A double ∞-categorical approach to formal ∞-category theory

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Many flavors of ∞-categories: Enriched

Let $\mathcal{V}$ be a monoidal ∞-category.

There is a notion of *$\mathcal{V}$-enriched ∞-categories* due to:

- Gepner-Haugseng [GH15],
- Hinich [Hin20],
- Lurie [Lur17],
- and more...

Examples of enriched ∞-categories

1. *Dg-categories* as ∞-categories enriched in the derived ∞-category of abelian groups
2. (∞, n)-Categories as ∞-categories enriched in $(\infty, n-1)$-categories (inductively)

Many flavors of ∞-categories: Internal

In the ∞-categorical context, the role of sets is played by the ∞-topos

$$\mathcal{S}$$

of ∞-groupoids/spaces/Kan complexes.

E.g. an ∞-category has a *space* of objects and *spaces* of maps between objects.

Idea: replace $\mathcal{S}$ by a different ∞-topos $\mathcal{E}$,
$\rightsquigarrow$ we obtain a notion of ∞-categories *internal to* $\mathcal{E}$.

(In fact: $\mathcal{E}$ may be a finitely complete ∞-category)

Many flavors of ∞-categories: Internal

This has been extensively studied by:

- Martini [Mar21], and Martini-Wolf [MW24],
- Rasekh [Ras22],
- Shah [Sha23] [Sha21],
- ...

Uses of internal ∞-categories

Internal categories have found many applications in homotopy theory:

- in *equivariant homotopy theory* [BDG$^+$16],
- in *condensed mathematics* [BGH20] [Wol22],
- in *motivic homotopy theory* [BH21] [BEH22].

The goal

We have seen two flavors: enriched and internal ∞-categories. There are certainly more!

Most concepts from category theory (should) have suitable adaptations for these flavors:

- (co)limits,
- Kan extensions,
- (co)limit completions,
- fibrations.

Goal: to produce these specialized category theories uniformly for all flavors of ∞-categories.

We will follow ideas of Street-Walters [SW78], Wood [Woo82], Verity [Ver92], Shulman [Shu08], and Riehl-Verity [RV22].

The plan

1. Double ∞-categories
2. ∞-Equipments
3. Formal ∞-category theory internal to ∞-equipments

Double ∞-categories: Motivation

We want some ambient playing field in which to develop a category theory.
Let's look at the following **guiding example**.

Most aspects of ∞-category theory can be purely phrased in terms of:

- functors,
- *profunctors*.

Definition. A profunctor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories is a functor

$$\mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}.$$

Example. Every ∞-category $\mathcal{C}$ admits a canonical profunctor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S} : (x, y) \mapsto \mathrm{Map}_{\mathcal{C}}(x, y),$$

adjunct to the Yoneda embedding.

Double ∞-categories: Motivation

Example. Let $\mathcal{C}$ be an ∞-category. Let $f : I \to \mathcal{C}$ be a diagram. A functor $g : J \to \mathcal{C}$ is the left Kan extension of f along a functor $w : I \to J$ if and only if the profunctor

$$C : J^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S} : (j, c) \mapsto \int_{i \in I} \mathrm{Map}_{\mathcal{S}}(\mathrm{Map}_J(w(i), j), \mathrm{Map}_{\mathcal{C}}(f(i), c))$$

is equivalent to the profunctor

$$\mathrm{Map}_{\mathcal{C}}(g(-), -).$$

I.e. C is the profunctor 'corepresented' by the functor $g : J \to \mathcal{C}$.

Take-away. A candidate ambient structure in which we could develop a category theory should have:

- two directions for arrows: functor and profunctor directions,
- a way in which functors may (co)represent profunctors.

Double ∞-categories: Definition

This brings us to double ∞-categories.

Imprecise description. A double ∞-category is ‘like’ a double category, but there are now **spaces** of:

- objects,
- vertical arrows,
- horizontal arrows,
- 2-cells, pictured as squares

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & \Downarrow & \downarrow \\ c & \longrightarrow & d. \end{array}$$

Compositions of arrows/2-cells are now unique up to *contractible choice*.

Double ∞-categories: Definition

Definition. A *double ∞-category* $\mathcal{P}$ is a categorical object in the ∞-category Cat_∞ of ∞-categories. I.e., it is a simplicial object

$$\mathcal{P} : \Delta^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

so that the *Segal condition* holds: restricting along the maps $\{i \leq i+1\} : [1] \to [n]$ induces an equivalence

$$\mathcal{P}_n \xrightarrow{\simeq} \mathcal{P}_1 \times_{\mathcal{P}_0} \cdots \times_{\mathcal{P}_0} \mathcal{P}_1$$

of ∞-categories.

Notation. The objects and arrows of $\mathcal{P}_0$ are the objects and vertical arrows of $\mathcal{P}$. The objects and arrows of $\mathcal{P}_1$ are the horizontal arrows and 2-cells of $\mathcal{P}$.

Double ∞-categories: Examples

Let $\mathcal{C}$ be an ∞-category with pullbacks.

Example ([Hau18]). There is a double ∞-category

$$\mathbb{S}\text{pan}(\mathcal{C})$$

of spans in $\mathcal{C}$, with:

- objects given by objects of $\mathcal{C}$,
- vertical arrows given by arrows of $\mathcal{C}$,
- horizontal arrows given by spans

  $$x \leftarrow e \rightarrow y$$

  in $\mathcal{C}$.

Double ∞-categories: Examples

Example ([AF20], [Rui23]). There is a double ∞-category

$$\mathbb{C}\mathrm{at}_\infty$$

of ∞-categories, with:

- objects given by ∞-categories,
- vertical arrows given by functors,
- horizontal arrows given by profunctors.

Equivalently, a profunctor is described by a *correspondence*, a functor

$$E \to [1] = \{0 \leq 1\}$$

with equivalences $E_1 \simeq \mathcal{C}$ and $E_0 \simeq \mathcal{D}$.

Double ∞-categories: Examples

This example can be generalized.

Example ([Rui23]). For suitable $\mathcal{E}$ (e.g. ∞-topos or (co)complete l.c.c.), there is a double ∞-category

$$\mathbb{C}\mathrm{at}_\infty(\mathcal{E})$$

of ∞-categories, functors and profunctors **internal** to $\mathcal{E}$.

Example ([Hau16]). For suitably monoidal $\mathcal{V}$, there is a double ∞-category

$$\mathbb{C}\mathrm{at}_\infty^{\mathcal{V}}$$

of ∞-categories, functors and profunctors **enriched** in $\mathcal{V}$.

∞-Equipments: Companions and conjoints

Let $\mathcal{P}$ be a double ∞-category.

To build a category theory internal to $\mathcal{P}$, we will need a way in which vertical arrows may (co)represent horizontal arrows in $\mathcal{P}$.

This goes via the notion of companions and conjoints. These can be viewed as a double categorical analogs of adjunctions.

∞-Equipments: Companions and conjoints

Definition. Let $f : x \to y$ be a vertical arrow in a double ∞-category $\mathcal{P}$. A horizontal arrow $F : x \to y$ in $\mathcal{P}$ is called the *companion* of f if there exist two 2-cells

$$\eta = \begin{array}{ccc} x & = & x \\ \| & \Downarrow & \downarrow{\scriptstyle f} \\ x & \xrightarrow[F]{} & y \end{array} \quad \text{and} \quad \epsilon = \begin{array}{ccc} x & \xrightarrow{F} & y \\ {\scriptstyle f}\downarrow & \Downarrow & \| \\ y & = & y \end{array}$$

that satisfy the following two identities:

$$\begin{array}{ccc} x & = & x \\ \| & \eta & \downarrow{\scriptstyle f} \\ x & \longrightarrow & y \\ {\scriptstyle f}\downarrow & \epsilon & \| \\ y & = & y \end{array} \simeq \begin{array}{ccc} x & = & x \\ {\scriptstyle f}\downarrow & = & \downarrow{\scriptstyle f} \\ y & = & y \end{array}, \qquad \begin{array}{ccccc} x & = & x & \xrightarrow{F} & y \\ \| & \eta & \downarrow & \epsilon & \| \\ x & \xrightarrow[F]{} & y & = & y \end{array} \simeq \begin{array}{ccc} x & \xrightarrow{F} & y \\ \| & \| & \| \\ x & \xrightarrow[F]{} & y \end{array}.$$

∞-Equipments: Companions and conjoints

Definition. Let $f : x \to y$ be a vertical arrow in a double ∞-category $\mathcal{P}$. Dually, a horizontal arrow $F' : y \to x$ is called the *conjoint* of f when there exist two 2-cells in $\mathcal{P}$

$$\eta' = \begin{array}{ccc} x & = & x \\ {\scriptstyle f}\downarrow & \Downarrow & \| \\ y & \xrightarrow[F']{} & x \end{array} \quad \text{and} \quad \epsilon' = \begin{array}{ccc} y & \xrightarrow{F'} & x \\ \| & \Downarrow & \downarrow{\scriptstyle f} \\ y & = & y \end{array}$$

that compose as follows:

$$\begin{array}{ccc} x & = & x \\ {\scriptstyle f}\downarrow & \eta' & \| \\ y & \longrightarrow & x \\ \| & \epsilon' & \downarrow{\scriptstyle f} \\ y & = & y \end{array} \simeq \begin{array}{ccc} x & = & x \\ {\scriptstyle f}\downarrow & = & \downarrow{\scriptstyle f} \\ y & = & y, \end{array} \qquad \begin{array}{ccccc} y & \xrightarrow{F'} & x & = & x \\ \| & \eta' & \downarrow & \epsilon' & \| \\ y & = & y & \xrightarrow[F']{} & x \end{array} \simeq \begin{array}{ccc} y & \xrightarrow{F'} & x \\ \| & \| & \| \\ y & \xrightarrow[F']{} & x. \end{array}$$

∞-Equipments: Definition

Proposition ([Rui23]). For a double ∞-category $\mathcal{P}$, the following statements are equivalent:

1. $\mathcal{P}$ has all companions and conjoints,
2. the source-target functor $\mathcal{P}_1 \to \mathcal{P}_0^{\times 2}$ is a cartesian fibration,
3. the source-target functor $\mathcal{P}_1 \to \mathcal{P}_0^{\times 2}$ is a cocartesian fibration.

Definition ([Rui23]). An ∞-*equipment* is a double ∞-category $\mathcal{P}$ so that the equivalent conditions of above are met.

Convention. If $f : x \to y$ is a vertical arrow in an ∞-equipment, we will write

$$f_{\circledast} : x \to y$$

for its companion, and

$$f^{\circledast} : y \to x$$

for its conjoint.

∞-Equipments: Examples

Example ([Rui23]). The double ∞-category $\mathbb{C}\mathrm{at}_\infty$ is an ∞-equipment.

Let $f : \mathcal{C} \to \mathcal{D}$ be a functor.

Then the companion of f is given by the profunctor

$$f_{\circledast} : \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S} : (d, c) \mapsto \mathrm{Map}_{\mathcal{D}}(d, fc).$$

The conjoint of f is given by the profunctor

$$f^{\circledast} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S} : (c, d) \mapsto \mathrm{Map}_{\mathcal{D}}(fc, d).$$

Example ([Rui23]). The double ∞-category $\mathbb{C}\mathrm{at}_\infty(\mathcal{E})$ is an ∞-equipment for suitable $\mathcal{E}$.

Formal category theory: Overview

The following aspects can be phrased/developed internal to an ∞-equipment:

- fully faithful arrows,
- weighted colimits,
- point-wise Kan extensions,
- exact squares,
- two-sided discrete fibrations,
- ...

Formal category theory: Weighted colimits

Definition ([Rui23]). Let $W : j \to i$ be a horizontal arrow in an ∞-equipment $\mathcal{P}$. Suppose that $f : i \to x$ is a vertical arrow of $\mathcal{P}$. A horizontal arrow $C : x \to j$ is called a *proarrow of W-weighted cones under f* if for every horizontal arrow $F : x \to j$

$$\left\{ \begin{array}{c} \text{the space of 2-cells} \\ \begin{array}{ccccc} x & \xrightarrow{F} & j & \xrightarrow{W} & i \\ \| & & \Downarrow & & \| \\ x & & \xrightarrow[f^{\circledast}]{\hspace{4em}} & & i \end{array} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{the space of 2-cells} \\ \begin{array}{ccc} x & \xrightarrow{F} & j \\ \| & \Downarrow & \| \\ x & \xrightarrow[C]{} & j \end{array} \end{array} \right\}$$

naturally.

Such a C is also known as a *right lifting of $f^{\circledast}$ through W* [RV22, Section 9.1].

Formal category theory: Weighted colimits

Definition ([Rui23]). Let $W : j \to i$ be a horizontal arrow in an ∞-equipment $\mathcal{P}$. Suppose that $f : i \to x$ is a vertical arrow of $\mathcal{P}$. A vertical arrow $g : j \to x$ is called the W-weighted colimit of f if its conjoint $g^{\circledast} : x \to j$ is the proarrow of W-cones under f.

Left Kan extensions are colimits weighted by conjoints $w^{\circledast} : j \to i$ of vertical arrows $w : i \to j$.

Fact ([Rui23]). This recovers the concept of left Kan extensions for ∞-categories [Lur09] for $\mathcal{P} = \mathbb{C}\mathrm{at}_{\infty}$.

Fact ([Rui24]). This recovers the concept of left Kan extensions for *internal* ∞-categories [MW24] for $\mathcal{P} = \mathbb{C}\mathrm{at}_{\infty}(\mathcal{E})$.

Formal category theory: Kan extension arrows

There is a definition of closed monoidal ∞-equipments in [Rui24].

Let $\mathcal{P}$ be a closed monoidal ∞-equipment with internal hom $[-,-]$.

Suppose that $x \in \mathcal{P}$, and let $w : i \to j$ be a vertical arrow.

Then there is a restriction vertical arrow

$$w^* : [j, x] \to [i, x].$$

Proposition ([Rui23]). Suppose that w^* admits a left adjoint $w_!$. If $p : y \to [i, x]$ is a vertical arrow, then $w_! p : y \to [j, x]$ is adjunct to the left Kan extension of $y \otimes i \to x$ along $y \otimes w$.

Under reasonable conditions, one can deduce the existence of a left adjoint $w_!$ when enough left Kan extensions exist.

Formal category theory: Fibrations

Let $\mathcal{P}$ be an ∞-equipment.

A theory of fibrations in $\mathcal{P}$ can be developed using an ∞-categorical analog of Grandis' and Paré's *tabulations/tabulators*.

Tabulations are examples of double categorical limits.

If $\mathcal{P}$ has all tabulations, it is called *tabular*.

Theorem ([Rui24]). If $\mathcal{P}$ is tabular, there exists a lax functor

$$\rho : \mathcal{P} \to \mathbb{S}\mathrm{pan}(\mathcal{P}_0),$$

called the *span representation* of $\mathcal{P}$.

Formal category theory: Fibrations

Suppose that $\mathcal{P}$ is tabular with span representation $\rho : \mathcal{P} \to \mathbb{S}\text{pan}(\mathcal{P}_0)$.

Definition. A span $(p, q) : e \to x \times y$ in $\mathcal{P}_0$ is called a *two-sided discrete fibration* (w.r.t. $\mathcal{P}$) if there exists a horizontal arrow $F : x \to y$ in $\mathcal{P}$ so that $\rho(F) = (p, q)$.

Theorem ([Rui24]). Let $(p, q) : e \to x \times y$ be a two-sided discrete fibration in $\mathcal{P}_0$. Then the induced functor

$$(p, q)_* : \text{Vert}(\mathcal{P})(z, e) \to \text{Vert}(\mathcal{P})(z, x) \times \text{Vert}(\mathcal{P})(z, y)$$

is a two-sided discrete fibration of ∞-categories.

Here $\text{Vert}(\mathcal{P})$ denotes the vertical $(\infty, 2)$-category of $\mathcal{P}$, and $\text{Vert}(\mathcal{P})(x, y)$ denotes the mapping ∞-category of arrows $x \to y$ in $\text{Vert}(\mathcal{P})$.

Formal category theory: Fibrations

One may characterize the tabular equipments $\mathcal{P}$ for which ρ is *horizontally locally reflective* [Rui24]. These are called the *fibrational ∞-equipments.*

For a fibrational ∞-equipment $\mathcal{P}$, the horizontal arrows of $\mathcal{P}$ may be viewed as two-sided discrete fibrations. Precisely, the functor

$$\rho_{x,y} : \mathrm{Hor}(\mathcal{P})(x,y) \to \mathrm{Hor}(\mathbb{S}\mathrm{pan}(\mathcal{P}_0))(x,y) = \mathcal{P}_0/(x \times y)$$

is a fully faithful right adjoint for all $x, y \in \mathcal{P}$.

The cartesian closed fibrational ∞-equipments support an excellent theory of two-sided discrete/left/right fibrations.

Example. The equipment $\mathbb{C}\mathrm{at}_\infty(\mathcal{E})$ is fibrational and cartesian closed for every ∞-topos $\mathcal{E}$.

Formal category theory: Fibrations

We may also characterize span double ∞-categories using this machinery.

This is a double ∞-categorical analog of a result by Carboni-Kasangian-Street for 2-categories [CKS84].

Theorem ([Rui24]). Let $\mathcal{P}$ be a double ∞-category. Then the following are equivalent:

- $\mathcal{P}$ is equivalent to a double ∞-category of spans,
- $\mathcal{P}$ is fibrational and every span in $\mathcal{P}_0$ is a two-sided discrete fibration (w.r.t. $\mathcal{P}$),
- the span representation for $\mathcal{P}$ is an equivalence.

What's next?

- Relation between ∞-equipments and ∞-cosmoses of Riehl and Verity [RV22]
- Colimit completions internal to ∞-equipments (WIP)
- Lurie's *operadic Kan extensions* [Lur17] from an ∞-equipmental perspective (WIP)
- Enriched ∞-category theory via ∞-equipments
  There is existing work of Hinich on weighted colimits [Hin21].
  Recently, Heine has developed an extensive theory of enriched ∞-categories [Hei24].
- A type theory for ∞-equipments?
  The recent work of New and Licata [NL23] is relevant.

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