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# A double $\infty$ -categorical approach to formal $\infty$ -category theory

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# Many flavors of $\infty$ -categories: Enriched

Let  ${\mathcal V}$  be a monoidal  $\infty\text{-category.}$ 

There is a notion of  $\mathcal{V}$ -enriched  $\infty$ -categories due to:

- Gepner-Haugseng [GH15],
- Hinich [Hin20],
- Lurie [Lur17],
- and more...

#### Examples of enriched $\infty$ -categories

- 1. Dg-categories as  $\infty$ -categories enriched in the derived  $\infty$ -category of abelian groups
- 2.  $(\infty, n)$ -Categories as  $\infty$ -categories enriched in  $(\infty, n 1)$ -categories (inductively)



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# Many flavors of $\infty$ -categories: Internal

In the  $\infty$ -categorical context, the role of sets is played by the  $\infty$ -topos

S

of  $\infty$ -groupoids/spaces/Kan complexes.

E.g. an  $\infty$ -category has a *space* of objects and *spaces* of maps between objects.

Idea: replace *S* by a different  $\infty$ -topos *E*,  $\rightsquigarrow$  we obtain a notion of  $\infty$ -categories *internal to E*.

(In fact:  $\mathcal{E}$  may be a finitely complete  $\infty$ -category)



# Many flavors of $\infty$ -categories: Internal

This has been extensively studied by:

- Martini [Mar21], and Martini-Wolf [MW24],
- Rasekh [Ras22],
- Shah [Sha23] [Sha21],
- ...

#### Uses of internal $\infty$ -categories

Internal categories have found many applications in homotopy theory:

- in equivariant homotopy theory [BDG<sup>+</sup>16],
- in condensed mathematics [BGH20] [Wol22],
- in *motivic homotopy theory* [BH21] [BEH22].



# The goal

We have seen two flavors: enriched and internal  $\infty$ -categories. There are certainly more!

Most concepts from category theory (should) have suitable adaptations for these flavors:

- (co)limits,
- Kan extensions,
- (co)limit completions,
- fibrations.

**Goal:** to produce these specialized category theories uniformly for all flavors of  $\infty$ -categories.

We will follow ideas of Street-Walters [SW78], Wood [Woo82], Verity [Ver92], Shulman [Shu08], and Riehl-Verity [RV22].



# The plan

- 1. Double  $\infty$ -categories
- 2.  $\infty$ -Equipments
- 3. Formal  $\infty$ -category theory internal to  $\infty$ -equipments



#### Double $\infty$ -categories: Motivation

We want some ambient playing field in which to develop a category theory. Let's look at the following **guiding example**.

Most aspects of  $\infty$ -category theory can be purely phrased in terms of:

- functors,
- profunctors.

**Definition.** A profunctor  $F : \mathcal{C} \to \mathcal{D}$  between  $\infty$ -categories is a functor

 $\mathfrak{D}^{\mathrm{op}}\times \mathfrak{C}\to \mathbb{S}.$ 

**Example.** Every  $\infty$ -category  $\mathcal{C}$  admits a canonical profunctor

 $\mathfrak{C}^{\mathsf{op}} \times \mathfrak{C} \to \mathfrak{S} : (x, y) \mapsto \mathsf{Map}_{\mathfrak{C}}(x, y),$ 

adjunct to the Yoneda embedding.



#### Double $\infty$ -categories: Motivation

**Example.** Let  $\mathcal{C}$  be an  $\infty$ -category. Let  $f : I \to \mathcal{C}$  be a diagram. A functor  $g : J \to \mathcal{C}$  is the left Kan extension of f along a functor  $w : I \to J$  if and only if the profunctor

$$C: J^{\mathrm{op}} \times \mathfrak{C} \to \mathfrak{S}: (j, c) \mapsto \int_{i \in I} \mathrm{Map}_{\mathfrak{S}}(\mathrm{Map}_{J}(w(i), j), \mathrm{Map}_{\mathfrak{C}}(f(i), c))$$

is equivalent to the profunctor

$$\operatorname{Map}_{\mathfrak{C}}(g(-),-).$$

I.e. *C* is the profunctor 'corepresented' by the functor  $g: J \to \mathbb{C}$ .

**Take-away.** A candidate ambient structure in which we could develop a category theory should have:

- two directions for arrows: functor and profunctor directions,
- a way in which functors may (co)represent profunctors.



# Double $\infty$ -categories: Definition

This brings us to double  $\infty$ -categories.

Imprecise description. A double  $\infty$ -category is 'like' a double category, but there are now **spaces** of:

- objects,
- vertical arrows,
- horizontal arrows,
- 2-cells, pictured as squares



Compositions of arrows/2-cells are now unique up to contractible choice.



#### Double $\infty$ -categories: Definition

**Definition.** A *double*  $\infty$ -*category*  $\mathcal{P}$  is a categorical object in the  $\infty$ -category Cat<sub> $\infty$ </sub> of  $\infty$ -categories. I.e., it is a simplicial object

 $\mathfrak{P}:\Delta^{op}\to Cat_\infty$ 

so that the *Segal condition* holds: restricting along the maps  $\{i \le i + 1\} : [1] \rightarrow [n]$  induces an equivalence

$$\mathfrak{P}_n \xrightarrow{\simeq} \mathfrak{P}_1 \times_{\mathfrak{P}_0} \cdots \times_{\mathfrak{P}_0} \mathfrak{P}_1$$

of  $\infty$ -categories.

**Notation.** The objects and arrows of  $\mathcal{P}_0$  are the objects and vertical arrows of  $\mathcal{P}$ . The objects and arrows of  $\mathcal{P}_1$  are the horizontal arrows and 2-cells of  $\mathcal{P}$ .



#### Double $\infty$ -categories: Examples

Let  $\mathcal{C}$  be an  $\infty$ -category with pullbacks. **Example** ([Hau18]). There is a double  $\infty$ -category

 $Span(\mathcal{C})$ 

of spans in  $\mathcal{C}$ , with:

- objects given by objects of C,
- vertical arrows given by arrows of C,
- horizontal arrows given by spans

$$x \leftarrow e \rightarrow y$$

in C.



#### Double $\infty$ -categories: Examples

**Example** ([AF20], [Rui23]). There is a double  $\infty$ -category

 $\mathbb{C}\text{at}_\infty$ 

of  $\infty$ -categories, with:

- objects given by  $\infty$ -categories,
- vertical arrows given by functors,
- horizontal arrows given by profunctors.

Equivalently, a profunctor is described by a correspondence, a functor

 $E \rightarrow [1] = \{0 \leq 1\}$ 

with equivalences  $E_1 \simeq \mathbb{C}$  and  $E_0 \simeq \mathbb{D}$ .



#### Double $\infty$ -categories: Examples

This example can be generalized.

**Example** ([Rui23]). For suitable  $\pounds$  (e.g.  $\infty$ -topos or (co)complete l.c.c.), there is a double  $\infty$ -category

#### $\mathbb{C}\mathsf{at}_\infty(\mathcal{E})$

of  $\infty$ -categories, functors and profunctors **internal** to  $\mathcal{E}$ .

**Example** ([Hau16]). For suitably monoidal  $\mathcal{V}$ , there is a double  $\infty$ -category

#### $\mathbb{C}\text{at}_\infty^{\mathcal{V}}$

of  $\infty$ -categories, functors and profunctors **enriched** in  $\mathcal{V}$ .



#### $\infty$ -Equipments: Companions and conjoints

Let  $\mathcal{P}$  be a double  $\infty$ -category.

To build a category theory internal to  $\mathcal{P}$ , we will need a way in which vertical arrows may (co)represent horizontal arrows in  $\mathcal{P}$ .

This goes via the notion of companions and conjoints. These can be viewed as a double categorical analogs of adjunctions.

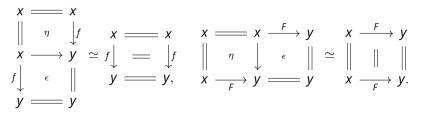


## $\infty$ -Equipments: Companions and conjoints

**Definition**. Let  $f : x \to y$  be a vertical arrow in a double  $\infty$ -category  $\mathcal{P}$ . A horizontal arrow  $F : x \to y$  in  $\mathcal{P}$  is called the *companion* of f if there exist two 2-cells

$$\eta = \iint_{X \longrightarrow F} \begin{array}{c} x & \xrightarrow{x} & x & \xrightarrow{F} & y \\ \downarrow & \downarrow & \downarrow f & \text{and} & \epsilon = f \downarrow & \downarrow & \downarrow \\ x & \xrightarrow{F} & y & & y & \xrightarrow{Y} \end{array}$$

that satisfy the following two identities:





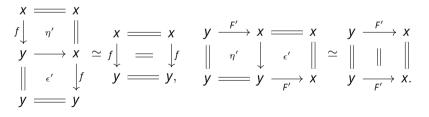
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### $\infty$ -Equipments: Companions and conjoints

**Definition**. Let  $f : x \to y$  be a vertical arrow in a double  $\infty$ -category  $\mathcal{P}$ . Dually, a horizontal arrow  $F' : y \to x$  is called the *conjoint* of f when there exist two 2-cells in  $\mathcal{P}$ 

$$\eta' = \begin{array}{cccc} x & & & & y & \xrightarrow{F'} & x \\ f & & & & \\ y & \xrightarrow{F'} & x & & & y & \xrightarrow{F'} & y \\ & & & & & & y & \xrightarrow{F'} & y \end{array}$$

that compose as follows:





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# $\infty$ -Equipments: Definition

**Proposition** ([Rui23]). For a double  $\infty$ -category  $\mathcal{P}$ , the following statements are equivalent:

- 1.  $\mathcal{P}$  has all companions and conjoints,
- 2. the source-target functor  $\mathcal{P}_1 \to \mathcal{P}_0^{\times 2}$  is a cartesian fibration,
- 3. the source-target functor  $\mathcal{P}_1 \to \mathcal{P}_0^{\times 2}$  is a cocartesian fibration.

**Definition** ([Rui23]). An  $\infty$ -*equipment* is a double  $\infty$ -category  $\mathcal{P}$  so that the equivalent conditions of above are met.

**Convention.** If  $f : x \to y$  is a vertical arrow in an  $\infty$ -equipment, we will write

$$f_{\circledast}: x \to y$$

for its companion, and

$$f^{\circledast}: y \to x$$

for its conjoint.



#### $\infty$ -Equipments: Examples

**Example** ([Rui23]). The double  $\infty$ -category  $\mathbb{C}at_{\infty}$  is an  $\infty$ -equipment. Let  $f : \mathbb{C} \to \mathcal{D}$  be a functor.

Then the companion of f is given by the profunctor

 $f_{\circledast}: \mathfrak{D}^{\mathsf{op}} \times \mathfrak{C} \to \mathfrak{S}: (d, c) \mapsto \mathsf{Map}_{\mathfrak{D}}(d, fc).$ 

The conjoint of f is given by the profunctor

 $f^{\circledast}: \mathbb{C}^{\mathsf{op}} \times \mathcal{D} \to \mathbb{S}: (c, d) \mapsto \mathsf{Map}_{\mathcal{D}}(fc, d).$ 

**Example** ([Rui23]). The double  $\infty$ -category  $\mathbb{C}at_{\infty}(\mathcal{E})$  is an  $\infty$ -equipment for suitable  $\mathcal{E}$ .



# Formal category theory: Overview

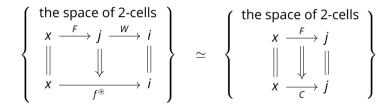
The following aspects can be phrased/developed internal to an  $\infty$ -equipment:

- fully faithful arrows,
- weighted colimits,
- point-wise Kan extensions,
- exact squares,
- two-sided discrete fibrations,
- ...



# Formal category theory: Weighted colimits

**Definition** ([Rui23]). Let  $W : j \to i$  be a horizontal arrow in an  $\infty$ -equipment  $\mathcal{P}$ . Suppose that  $f : i \to x$  is a vertical arrow of  $\mathcal{P}$ . A horizontal arrow  $C : x \to j$  is called a *proarrow of W-weighted cones under f* if for every horizontal arrow  $F : x \to j$ 



naturally.

Such a *C* is also known as a *right lifting of*  $f^{\circledast}$  *through W* [RV22, Section 9.1].



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# Formal category theory: Weighted colimits

**Definition** ([Rui23]). Let  $W : j \to i$  be a horizontal arrow in an  $\infty$ -equipment  $\mathcal{P}$ . Suppose that  $f : i \to x$  is a vertical arrow of  $\mathcal{P}$ . A vertical arrow  $g : j \to x$  is called the *W*-weighted colimit of f if its conjoint  $g^{\circledast} : x \to j$  is the proarrow of *W*-cones under f.

Left Kan extensions are colimits weighted by conjoints  $w^{\circledast} : j \to i$  of vertical arrows  $w : i \to j$ .

**Fact** ([Rui23]). This recovers the concept of left Kan extensions for  $\infty$ -categories [Lur09] for  $\mathcal{P} = \mathbb{C}at_{\infty}$ .

**Fact** ([Rui24]). This recovers the concept of left Kan extensions for *internal*  $\infty$ -categories [MW24] for  $\mathcal{P} = \mathbb{C}at_{\infty}(\mathcal{E})$ .



# Formal category theory: Kan extension arrows

There is a definition of closed monoidal  $\infty$ -equipments in [Rui24]. Let  $\mathcal{P}$  be a closed monoidal  $\infty$ -equipment with internal hom [-, -]. Suppose that  $x \in \mathcal{P}$ , and let  $w : i \to j$  be a vertical arrow.

Then there is a restriction vertical arrow

 $w^*:[j,x]\to [i,x].$ 

**Proposition** ([Rui23]). Suppose that  $w^*$  admits a left adjoint  $w_i$ . If  $p : y \to [i, x]$  is a vertical arrow, then  $w_i p : y \to [j, x]$  is adjunct to the left Kan extension of  $y \otimes i \to x$  along  $y \otimes w$ .

Under reasonable conditions, one can deduce the existence of a left adjoint  $w_1$  when enough left Kan extensions exist.



Let  $\mathcal{P}$  be an  $\infty$ -equipment.

A theory of fibrations in  $\mathcal{P}$  can be developed using an  $\infty$ -categorical analog of Grandis' and Paré's *tabulations/tabulators*.

Tabulations are examples of double categorical limits.

If  $\mathcal{P}$  has all tabulations, it is called *tabular*.

**Theorem** ([Rui24]). If  $\mathcal{P}$  is tabular, there exists a lax functor

 $\rho: \mathcal{P} \to \mathbb{S}pan(\mathcal{P}_0),$ 

called the *span representation* of  $\mathcal{P}$ .



Suppose that  $\mathcal{P}$  is tabular with span representation  $\rho : \mathcal{P} \to \mathbb{S}pan(\mathcal{P}_0)$ .

**Definition**. A span  $(p,q) : e \to x \times y$  in  $\mathcal{P}_0$  is called a *two-sided discrete fibration* (w.r.t.  $\mathcal{P}$ ) if there exists a horizontal arrow  $F : x \to y$  in  $\mathcal{P}$  so that  $\rho(F) = (p,q)$ .

**Theorem** ([Rui24]). Let  $(p,q) : e \to x \times y$  be a two-sided discrete fibration in  $\mathcal{P}_0$ . Then the induced functor

 $(p,q)_*$ : Vert $(\mathcal{P})(z,e) \rightarrow$  Vert $(\mathcal{P})(z,x) \times$  Vert $(\mathcal{P})(z,y)$ 

is a two-sided discrete fibration of  $\infty$ -categories.

Here  $Vert(\mathcal{P})$  denotes the vertical  $(\infty, 2)$ -category of  $\mathcal{P}$ , and  $Vert(\mathcal{P})(x, y)$  denotes the mapping  $\infty$ -category of arrows  $x \to y$  in  $Vert(\mathcal{P})$ .



One may characterize the tabular equipments  $\mathcal{P}$  for which  $\rho$  is *horizontally locally reflective* [Rui24]. These are called the *fibrational*  $\infty$ -*equipments*.

For a fibrational  $\infty$ -equipment  $\mathfrak{P}$ , the horizontal arrows of  $\mathfrak{P}$  may be viewed as two-sided discrete fibrations. Precisely, the functor

 $\rho_{x,y}$ : Hor( $\mathfrak{P}$ )(x, y)  $\rightarrow$  Hor( $\mathbb{S}$ pan( $\mathfrak{P}_0$ ))(x, y) =  $\mathfrak{P}_0/(x \times y)$ 

is a fully faithful right adjoint for all  $x, y \in \mathcal{P}$ .

The cartesian closed fibrational  $\infty$ -equipments support an excellent theory of two-sided discrete/left/right fibrations.

**Example.** The equipment  $\mathbb{C}at_{\infty}(\mathcal{E})$  is fibrational and cartesian closed for every  $\infty$ -topos  $\mathcal{E}$ .



We may also characterize span double  $\infty$ -categories using this machinery.

This is a double  $\infty$ -categorical analog of a result by Carboni-Kasangian-Street for 2-categories [CKS84].

**Theorem** ([Rui24]). Let  $\mathcal{P}$  be a double  $\infty$ -category. Then the following are equivalent:

- $\mathcal{P}$  is equivalent to a double  $\infty$ -category of spans,
- ${\mathfrak P}$  is fibrational and every span in  ${\mathfrak P}_0$  is a two-sided discrete fibration (w.r.t.  ${\mathfrak P}),$
- the span representation for  $\mathcal{P}$  is an equivalence.



#### What's next?

- Relation between  $\infty\text{-equipments}$  and  $\infty\text{-cosmoses}$  of Riehl and Verity [RV22]
- Colimit completions internal to  $\infty$ -equipments (WIP)
- Lurie's operadic Kan extensions [Lur17] from an  $\infty$ -equipmental perspective (WIP)
- Enriched  $\infty$ -category theory via  $\infty$ -equipments There is existing work of Hinich on weighted colimits [Hin21]. Recently, Heine has developed an extensive theory of enriched  $\infty$ -categories [Hei24].
- A type theory for ∞-equipments? The recent work of New and Licata [NL23] is relevant.



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#### Thank you for your attention.

