Double cats as pseudomonads w app. double profunctors

Plan

1. Review 1D

- categories : monads
- profunctors : bimodules
- functors / cofunctors : certain special cases

2. Extend definitions to 2D

- PsDblCat (recover notions known in the literature)
- PsDblCof (introduced M. 2023)
- PsDblProf (WIP jw. N. Gambino)

3. Grothendieck construction

profunctors $A^{op} \times B \longrightarrow \mathrm{set}$

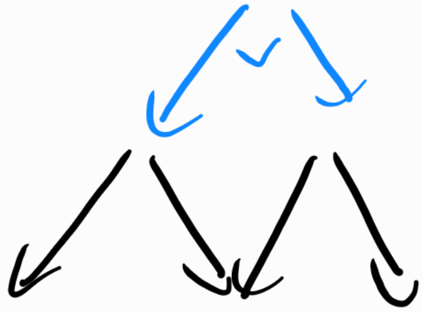
double profunctors ???

4. Strictification

- double functors (Grandis, Paré 1999; Campbell 2019)
- double cofunctors (M. 2023)
- double profunctors (M. 2024.)

Part 1 Review 1D case

Bénabou 1967 / 1973 Span



Composition is via pullback

monads are categories

$$\begin{tikzcd}
& A_2 \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[d, "\mu"] & \\
A_0 & A_1 \arrow[l, "s"'] \arrow[r, "t"] & A_0 \\
& A_0 \arrow[ul, "1_{A_0}"] \arrow[ur, "1_{A_0}"'] \arrow[u, "\eta"'] &
\end{tikzcd}$$

Street 1972 Formal Theory of Monads

Morphisms of monads $\leftrightsquigarrow$ cofunctors

$$A_0 \xleftarrow{1_{A_0}} A_0 \xrightarrow{F_0} B_0$$

Co-morphisms of monads $\leftrightsquigarrow$ functors

2-cells $\leftrightsquigarrow$ equalities

Lack-Street 2002 KL & EM

(co)completions

More general 2-cells which correspond to natural transformations (& 2-cells b/w cofunctors)

Garner Shulman 2016

Joyal Gambino 2017

Bimod($\mathcal{K}$)

Bimodules of monads $\rightsquigarrow$ profunctors between cats

Morphisms of bimodules $\rightsquigarrow$ Natural transf. between profunctors

Bimodules

$$A_0 \leftarrow P_1 \rightarrow B_0 \qquad \begin{matrix} P_1(x,y) \\ \text{hetromorphisms} \end{matrix}$$

$$\begin{tikzcd} A_0 \arrow[r, "P_1", "|" marking] \arrow[d, "\mathbb{A}"', "|" marking] \arrow[dr, phantom, "\Uparrow \rho"] & B_0 \\ A_0 \arrow[ur, "P_1"', "|" marking] & \end{tikzcd} \qquad P_1 \times_{A_0} A_1 \xrightarrow{\rho} P_1$$

$$\begin{tikzcd} A_0 \arrow[r, "P_1", "|" marking] \arrow[dr, "P_1"', "|" marking] \arrow[r, phantom, shift right=4ex, "\Downarrow \lambda"] & B_0 \arrow[d, "\mathbb{B}", "|" marking] \\ & B_0 \end{tikzcd} \qquad B_1 \times_{B_0} P_1 \xrightarrow{\lambda} P_1$$

$$\mathbb{A}^{op} \times \mathbb{B} \xrightarrow{P} \mathrm{Set}$$

algebras for the distributive law between $\mathbb{B} \circ (-)$ & $(-) \circ \mathbb{A}_{,}$ on $\mathrm{Span}(\mathrm{Set})(A_0, B_0)$

Composition of bimodules

$$\mathbb{A} \overset{P}{\nrightarrow} \mathbb{B} \overset{Q}{\nrightarrow} \mathbb{C}$$

... is given by the reflexive coequaliser

$$Q \circ \mathbb{B} \circ P \underset{1 \circ \lambda^P}{\overset{\rho^Q \circ 1}{\rightrightarrows}} Q \circ P \longrightarrow Q \circ_{\mathbb{B}} P$$

Cofunctors and functors

ρ free on $A_0 = A_0 \xrightarrow{F_0} B_0$

λ free on $A_0 = A_0 \xrightarrow{F_0} B_0$

Part 2 Tricategory Span(Cat)

$$\begin{tikzcd} & \cdot \arrow[dl, "L"'] \arrow[dr, "R"] \arrow[dd, bend right=40] \arrow[dd, bend left=40] \arrow[dd, phantom, "\overset{\phi}{\Rightarrow}"] & \\ A & & B \\ & \cdot \arrow[ul, "L'"] \arrow[ur, "R'"'] & \end{tikzcd}$$

$R'\phi = R$

$L'\phi = L$

Gambino, Lobbia 2021

Formal Theory of Pseudomonads pt 1.

pseudomonads in Span(Cat) $\longleftrightarrow$ (loosely pseudo) double cat's

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "(hg)f"'] \arrow[dr, phantom, "\cong"] & \cdot \arrow[d, "h(gf)"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "f1"'] \arrow[dr, phantom, "\cong"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & \cdot \arrow[d, "1f"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

Co-morphisms $\rightsquigarrow$ (loosely pseudo) double functors

$$A_0 \xleftarrow{1_{A_0}} A_0 \xrightarrow{F_0} B_0$$

(tightly natural, loosely pseudonat.)

2-cells $\rightsquigarrow$ double n.t w/ identity components.

3-cells $\rightsquigarrow$ equalities

M. 2023 Formal Theory of Pseudomonads pt 2.

More general 2-cells $\rightsquigarrow$ loose ps. nat transformations

More general 3-cells $\rightsquigarrow$ double modifications

Morphisms of pseudomonads give a notion of (loosely pseudo) double cofunctors.

$$\mathbb{A}_0 \xrightarrow[\text{functor}]{F_0} \mathbb{B}_0$$

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "F_x f"'] \arrow[dr, phantom, "F_x \beta"] & X' \arrow[d, "F_{x'} f'"] & & F_0 X \arrow[r, "F_0 x"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \beta"] & F_0 X' \arrow[d, "f'"] \\
\underline{Y} \arrow[r, "\underset{\sim}{y}"'] & \underline{Y'} & & Y \arrow[r, "y"'] & Y' \arrow[lll, phantom, "\longmapsfrom"]
\end{tikzcd}$$

respecting loose composition + identities up to coherent inv. double cells.

General 2-cells / 3-cells give a notion of double pseudonatural cotransformation / double comodification

$$
\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\Downarrow \phi_x"] & Y \arrow[d, "\phi_Y"] \\
X' \arrow[r, "x'"'] & Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "F_X f"'] & X \arrow[d, "\phi_X"] \\
\underline{Y} \arrow[d, "\phi_{\underline{Y}}"'] \arrow[r, phantom, "\underset{\sim}{\phi}_{x,f} \cong"] & X' \arrow[d, "G_{X'} f"] \\
\underline{Y}' \arrow[r, equal] & \underline{Y}'
\end{tikzcd}
$$

Thm (M. 2023)

These data assemble into a tricategory PsDblCof.

N.B. New even for bicategories.

What about double profunctors ?

Problem :

$$\mathrm{Cat}/B \xrightarrow{F^*} \mathrm{Cat}/A$$

does not preserve coequalisers, but

Prop^n (M. 2024)

F^* does preserve codescent objects.

WIP j.w N. Gambino:

If $\mathcal{K}$ is a Gray-cat whose homs have codescent objects seperately preserved by composition then there is a tricategory

$\mathrm{Bimod}(\mathcal{K})$.

objects: pseudomonads,

homs: pseudoalgebras for pseudo-distributive law,

Composition: codescent objects.

Part 3 Grothendieck construction.

$$\mathbb{A}_0 \longleftarrow P_1 \longrightarrow \mathbb{B}_0$$

(loose) hetromorphisms,

hetero-doublecells

$$\begin{tikzcd} X \arrow[r, "P", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow \tilde{p}"] & Y \arrow[d, "g"] \\ X' \arrow[r, "P'"', "\shortmid" marking] & Y' \end{tikzcd}$$

$$\frac{P_1 \longrightarrow \mathbb{A}_0 \times \mathbb{B}_0}{\mathbb{A}_0 \times \mathbb{B}_0 \rightsquigarrow \mathrm{Prof}}$$

normal

lax

P_1 has actions on either side by $\mathbb{A}$ & $\mathbb{B}$:

$$\left.\begin{array}{l} P_1 \times_{\mathbb{A}_0} \mathbb{A}_1 \xrightarrow{r} P_1 \\[2em] \mathbb{B}_1 \times_{\mathbb{B}_0} P_1 \xrightarrow{\ell} P_1 \end{array}\right\} \text{functors}$$

$$\left(\begin{array}{c} X \\ f\downarrow \\ X' \end{array}, Y \right) \longmapsto P(f,Y) \; \begin{array}{c} P(X,Y) \\ \uparrow \\ P(X',Y) \end{array} \quad \text{functor}$$

$$\left(X, \begin{array}{c} Y \\ \downarrow g \\ Y' \end{array} \right) \longmapsto P(X,g) \; \begin{array}{c} P(X,Y) \\ \downarrow \\ P(X,Y) \end{array} \quad \text{functor}$$

The 2-cells of spans r & ℓ underlie pseudoalgebras for

$(-)\circ\mathbb{A}$, $\mathbb{B}\circ(-)$.

$$
\begin{tikzcd}
P_1 \times_{\mathbb{A}_0} \mathbb{A}_1 \times_{\mathbb{A}_0} \mathbb{A}_1 \arrow[r, "r\times 1"] \arrow[d, "1\times\mu"'] \arrow[dr, phantom, "\cong\chi"] & P_1\times\mathbb{A}_1 \arrow[d, "r"] \\
P_1\times\mathbb{A}_1 \arrow[r, "r"'] & P_1
\end{tikzcd}
\qquad
\begin{tikzcd}
P_1 \arrow[r, "1\times\eta"] \arrow[dr, "1_{P_1}"'] & P_1\times\mathbb{A}_1 \arrow[d, "r"] \arrow[dl, phantom, "\cong\iota"] \\
& P_1
\end{tikzcd}
$$

+ pseudoalgebra axioms

similar for ℓ.

Thus we have pseudofunctors

$$\mathbb{A}^{op}_{loose} \overset{P(-,Y)}{\rightsquigarrow} \mathrm{Cat} \quad , \quad \mathbb{B}_{loose} \overset{P(X,-)}{\rightsquigarrow} \mathrm{Cat}$$

The bimodularity condition is mediated by an invertible natural transformation.

$$\begin{tikzcd}
 & \mathbb{B}_1 \times_{\mathbb{B}_0} P_1 \times_{\mathbb{A}_0} \mathbb{A}_1 \arrow[dl, "\ell \times 1"'] \arrow[dr, "1 \times r"] & \\
P_1 \times_{\mathbb{A}_0} \mathbb{A}_1 \arrow[dr, "r"'] & \cong & \mathbb{B}_1 \times P_1 \arrow[dl, "\ell"] \\
 & P_1 &
\end{tikzcd}$$

Thm (M. 2024)

These data comprise a double functor

$$\left(\mathbb{A}^{op} \times \mathbb{B}\right)^{\mathrm{loose}\,T} \rightsquigarrow \underline{\mathbb{P}\mathrm{rof}}$$

that is (loosely) normal lax,

and (tightly) pseudo.

N.B. This formulation is hard to guess without the pseudomonad perspective on double categories!

Composition of 1D profunctors

$$\frac{\left\{\left(X \overset{p}{\nrightarrow} Y,\ Y \overset{q}{\nrightarrow} Z\right)\right\}}{\left\langle (p,q) \sim (p',q') \;\middle|\; \exists\, Y \xrightarrow{y} Y' \quad (yp, q) = (p', q'y) \right\rangle}$$

Composition of double profunctors

$$\begin{tikzcd}
X \arrow[r, "yp", "\mid"{marking}] \arrow[d, equal] & Y' \arrow[r, "q", "\mid"{marking}] \arrow[d, phantom, "\cong"] & Z \arrow[d, equal] \\
X \arrow[r, "p"', "\mid"{marking}] & Y \arrow[r, "qy"', "\mid"{marking}] & Z
\end{tikzcd}$$

Part 4: Strictification

(Grandis, Paré 1999)

$$\begin{tikzcd} \mathbb{A} \arrow[d, squiggly, "\eta_{\mathbb{A}}"] \\ st_2(\mathbb{A}) \end{tikzcd}$$

weak in the loose direction

strict in both directions

$$\begin{tikzcd} \mathrm{DblCat} \arrow[r, bend right, "\mathrm{Inc}"'] \arrow[r, phantom, "\perp"] & \mathrm{PsDblCat} \arrow[l, bend right, "st_2(-)"'] \end{tikzcd}$$

(Campbell 2019)

<u>Th</u>^{M} (Grandis / Paré, M., Gambino / M.) Let $\mathbb{A}, \mathbb{B}$ be strict double categories & $\mathbb{A} \overset{P}{\longrightarrow\!\!\!\!\!\!+\;} \mathbb{B}$ be a double profunctor,

1. $(\mathbb{A}_{loose})_1$ free on a graph, P a double pseudofunctor

$\Longrightarrow$ $P \cong P'$ a strict double functor (Grandis Paré)

2. $(\mathbb{B}_{loose})_1$ " " ", P a double pseudocofunctor

$\Longrightarrow$ $P \cong P'$ a strict double cofunctor (M. 2023)

Th^{m} (M. 2024)

$(\mathbb{A}_{loose})_1$, $(\mathbb{B}_{loose})_1$ are both free on graphs,

$\mathbb{A} \overset{P}{\longrightarrow\!\!\!\!\!\!\!\!/\;\;\;} \mathbb{B}$ a double profunctor

$\Longrightarrow \quad P \cong P'$ where

$$P_1 \times_{\mathbb{A}_0} \mathbb{A} \xrightarrow{r} P_1$$

$$\mathbb{B}_1 \times_{\mathbb{B}_0} P_1 \xrightarrow{\ell} P$$

strict algebras

... but still have a natural iso. mediating the usual

$$
\begin{tikzcd}
& \mathbb{B}_1 \times_{\mathbb{B}_0} P_1 \times_{\mathbb{A}_0} \mathbb{A}_1 \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\substack{\text{bimodularity}\\ \text{condition}\\ \cong}"] & \\
\mathbb{B}_1 \times_{\mathbb{B}_0} P_1 \arrow[dr] & & P_1 \times_{\mathbb{A}_0} \mathbb{A}_1 \arrow[dl] \\
& P_1 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(X \arrow[r, "x"] & X' \arrow[r, "p"] & Y') \arrow[r, "y"] & Y \\
X \arrow[u, equal] \arrow[r, "x"'] \arrow[urrr, phantom, "\cong"] & (X' \arrow[r, "p"'] & Y' \arrow[r, "y"'] & Y) \arrow[u, equal]
\end{tikzcd}
$$

These semi-strict double profunctors correspond to double functors

$$\left(\mathbb{A}^{\overset{\text{loose}}{op}} \times \mathbb{B}\right)^{T} \rightsquigarrow \mathbb{P}\mathrm{rof}$$

loose: normal lax

tight: seperately strict, jointly pseudo.

$\mathbb{A}$ strict $\Longrightarrow$ $\mathbb{A} \overset{1_{\mathbb{A}}}{\nrightarrow} \mathbb{A}$ strict

$$\mathbb{A} \xrightarrow{P} \mathbb{B} \xrightarrow{Q} \mathbb{C}$$

$\mathbb{A}, \mathbb{B}, \mathbb{C}$ strict

P, Q semi-strict

$\Longrightarrow$ $Q \circ P$ semi-strict.