot & \end{tikzcd}$$

$\mathcal{C}$: Category

$$\mathcal{C} : \text{discrete} \Longleftrightarrow \left(\begin{tikzcd}[column sep=small] & \exists\, \cdot \arrow[dl, "h"'] \arrow[dr, dash, "k"] \arrow[d, phantom, "\vee"] & \\ \forall\, \cdot \arrow[dr, "f"'] & {} & \cdot\, \forall \arrow[dl, "g"] \\ & \mathcal{C} & \end{tikzcd} \text{ in } \mathbb{P}\mathrm{rof} : \mathrm{B.C.} \right)$$

Theorem [HN24, Thm 3.3.16] $\mathbb{D}$: dbl cat'y.

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$\mathbb{D}$

$$\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "t"] & \\
X \arrow[rr, "P"', "|" marking] & {} & Y
\end{tikzcd}$$

exhibits A as a tabulator of P

$\overset{\text{def}}{\Longleftarrow}$ [Grandis & Paré]

$\mathbb{D}$

$$\begin{tikzcd}
 & \forall A' \arrow[dl, "\forall f'"'] \arrow[dr, "\forall g'"] \arrow[d, phantom, "\forall t'"] & \\
X \arrow[rr, "P"', "|" marking] & {} & Y
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$\exists ! u$

$$\begin{tikzcd}
 & A' \arrow[dl, "f'"'] \arrow[dr, "g'"] \arrow[d, phantom, "t'"] & \\
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\end{tikzcd}
=
\begin{tikzcd}
 & A' \arrow[ddl, "f'"'] \arrow[ddr, "g'"] \arrow[d, "u"] \arrow[dl, phantom, "\wr"] \arrow[dr, phantom, "a"] & \\
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A tabulator is strong if t is opcartesian ($f^{*} \odot g_{!} \cong P$)

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"Hor. arrows are spans"

$\mathbb{D}$
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[Grandis & Paré]

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$\exists ! u$
$$\begin{tikzcd} & A' \arrow[dl, "f'"'] \arrow[dr, "g'"] \arrow[d, phantom, "t'"] & \\ X \arrow[rr, "P"', "|"{marking}] & {} & Y \end{tikzcd} \quad = \quad \begin{tikzcd} & A' \arrow[d, "u"] \arrow[ddl, bend right, "f'"'] \arrow[ddr, bend left, "g'"] & \\ & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "t"] \arrow[l, phantom, "\cong"] \arrow[r, phantom, "a"] & \\ X \arrow[rr, "P"', "|"{marking}] & {} & Y \end{tikzcd}$$

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$$\mathbb{D} \quad \begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \text{ is a fibration } \overset{\text{def}}{\underset{\text{[HN]}}{\Longleftrightarrow}} \quad \mathbb{D} \quad \begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "!"] \arrow[d, phantom, "\exists \mathrm{tab}"] & \\ B \arrow[rr, "|" marking, "\exists p"'] & & 1 \end{tikzcd}$$

Theorem [HN24, Thm 3.3.16.] $\mathbb{D}$: dbl caty.

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$$\mathbb{D}\ \lrcorner \quad \begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \text{ is a fibration } \overset{\text{def}}{\underset{\text{[HN]}}{\Longleftarrow}} \quad \mathbb{D}\ \lrcorner \quad \begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "!"] \arrow[d, phantom, "\exists\,\mathrm{tab}"] & \\ B \arrow[rr, "\exists\, p"'] & & 1 \end{tikzcd}$$

$$\mathcal{M} = \{\text{fibrations}\}$$

$$\mathbb{D}\,\lrcorner \qquad \begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \quad \text{is final} \quad \overset{\text{def}}{\iff} \quad \begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "!"] & \\ B \arrow[dr, "!"'] & = & 1 \arrow[dl, equal] \\ & 1 & \end{tikzcd} = \begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "!"] & \\ B \arrow[rr, "|" marking, "\text{opc}", "\text{cart}"'] \arrow[dr, "!"'] & & 1 \arrow[dl, equal] \\ & 1 & \end{tikzcd}$$

$\mathbb{D}$

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd}$$ is <u>final</u> $\overset{\text{def}}{\Longleftrightarrow}$

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<u>Thm.</u> (Comprehensive Factorisation, Thm 3.3.11. in [HN])

$\mathbb{D}$ is an equipment w/ s. tabulators & composable fibrations

$\Rightarrow$ (Final, Fibration) form an OFS

<u>E.g.</u> In $\mathbb{P}\mathrm{rof}$, f is final iff $\operatorname{colim}_{a \in A} B(b, fa) = 1$,

which recovers final functors

$\rightsquigarrow$ The comprehensive factorisation system

(Street & Walters)

Defn. [Walters & Wood]

$\mathbb{D}$ is discrete $\overset{\text{def}}{\Longleftrightarrow}$

$$\begin{tikzcd} & A \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] \arrow[d, phantom, "\vee"] & \\ A\times A \arrow[dr, "\Delta\times id"'] & {} & A\times A \arrow[dl, "id\times\Delta"] \\ & A\times A\times A & \end{tikzcd}$$,

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\vee"] & \\ A \arrow[dr, "\Delta"'] & {} & A \arrow[dl, "\Delta"] \\ & A\times A & \end{tikzcd}$$

are Beck-Chevalley

(**Rmk.** $\mathbb{P}$rof is not discrete)

Defn. [Walters & Wood]

$\mathbb{D}$ is discrete $\overset{\text{def}}{\iff}$

$$\begin{tikzcd} & A \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ A\times A \arrow[dr, "\Delta\times id"'] & \vee & A\times A \arrow[dl, "id\times\Delta"] \\ & A\times A\times A & \end{tikzcd} , \qquad \begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, equal] & \\ A \arrow[dr, "\Delta"'] & \vee & A \arrow[dl, "\Delta"] \\ & A\times A & \end{tikzcd}$$

are Beck-Chevalley

(Rmk. $\mathbb{P}$rof is not discrete)

Prop. · A discrete cartesian equipment is $\underbrace{\text{dagger compact}}$

(Every object is horizontally self-dual)

Defn. [Walters & Wood]

$\mathbb{D}$ is discrete $\overset{\text{def}}{\iff}$

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are Beck-Chevalley

(Rmk. $\mathbb{P}$rof is not discrete)

Prop. • A discrete cartesian equipment is dagger compact

(Every object is horizontally self-dual)

• There is a duality

$$\begin{tikzcd} & A \arrow[dl, "{\langle f,g\rangle}"'] \arrow[dr, "h"] \arrow[d, phantom, "\hat{\alpha}"] & \\ X\times Y \arrow[rr, "\hat{p}"', "|" marking] & {} & Z \end{tikzcd} \quad \Big\| \quad \begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "{\langle g,h\rangle}"] \arrow[d, phantom, "\check{\alpha}"] & \\ X \arrow[rr, "\check{p}"', "|" marking] & {} & Y\times Z \end{tikzcd}$$

$\mathbb{D}$ $\quad A \xrightarrow{f} B$ is an

- inclusion iff $$\begin{tikzcd} A \arrow[rr, equals, "\shortmid" marking] \arrow[dr, "f"'] & {\scriptstyle \text{cart}} & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$ (fully faithful in $\mathbb{P}\mathrm{rof}$)

- cover iff $$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "f"] & \\ B \arrow[rr, equals, "\shortmid" marking] & {\scriptstyle \text{opc}} & B \end{tikzcd}$$ (absolutely dense in $\mathbb{P}\mathrm{rof}$)

Prop.

- In a discrete cartesian equipment $\mathbb{D}$,

  Cover $=$ Final ($= \mathcal{E}$ for $\mathbb{R}\mathrm{el}(\mathcal{E}, \mathcal{M})$)

- If $\mathbb{D}$ has all Beck-Chevalley pullbacks,

  final morphisms are stable.

$\mathbb{D}\lrcorner$ $\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}$ is an

- inclusion iff $$\begin{tikzcd} A \arrow[rr, equal, "|" marking] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\mathrm{cart}"] & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$ (fully faithful in $\mathbb{P}\mathrm{rof}$)

- cover iff $$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "f"] & \\ B \arrow[rr, equal, "|" marking] & {} \arrow[u, phantom, "\mathrm{opc}"] & B \end{tikzcd}$$ (lax epi. in $\mathbb{P}\mathrm{rof}$)

Prop.

- In a discrete cartesian equipment $\mathbb{D}$,

  Cover $=$ Final ($= \mathcal{E}$ for $\mathbb{R}\mathrm{el}(\mathcal{E}, \mathcal{M})$)

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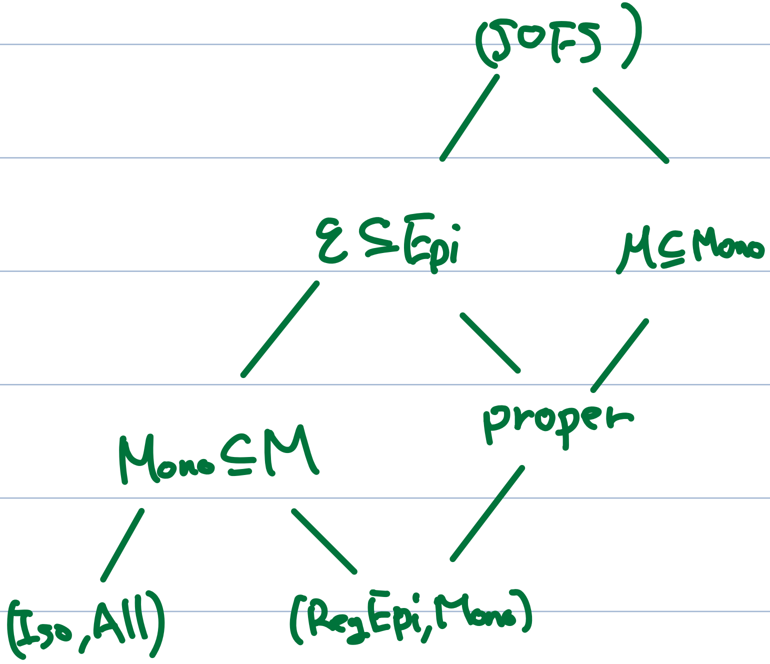
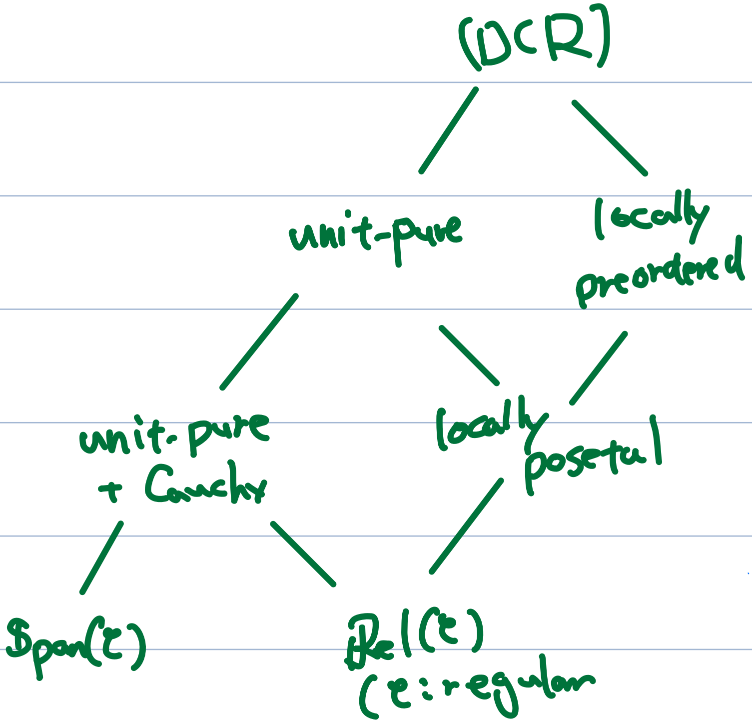
$\rightsquigarrow$ (Final, Fibration) : SOFS

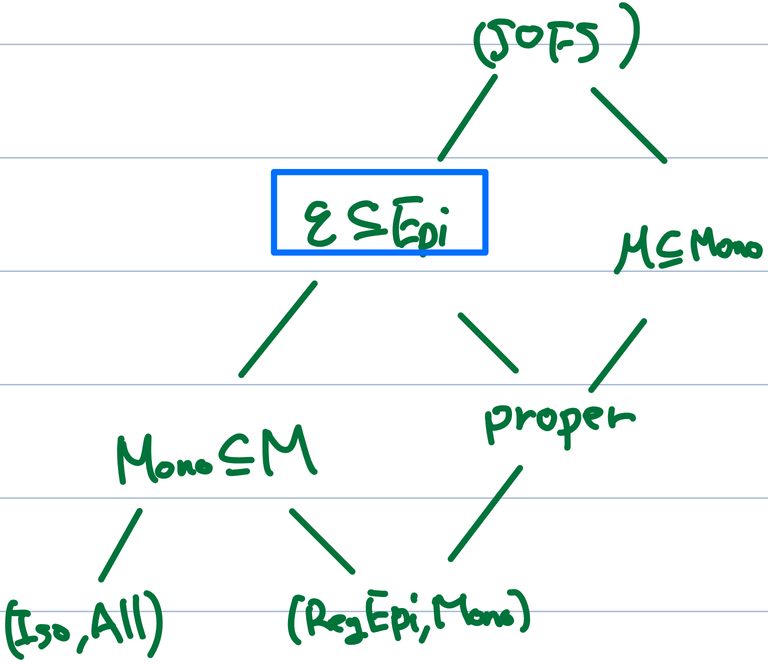
VWDC 2024 _ DCR

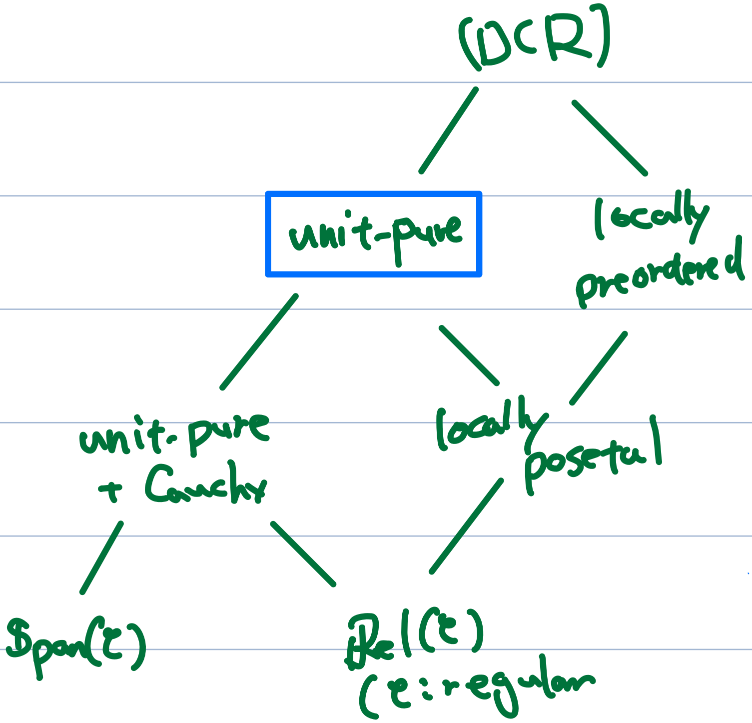
1. Double category of relations

2. Double categories vs factorisation systems


(JOFS)
ℰ ⊆ Epi
ℳ ⊆ Mono
Mono ⊆ ℳ
proper
(Iso, All)
(RegEpi, Mono)


(DCR)
unit-pure
locally preordered
unit-pure + Cauchy
locally posetal
Span(𝒞)
Rel(𝒞)
(𝒞: regular


(SOFS)
$\mathcal{E} \subseteq Epi$
$\mathcal{M} \subseteq Mono$
$Mono \subseteq \mathcal{M}$
proper
(Iso, All)
(RegEpi, Mono)


(DCR)
unit-pure
locally preordered
unit-pure + Cauchy
locally posetal
Span(𝒞)
Rel(𝒞)
(𝒞: regular

$\mathbb{D}$ is unit-pure

$\overset{\text{def}}{\Longleftrightarrow}$
$$\begin{tikzcd} x \arrow[d, bend right=60, "u"'] \arrow[d, bend left=60, "v"] \arrow[d, phantom, "\alpha"] \\ x \end{tikzcd} \Rightarrow u = v, \ \alpha = id_u$$

(There is no non-trivial cell of the form $\begin{tikzcd} \cdot \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\cdot"] \\ \cdot \end{tikzcd}$)

i.e.

$\underbrace{\mathcal{V}(\mathbb{D})}$ is locally discrete

└ the 2-cat'y of vertical arrows & $\begin{tikzcd} \cdot \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\cdot"] \\ \cdot \end{tikzcd}$

- $\mathbb{D}$ is unit-pure

def $\Longleftrightarrow$

$$\begin{tikzcd} X \arrow[d, bend right=60, "u"'] \arrow[d, bend left=60, "v"] \arrow[d, phantom, "\exists\alpha"] \\ Y \end{tikzcd} \Rightarrow u = v,\ \alpha = \mathrm{id}_u$$

(There is no non-trivial cell of the form $\begin{tikzcd} {} \arrow[d, bend right=60] \arrow[d, bend left=60] \arrow[d, phantom, "\cdot"] \\ {} \end{tikzcd}$)

Prop. $\mathbb{D}$ is a unit-pure equipment w/ strong tabulators

$\Rightarrow$ $\mathbb{D}$ has Beck-Chevalley pullbacks

Proof)

- $\mathbb{D}$ is unit-pure

def $\Longleftrightarrow$

$$\begin{tikzcd} X \arrow[d, bend right=50, "u"'] \arrow[d, bend left=50, "v"] \arrow[d, phantom, "\exists\alpha"] \\ Y \end{tikzcd} \Rightarrow u=v, \ \alpha = id_u$$

(There is no non-trivial cell of the form $\begin{tikzcd}[column sep=small] \cdot \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\cdot"] \\ \cdot \end{tikzcd}$)

<u>Prop.</u> $\mathbb{D}$ is a unit-pure equipment w/ strong tabulators

$\Rightarrow$ $\mathbb{D}$ has Beck-Chevalley pullbacks

<u>Proof)</u>

$$\begin{tikzcd} & T(C(f,g)) \arrow[dl, "l"'] \arrow[dr, "r"] \arrow[d, phantom, "tab"] & \\ A \arrow[rr, "|" marking] \arrow[dr, "f"'] & {} \arrow[d, phantom, "cort"] & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

is a pullback square.

□.

$\begin{array}{c} X \\ \downarrow{\scriptstyle f} \\ Y \end{array}$ is a cover $\Rightarrow$ f is co-fully faithful in $\mathcal{V}(\mathbb{D})$

$$\left(\forall A,\ \mathcal{V}(\mathbb{D})(Y,A) \xrightarrow{-\circ f} \mathcal{V}(\mathbb{D})(X,A) : \text{f.f.}\right)$$

$\underset{\mathbb{D}:\text{unit-pure}}{\Rightarrow}$ f is epi

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$$ is a cover $\Rightarrow$ f is co-fully faithful in $\mathcal{V}(\mathbb{D})$

$\left(\forall A,\ \mathcal{V}(\mathbb{D})(Y, A) \xrightarrow{- \circ f} \mathcal{V}(\mathbb{D})(X, A) : \text{f.f.} \right)$

$\underset{\mathbb{D}:\text{unit-pure}}{\Rightarrow}$ f is epi

Prop ([HN, 4.1.2., 3.3.10])

For a DCR $\mathbb{D}$,

$\mathbb{D}$ is unit-pure $\Longleftrightarrow$ Final(=Cover) $\subseteq$ Epi

Thm TFAE

(i) $\mathbb{D} \cong \mathbb{R}\mathrm{el}(\overset{\exists}{\mathcal{E}}, \overset{\exists}{\mathcal{M}})$, $\mathcal{E} \subseteq \mathrm{Epi}$

(ii) $\mathbb{D}$: unit-pure cartesian equipment w/ st.tab.

& fibrations are composable

Thm TFAE

(i) $\mathbb{D} \cong \mathbb{R}\mathrm{el}(\overset{\exists}{\mathcal{E}}, \overset{\exists}{\mathcal{M}})$, $\mathcal{E} \subseteq \mathrm{Epi}$

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& fibrations are composable

- Inclusion $\subseteq$ Mono if $\mathbb{D}$: unit-pure (the dual of the previous claim)
- Furthermore

Thm TFAE

(i) $\mathbb{D} \cong \mathbb{R}\mathrm{el}(\overset{\exists}{\mathcal{E}}, \overset{\exists}{\mathcal{M}})$, $\mathcal{E} \subseteq \mathrm{Epi}$

(ii) $\mathbb{D}$: unit-pure cartesian equipment w/ st. tab.

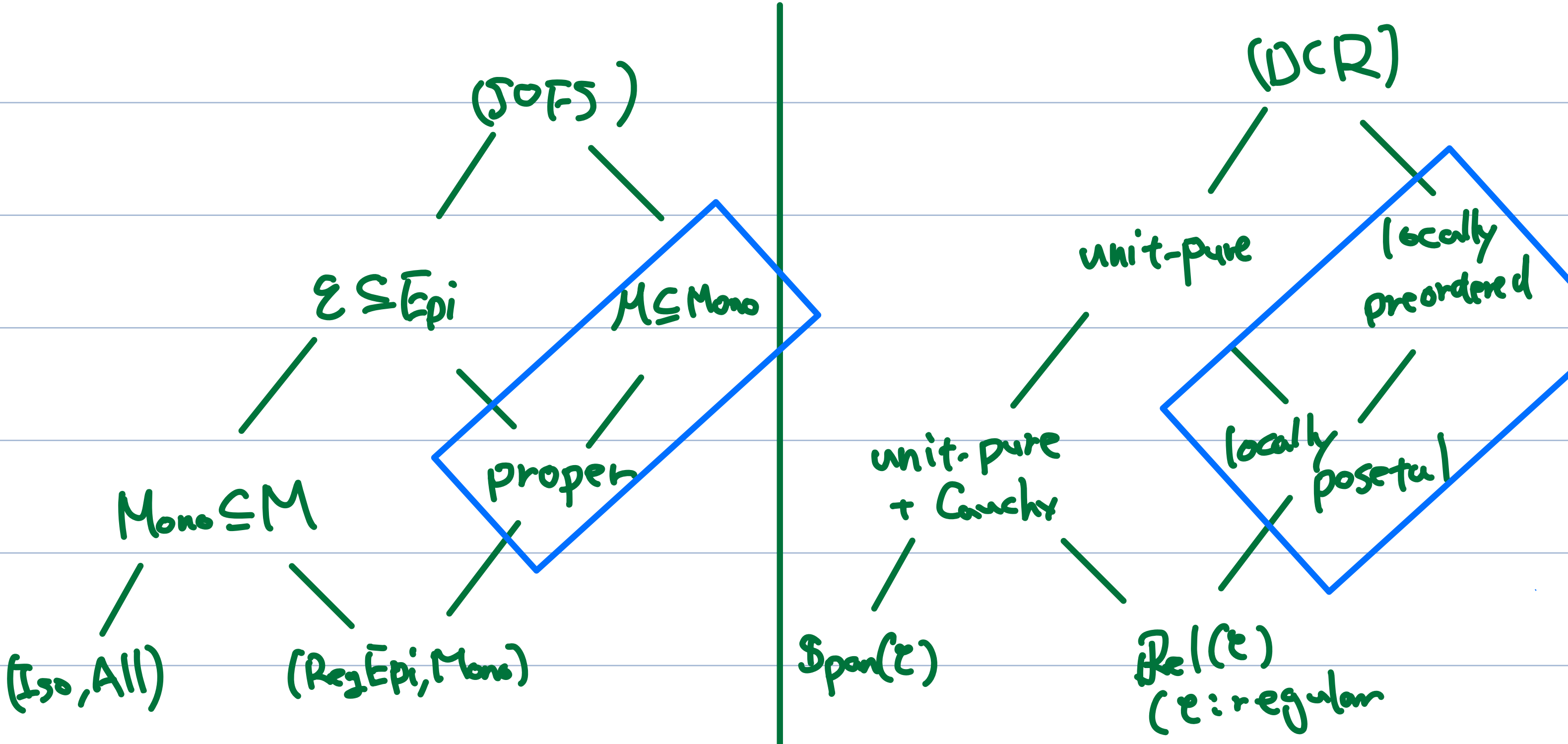
& fibrations are composable

- Inclusion $\subseteq$ Mono if $\mathbb{D}$: unit-pure

- Furthermore

Inclusion $=$ Mono when $\mathbb{D}$: unit-pure DCR

(m : mono $\Leftrightarrow$
$$\begin{tikzcd}[column sep=small] & \cdot \arrow[dl, equal] \arrow[dr, equal] \arrow[dd, phantom, "\lrcorner"] & \\ \cdot \arrow[dr, "m"'] & & \cdot \arrow[dl, "m"] \\ & \cdot & \end{tikzcd}$$
$\overset{\text{B.C.}}{\Longrightarrow}$
$$\begin{tikzcd}[column sep=small] & \cdot \arrow[dl, equal] \arrow[dr, equal] & \\ \cdot \arrow[rr, equal, "|" marking] \arrow[dr, "m"'] \arrow[ur, phantom, "\text{opc}" near end, shift right=2ex] & & \cdot \arrow[dl, "m"] \\ & \cdot \arrow[u, phantom, "\text{cart}"] & \end{tikzcd}$$
$\Longleftrightarrow$ m : inclusion)


(SOFS)
ℰ ⊆ Epi
ℳ ⊆ Mono
Mono ⊆ ℳ
proper
(Iso, All)
(RegEpi, Mono)
(DCR)
unit-pure
locally preordered
unit-pure + Cauchy
locally posetal
Span(𝒞)
Rel(𝒞)
(𝒞 : regular

Defn.

- $\mathbb{D}$ is locally preordered (flat in Grandis & Paré)

iff
$$\begin{tikzcd} A \arrow[r, "p", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "|" marking] & D \end{tikzcd}, \quad \begin{tikzcd} A \arrow[r, "p", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\beta"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "|" marking] & D \end{tikzcd} \quad \Rightarrow \quad \alpha = \beta .$$

Observe that in any DCR,

$$\begin{tikzcd} A \arrow[r, "p", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "|" marking] & D \end{tikzcd} \quad \Vert \quad \begin{tikzcd} A\times B \arrow[r, "\check{p}", "|" marking] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\alpha"] & 1 \arrow[d, equal] \\ C\times D \arrow[r, "\check{q}"', "|" marking] & 1 \end{tikzcd} \quad \Vert \quad \begin{tikzcd} A\times B \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & {|\check{p}|} \arrow[l] \arrow[d, "|\alpha|"] \\ C\times D & {|\check{q}|} \arrow[l] \end{tikzcd}$$

dagger compact

strong tabulator

fibration

Observe that in any DCR,

$$\begin{tikzcd}
A \arrow[r, "p", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] & \Big\| & A\times B \arrow[r, "\check{p}", "|"{marking}] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\alpha"] & 1 \arrow[d, equal] & \Big\| & A\times B \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & {|\check{p}|} \arrow[l] \arrow[d, "|\alpha|"] \\
C \arrow[r, "q"', "|"{marking}] & D & & C\times D \arrow[r, "\check{q}"', "|"{marking}] & 1 & & C\times D & {|\check{q}|} \arrow[l, "\text{fibration}"']
\end{tikzcd}$$

- Fibration ⊆ Mono ⇒ locally preordered

Observe that in any DCR,

$$\begin{tikzcd} A \arrow[r, "p", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "|" marking] & D \end{tikzcd} = \begin{tikzcd} A\times B \arrow[r, "\check{p}", "|" marking] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\alpha"] & 1 \arrow[d, equal] \\ C\times D \arrow[r, "\check{q}"', "|" marking] & 1 \end{tikzcd} = \begin{tikzcd} A\times B \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & {|\check{p}|} \arrow[l] \arrow[d, "|\alpha|"] \\ C\times D & {|\check{q}|} \arrow[l] \end{tikzcd}$$

- Fibration $\subseteq$ Mono $\Rightarrow$ locally preordered

Prop. For a DCR,

locally preorderd $\Leftarrow$ Fibration $\subseteq$ Mono

Defn.

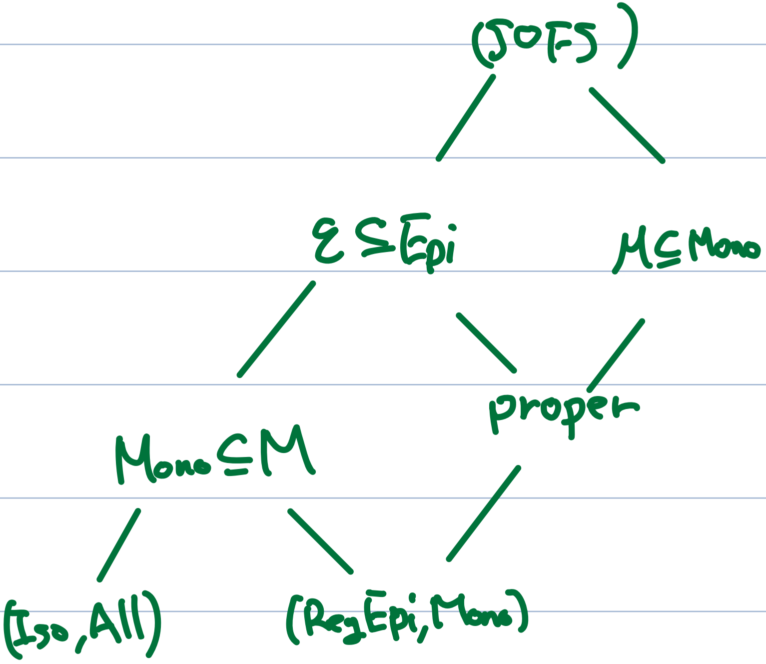
- $\mathbb{D}$ is locally posetal iff it is locally preordered and $\underbrace{\mathcal{V}(\mathbb{D})}_{\text{the 2-caty of vertical arrows.}}$ is locally posetal

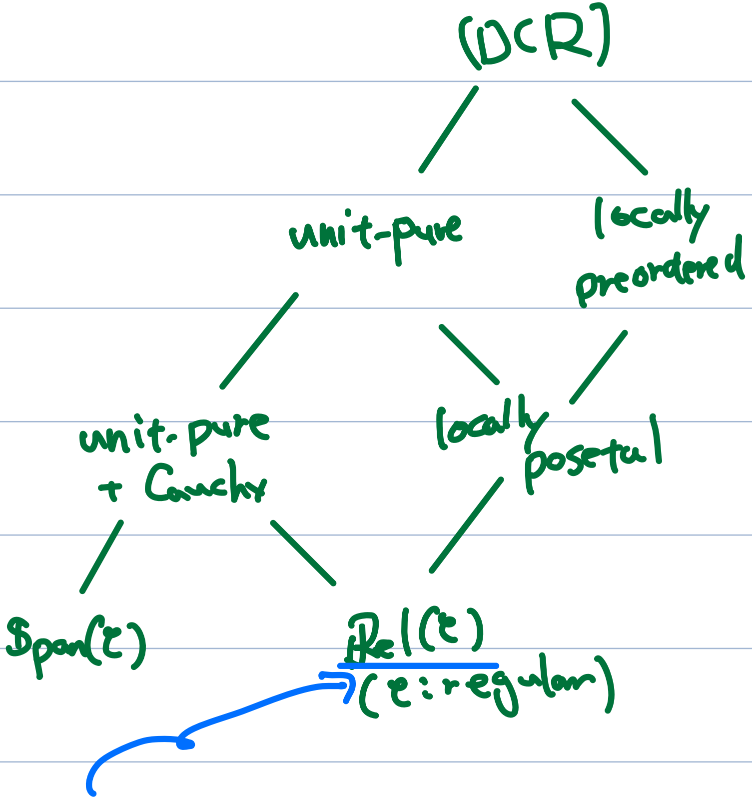
Prop. For a discrete cartesian equipment,

$$\text{locally posetal} \iff \begin{array}{l}\text{locally preordered}\\ + \text{unit-pure}\end{array}$$

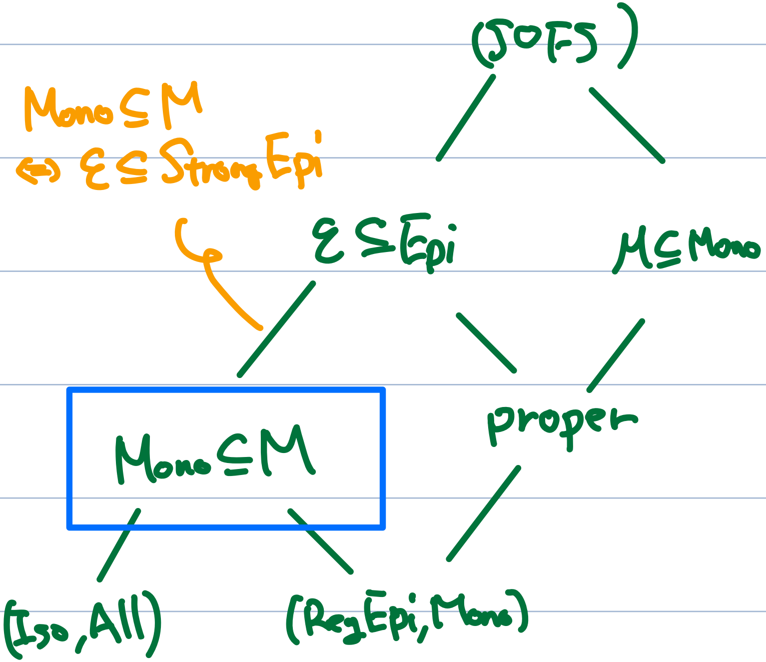
Prop. For a DCR,

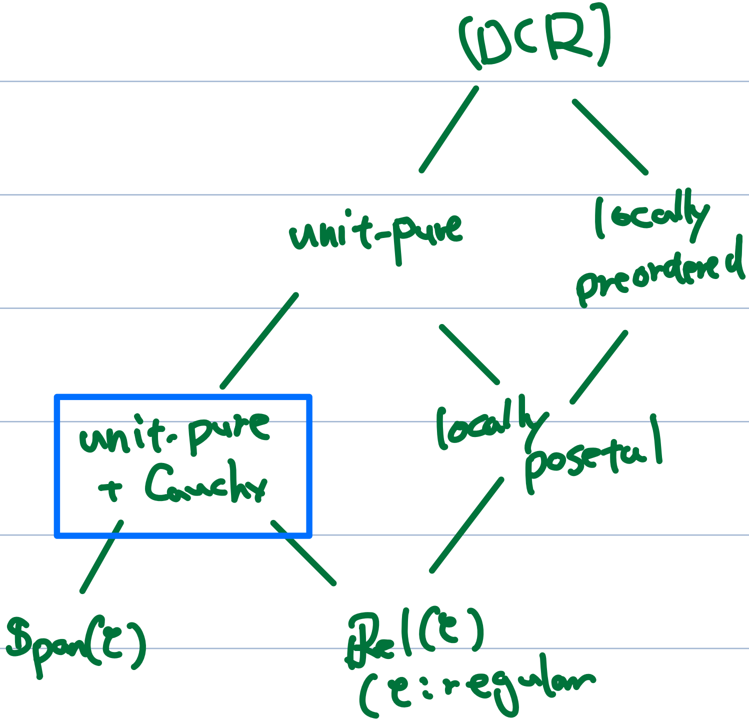
$$\text{locally posetal} \iff \begin{array}{l}\text{proper}\\ (\mathcal{M} \subseteq \mathrm{Mono},\ \mathcal{E} \subseteq \mathrm{Epi})\end{array}$$





- Carboni, Walters
  by functional completeness
- Lambert


(JOFS)
Mono ⊆ M
⟺ ε ⊆ StrongEpi
ε ⊆ Epi
M ⊆ Mono
Mono ⊆ M
proper
(Iso, All)
(RegEpi, Mono)


(DCR)
unit-pure
locally preordered
unit-pure + Cauchy
locally posetal
Span(C)
Rel(C)
(C: regular

In $\mathbf{Prof}$, $\left(P : \mathcal{A} \nrightarrow \mathcal{B} \iff P : \mathcal{A}^{op} \times \mathcal{B} \to \mathbf{Set} \right)$

- $\mathcal{C}$ is Cauchy-complete $\iff$

$$\left(\begin{tikzcd} \forall \mathcal{A} \arrow[r, bend left, "\forall P", "\shortmid"{marking}] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "\forall Q"] \end{tikzcd} \Rightarrow P = f_* \text{ for some } f : \mathcal{A} \to \mathcal{C} \right).$$

In $\mathbb{P}\mathrm{rof}$, $\quad (P : \mathcal{A} \nrightarrow \mathcal{B} \iff P : \mathcal{A}^{op} \times \mathcal{B} \to \mathrm{Set})$

- $\mathcal{C}$ is Cauchy-complete $\iff$

$$\left(\begin{tikzcd} \forall \mathcal{A} \arrow[r, shift left=2, "\forall P", "\mid" marking] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left=2, "\forall Q", "\mid" marking] \end{tikzcd} \implies P = f_* \text{ for some } f : \mathcal{A} \to \mathcal{C} \right).$$

Defn (Paré 2021)

$\mathbb{D}$ is Cauchy $\overset{\text{def}}{\iff}$ $\forall A \in \mathbb{D}$ $\forall$
$$\begin{tikzcd} B \arrow[r, shift left=2, "\forall P", "\mid" marking] \arrow[r, phantom, "\perp"] & A \arrow[l, shift left=2, "\forall Q", "\mid" marking] \end{tikzcd},$$
$\exists f : B \to A$, $P \cong f_*$.

In $\mathbb{P}\mathrm{rof}$, $\left(P : \mathcal{A} \nrightarrow \mathcal{B} \Leftrightarrow P : \mathcal{A}^{op} \times \mathcal{B} \to \mathrm{Set} \right)$

- $\mathcal{C}$ is Cauchy-complete $\Leftrightarrow$

$$\left(\begin{tikzcd} \forall \mathcal{A} \arrow[r, shift left=2, "\forall P", "|"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left=2, "\forall Q", "|"'] \end{tikzcd} \Rightarrow P = f_* \text{ for some } f : \mathcal{A} \to \mathcal{C} \right).$$

Defn (Paré 2021)

$\mathbb{D}$ is Cauchy $\overset{def}{\Longleftrightarrow}$ $\forall A \in \mathbb{D}$ $\forall$

$$\begin{tikzcd} B \arrow[r, shift left=2, "\forall P", "|"'] \arrow[r, phantom, "\perp"] & A \arrow[l, shift left=2, "\forall Q", "|"'] \end{tikzcd} \ \exists f : B \to A,\ P \cong f_*.$$

$\exists!$ in unit-pure

In $\mathbb{P}\mathrm{rof}$, $\quad (P : \mathcal{A} \nrightarrow \mathcal{B} \Leftrightarrow P : \mathcal{A}^{op} \times \mathcal{B} \to \mathrm{Set})$

- $\mathcal{C}$ is Cauchy-complete $\Leftrightarrow$ $\left(\begin{tikzcd} \forall\mathcal{A} \arrow[r, bend left, "\forall P", "|"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "\forall Q", "|"'] \end{tikzcd} \Rightarrow P = f_* \text{ for some } f : \mathcal{A} \to \mathcal{C}\right)$

Defn (Paré 2021)

$\mathbb{D}$ is Cauchy $\overset{\text{def}}{\Longleftrightarrow}$ $\forall A \in \mathbb{D}$ $\forall$ $\begin{tikzcd} B \arrow[r, bend left, "\forall P", "|"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "\forall Q", "|"'] \end{tikzcd}$, $\exists f : B \to A$, $P \cong f_*$.

$\exists!$ in unit-pure

Thm In a unit-pure Cauchy equipment,

$$\text{Cover} \perp \text{Inclusion}.$$

If $\mathbb{D}$ is moreover DCR,

(Inclusion $=$) Mono $\subseteq$ Fibration ($=$ Cover$^{\perp}$)

Thm (4.2.3. in HN)

In a unit-pure DCR,

$P : A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ is a left adjoint $\iff$

$$\begin{tikzcd}
 & \exists\, TP \arrow[dl, "\exists \ell"'] \arrow[dr, "\exists r"] & \\
A \arrow[rr, "P"', "|" marking] & \text{tab} & B
\end{tikzcd},$$

$\ell \in \mathrm{Mono} \cap \mathrm{Cover}$

(This generalise the fact that

a span $A \overset{P}{\relbar\joinrel\mapstochar\joinrel\rightarrow} B = (A \xleftarrow{\ell} |P| \xrightarrow{r} B)$ is a left adj.

iff ℓ is an isomorphism)

Thm (4.2.3. in HN)

In a unit-pure DCR,

$P : A \xrightarrow{+} B$ is a left adjoint $\iff$

$$\begin{tikzcd} & \exists TP \arrow[dl, "\exists l"'] \arrow[dr, "\exists r"] \arrow[d, phantom, "tab"] & \\ A \arrow[rr, "|" marking, "p"'] & {} & B \end{tikzcd}$$

, $l \in \mathrm{Mono} \cap \mathrm{Cover}$

(This generalise the fact that

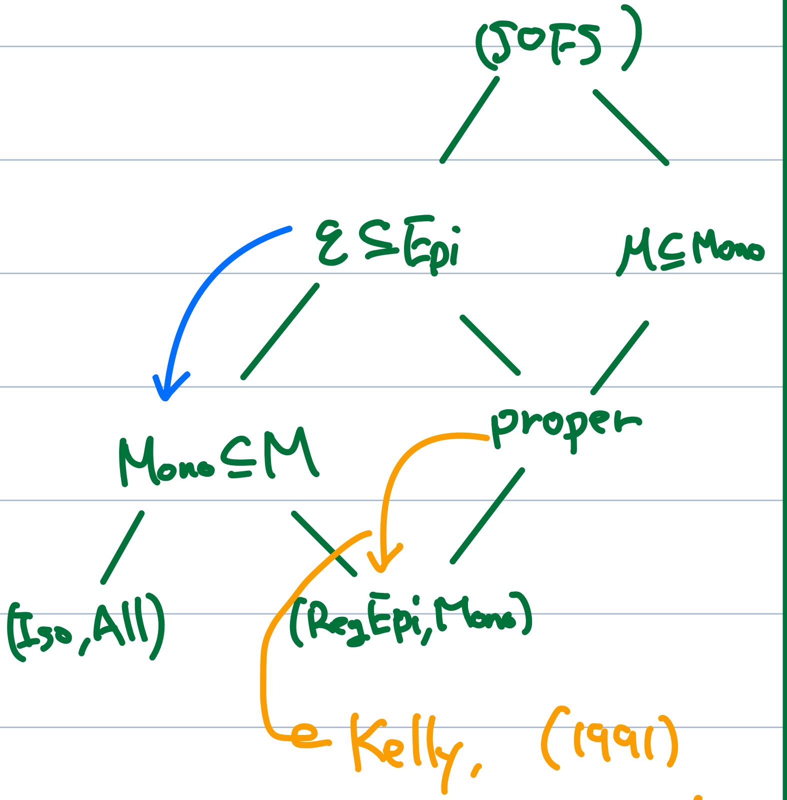
a span $A \xrightarrow{p} B = (A \xleftarrow{l} |P| \xrightarrow{r} B)$ is a left adj.

iff l is an isomorphism)

Thm. A DCR $\mathbb{D} \simeq \mathbb{R}\mathrm{el}(\mathcal{E}, \mathcal{M})$ is

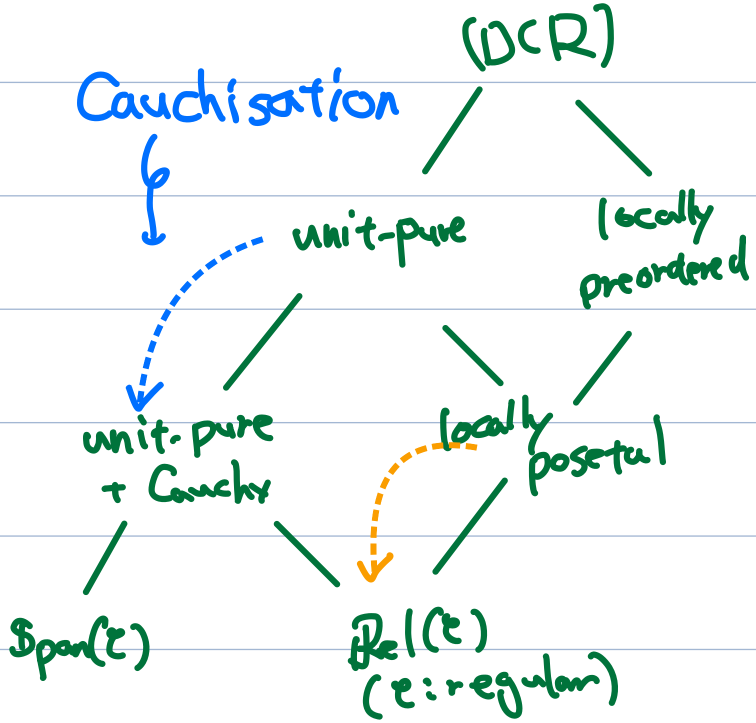
unit-pure $\wedge$ Cauchy iff $\mathrm{Mono} \subseteq \mathrm{Fibration}$

cf. [Pavlovic. Maps I. Relative to a factorisation system]



Kelly, (1991)

A note on relations relative to a factorisation system



Let $\mathbb{D}, \mathbb{E}$: unit-pure equipments.

- $\mathbb{E}$ is a Cauchisation of $\mathbb{D}$

$\overset{\text{def}}{\Longleftrightarrow}$ · $\mathcal{H}(\mathbb{E}) = \mathcal{H}(\mathbb{D})$

· $\mathbb{E}$: Cauchy & unit-pure.

Rmk. $\mathbb{E}$ is a free-object of

$$\left\{\begin{matrix}\text{unit-pure}\\ \text{Cauchy}\end{matrix} \text{ equipment}\right\} \hookrightarrow \left\{\text{unit-pure equipment}\right\} \left(\overset{\text{full}}{\subseteq} \mathcal{E}quip\right)$$

i.e.

$$\begin{tikzcd} & \mathbb{D} \arrow[dr, "U"] & \\ \mathbb{E} \arrow[ur, "\subset"'] \arrow[rr, "V"] & & \mathbb{D}' \end{tikzcd}$$

when $\mathbb{D}'$ is unit-pure & Cauchy

$f\big\downarrow \longmapsto Vf\big\downarrow$ s.t. $(Vf)_* \cong U(f_*)$

Let $\mathbb{D}, \mathbb{E}$: unit-pure equipments

- $\mathbb{E}$ is a Cauchisation of $\mathbb{D}$

  $\overset{\text{def}}{\iff}$ · $\mathcal{H}(\mathbb{E}) = \mathcal{H}(\mathbb{D})$

  · $\mathbb{E}$: Cauchy & unit-pure.

(Free object of $\left\{\begin{matrix}\text{unit-pure} \\ \text{Cauchy}\end{matrix} \text{ equipment}\right\} \hookrightarrow \left\{\text{unit-pure equipment}\right\}$

up-to equivalence.)

<u>Rmk.</u> Cauchisation is unique in Equip if exists.

Let $\mathbb{D}, \mathbb{E}$: unit-pure equipments

- $\mathbb{E}$ is a Cauchisation of $\mathbb{D}$ $\overset{\text{def}}{\Longleftrightarrow}$
  - $\mathcal{H}(\mathbb{E}) = \mathcal{H}(\mathbb{D})$
  - $\mathbb{E}$: Cauchy & unit-pure.

(Free object of $\left\{\begin{matrix}\text{unit-pure} \\ \text{Cauchy}\end{matrix} \text{ equipment}\right\} \hookrightarrow \{\text{unit-pure equipment}\}$ up-to equivalence.)

Rmk. Cauchisation is unique in $\mathcal{E}\mathrm{quip}$ if exists.

Thm. $\mathbb{D}$: unit-pure DCR has a cauchisation $\mathbb{C}\mathrm{au}(\mathbb{D})$:

- $\mathrm{Obj}(\mathbb{C}\mathrm{au}(\mathbb{D})) = \mathrm{Obj}(\mathbb{D})$
- $\mathbb{V}\,\mathbb{C}\mathrm{au}(\mathbb{D})(A, B) = \left\{ F : A \nrightarrow B \;\middle|\; \begin{tikzcd}[ampersand replacement=\&] \& {} \arrow[dl, "\exists \ell"'] \arrow[dr] \arrow[d, phantom, "\text{tab}"] \& \\ {} \arrow[rr, "F"'] \& {} \& {} \end{tikzcd} ,\ \ell \in \mathrm{Mono} \cap \mathrm{Cover} \right\} / \cong$

Let $\mathbb{D}, \mathbb{E}$: unit-pure equipments

- $\mathbb{E}$ is a Cauchisation of $\mathbb{D}$ $\overset{\text{def}}{\Longleftrightarrow}$
  - $\mathcal{H}(\mathbb{E}) = \mathcal{H}(\mathbb{D})$
  - $\mathbb{E}$: Cauchy & unit-pure.

(Free object of
$\left\{\begin{matrix}\text{unit-pure}\\ \text{Cauchy}\end{matrix}\ \text{equipment}\right\} \hookrightarrow \{\text{unit-pure equipment}\}$
up-to equivalence.)

Rmk. Cauchisation is unique in $\mathcal{E}quip$ if exists.

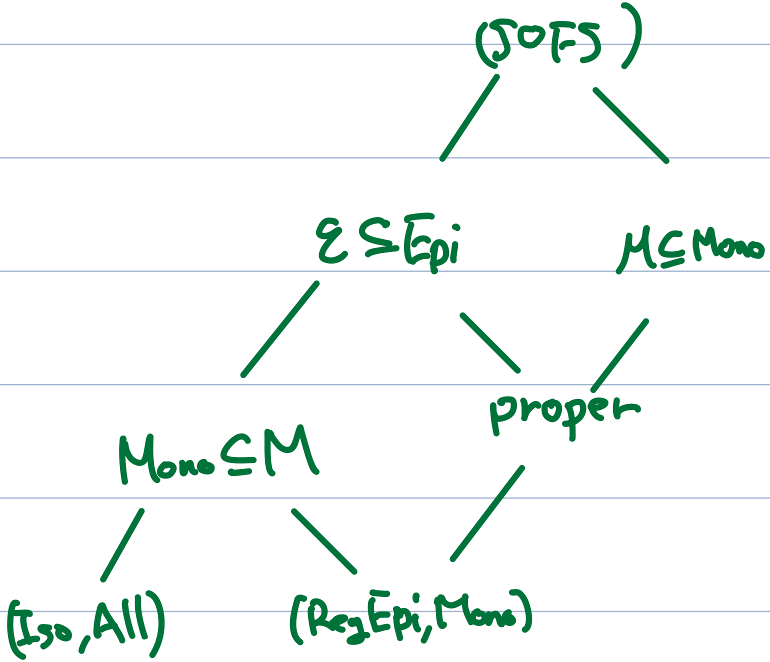
Thm. $\mathbb{D}$: unit-pure DCR has a cauchisation $\mathbb{C}au(\mathbb{D})$:

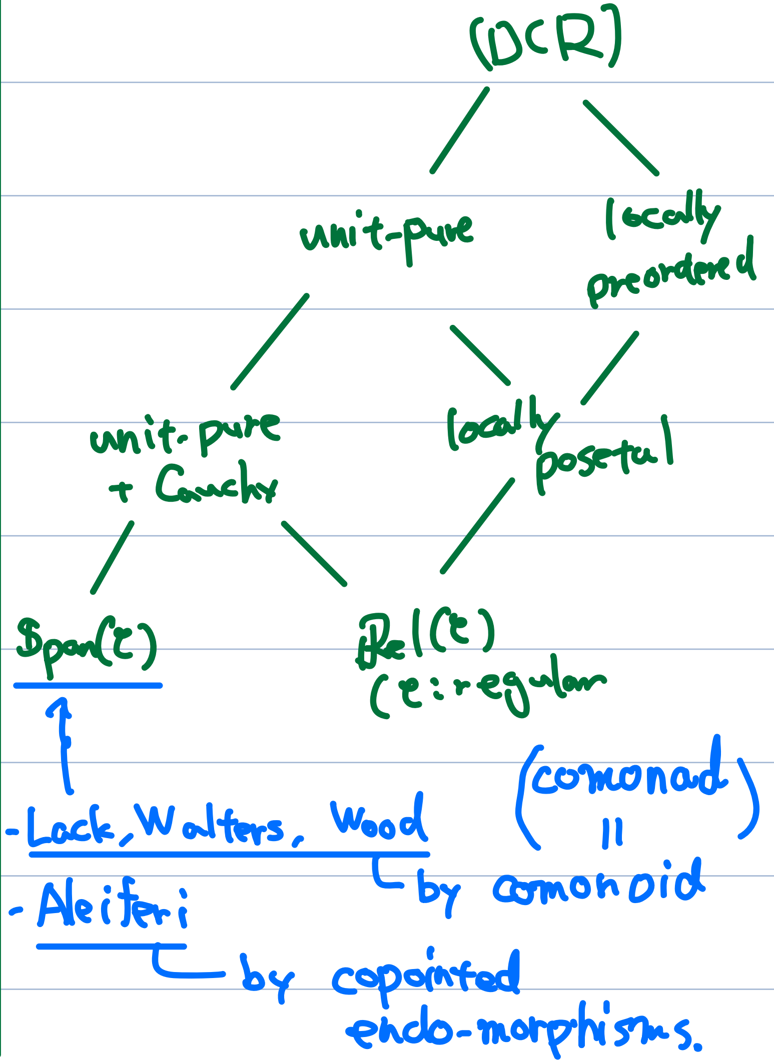
- $\mathrm{Obj}(\mathbb{C}au(\mathbb{D})) = \mathrm{Obj}(\mathbb{D})$
- $\mathbb{V}\,\mathbb{C}au(\mathbb{D})(A, B) = \left\{F : A \nrightarrow B \;\middle|\; \exists \ell\ \begin{tikzcd}[column sep=small] & \cdot \arrow[dl, "\ell"'] \arrow[dr] & \\ \cdot \arrow[rr, "F"', "|" marking] \arrow[rr, phantom, "\text{tab}", shift left=3ex] & & \cdot \end{tikzcd},\ \ell \in \mathrm{Mono} \cap \mathrm{Cover}\right\} / \cong$
- $$\begin{tikzcd} \cdot \arrow[r, "P", "|" marking] \arrow[d, "{[F]}"'] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, "{[F']}"] \\ \cdot \arrow[r, "P'"', "|" marking] & \cdot \end{tikzcd} \text{ in } \mathbb{C}au(\mathbb{D}) \quad \overset{\text{def}}{\Longleftrightarrow} \quad \begin{tikzcd} \cdot \arrow[r, "P", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\alpha"] & \cdot \arrow[r, "F'", "|" marking] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, "F"', "|" marking] & \cdot \arrow[r, "P'"', "|" marking] & \cdot \end{tikzcd} \text{ in } \mathbb{D}$$

Thm. If $\mathbb{D}$ is unit-pure DCR, so is $\mathbb{C}\mathrm{an}(\mathbb{D})$

E.g. When $\mathbb{D} \simeq \mathbb{R}\mathrm{el}(\mathcal{E},\mathcal{M})$ when $(\mathcal{E},\mathcal{M})$: proper.
Then $V\mathbb{C}\mathrm{an}(\mathbb{D})$ is the regularisation obtained in [Kelly, 1991].

E.g. When $\mathbb{D} \simeq \mathbb{R}\mathrm{el}(\mathrm{Epi}, \mathrm{RegMono})$ on a quasi-topos $\mathcal{C}$,
then $\mathbb{C}\mathrm{an}(\mathbb{D}) \simeq \mathbb{R}\mathrm{el}(C_s(\mathcal{C}))$
where $C_s(\mathcal{C})$: topos of coarce objects.





- Lack, Walters, Wood — by comonoid (comonad = comonoid)
- Aleiferi — by copointed endo-morphisms.

On copoint & comonoid aspects, observe the followings.

[HN], [Lam22]

- Tabulator = limit of $A \xrightarrow[p]{} B$

$$\begin{tikzcd} A \arrow[r, "p"', "|" marking] & B \end{tikzcd}$$

[Aleiferi]

- co-Eilenberg Moore for copointed endo-morphism = limit of

$$\begin{tikzcd} A \arrow[r, bend left, "p", "|" marking] \arrow[r, bend right, equal, "|" marking] \arrow[r, phantom, "\varepsilon"] & A \end{tikzcd}$$

cf. [Lack Walters, Wood]

- co-Eilenberg Moore for comonoid = limit of

$$\left(A,\ \begin{tikzcd} A \arrow[r, bend left, "p", "|" marking] \arrow[r, bend right, equal, "|" marking] \arrow[r, phantom, "\varepsilon"] & A \end{tikzcd},\ \begin{tikzcd} A \arrow[rr, bend left, "p", "|" marking] \arrow[dr, "p"', "|" marking] \arrow[rr, phantom, "\delta"] & & A \\ & {} \arrow[ur, "p"', "|" marking] & \end{tikzcd} \right)$$

Prop. In a discrete cartesian equipment;

(i) An endo-morphism $A \xrightarrow{f} A$ is copointed

iff there exists a (unique) structure of comonoid.

(ii) (co-Eilenberg Moore for copointed $A \xrightarrow{f} A$)

= (co-Eilenberg Moore for comonoid $A \xrightarrow{f} A$).

Prop (4.3.16. of HN) $\mathbb{D}$: unit-pure cart. equipment.

Then $\mathbb{D}$ has (strong) co-EM for comonoids

$\Leftrightarrow$ $\mathbb{D}$ has (strong) tabulators.

Thm. (4.3.19. of HN) TFAE

(i) $\mathbb{D} \simeq \mathbb{S}\mathrm{pan}(\overset{2}{\mathcal{C}})$ ($\mathcal{C}$: finitely complete)

(ii) $\mathbb{D}$ is a unit-pure cart. equipment with strong co-EM for copointed endo.s and every morphism is a leg of co-EM.

(iii) $\mathbb{D}$ is a unit-pure cart. equipment with strong co-EM for comonoids and every morphism is a leg of co-EM

(cf. Aleiferi 5.3.2. & Lack, Walters, Wood 5.2.)

★ [HN] Double categories of relations relative to factorisation systems

[CW87] Carboni, Walters. Cartesian bicategories I.

[Kel91] Kelly. A note on relations relative to factorisation systems

[Ver92] Verity. Enriched categories, internal categories, and change of base

[Pav95] Pavlovic. Maps I. Relative to a factorisation system.

[GP99] Grandis, Paré. Limits in double categories.

[Shu08] Shulman. Framed bicategories and monoidal fibrations

[WW08] Walters, Wood. Frobenius objects in Cartesian bicategories

[LWW10] Lack, Walters, Wood. Bicategories of spans as cartesian bicategories

[Ale18] Aleiferi. Cartesian double categories with an emphasis on characterizing spans

[Par21] Paré, Morphisms of Rings.

[Lam22] Lambert. Double categories of relations.

Thank You!!