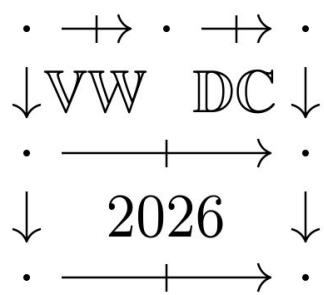
Double categories of profunctors

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Third Virtual Workshop on Double Categories



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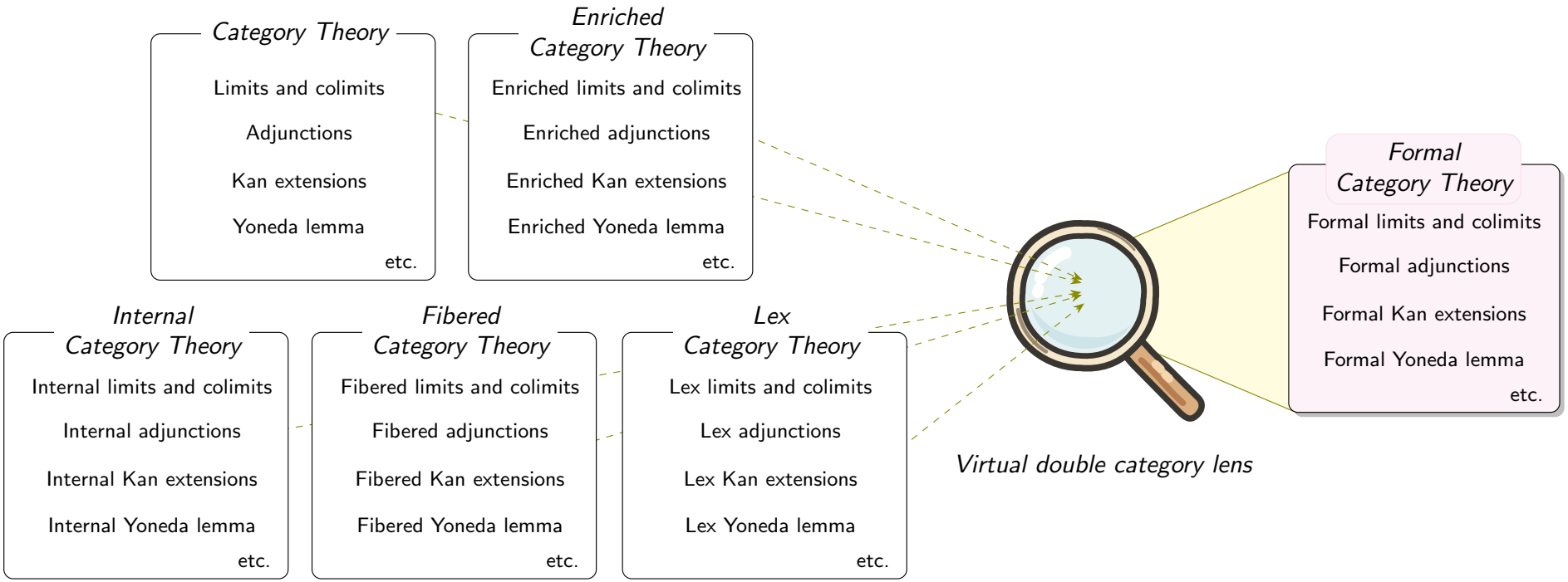
1 Background

2 Versatile colimits

3 Characterization

4 Bicolimits and future directions

What is formal category theory?



Slogan

$$\text{A category theory} \quad \coloneqq \quad \frac{\text{A structure consisting of categories,}}{\text{functors, nat.trans., and profunctors}}$$

An *(augmented) virtual double category*

The virtual double category $\mathbb{P}\mathrm{rof}$

- Objects: categories $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$;
- Tight arrows: functors $\mathbf{A} \xrightarrow{F} \mathbf{B}$, with their compositions and identities;
- Loose arrows: profunctors $\mathbf{A} \xrightarrow{P} \mathbf{B}$, i.e. $\mathbf{A}^{\mathrm{op}} \times \mathbf{B} \xrightarrow{P} \mathbf{Set}$;
- Cells: generalized natural transformations

$$\begin{array}{ccccccc} \mathbf{A}_0 & \overset{P_1}{\nrightarrow} & \mathbf{A}_1 & \overset{P_2}{\nrightarrow} & \cdots & \overset{P_n}{\nrightarrow} & \mathbf{A}_n \\ {\scriptstyle F}\downarrow & & \alpha & & & & \downarrow{\scriptstyle G} \\ \mathbf{B} & & \xrightarrow[Q]{} & & & & \mathbf{C} \end{array} \quad \Big\| \quad \begin{array}{c} P_1(x_0,x_1) \times P_2(x_1,x_2) \times \cdots \times P_n(x_{n-1},x_n) \\ \downarrow{\scriptstyle \alpha_{x_0,\ldots,x_n}} \\ Q(Fx_0, Gx_n) \end{array}$$

- and their compositions (+ identity cells)

$$\begin{array}{ccccccc} \mathbf{A}_0 & \overset{\vec{P}_1}{\nrightarrow} & \mathbf{A}_1 & \overset{\vec{P}_2}{\nrightarrow} & \cdots & \overset{\vec{P}_n}{\nrightarrow} & \mathbf{A}_n \\ {\scriptstyle F_0}\downarrow & \alpha_1 & {\scriptstyle F_1}\downarrow & \alpha_2 & \cdots & \alpha_n & \downarrow{\scriptstyle F_n} \\ \mathbf{B}_0 & \overset{}{\nrightarrow}_{Q_1} & \mathbf{B}_1 & \overset{}{\nrightarrow}_{Q_2} & \cdots & \overset{}{\nrightarrow}_{Q_n} & \mathbf{B}_n \\ {\scriptstyle G}\downarrow & & \beta & & & & \downarrow{\scriptstyle H} \\ \mathbf{C} & & \xrightarrow[R]{} & & & & \mathbf{D} \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccccc} \mathbf{A}_0 & \overset{\vec{P}_1}{\nrightarrow} & \mathbf{A}_1 & \overset{\vec{P}_2}{\nrightarrow} & \cdots & \overset{\vec{P}_n}{\nrightarrow} & \mathbf{A}_n \\ {\scriptstyle F_0 ; G}\downarrow & & \vec{\alpha} ; \beta & & & & \downarrow{\scriptstyle F_n ; H} \\ \mathbf{C} & & \xrightarrow[R]{} & & & & \mathbf{D} \end{array}$$

What is special about enriched category theory?

Question

Among all possible category theories, what is special about enriched category theory?

Goal

Characterize the virtual double categories $\mathbb{L}$ with

$$\mathbb{L} \simeq \mathbb{X}\text{-}\mathbb{P}\mathrm{rof} \qquad \text{for some enriching base } \mathbb{X}.$$

Here, $\mathbb{X}$-$\mathbb{P}$rof: $\mathbb{X}$-enriched categories, functors, natural transformations, and profunctors.

Enriching bases

$$\text{monoidal category} \hookrightarrow \text{bicategory} \hookrightarrow \text{virtual double category}$$

A monoidal category is a one-object bicategory.
A bicategory gives a virtual double category with only identity tight arrows.

Generalized enrichment

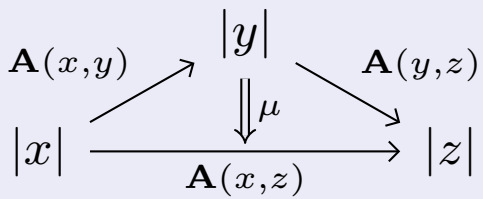
Enrichment in ~~a monoidal category $\mathcal{V}$~~ a bicategory $\mathcal{W}$

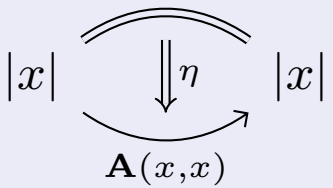
An enriched category $\mathbf{A}$ has:

Objects $x, y, \ldots$ with colors $|x|, |y|, \ldots \in \mathrm{Ob}\mathcal{W}$

Homs ~~$\mathbf{A}(x,y) \in \mathcal{V}$~~ $|x| \xrightarrow{\mathbf{A}(x,y)} |y|$ in $\mathcal{W}$

Compositions and units





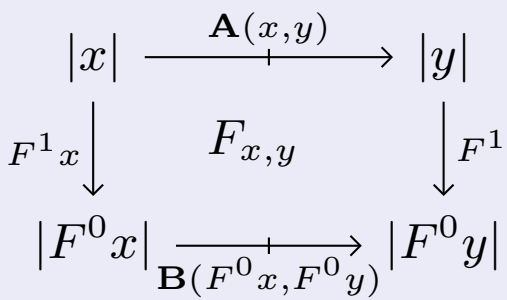
in $\mathcal{W}$

Enrichment in a virtual double category $\mathbb{X}$

An $\mathbb{X}$-category is defined in the same way as in the bicategory case.

An $\mathbb{X}$-functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$:

- an object map F^0;
- tight arrows $|x| \xrightarrow{F^1 x} |F^0 x|$;
- cells $F_{x,y}$.



in $\mathbb{X}$

Bicategory-enriched functors: $|F^0 x| = |x|$ and $F^1 x = \mathrm{id}_{|x|}$.

The embedding $\mathbb{X} \longhookrightarrow \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$

Suppose $\mathbb{X}$ has *loose units*.

Construction (full embedding)

$$\mathbb{X} \xhookrightarrow{Z} \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}, \qquad c \longmapsto \mathbf{Z}_c.$$

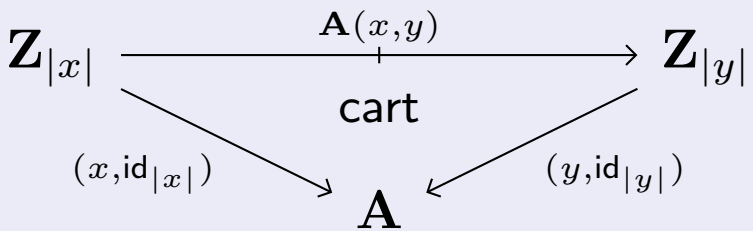
The $\mathbb{X}$-category $\mathbf{Z}_c = \{*\}$ with $|*| = c$, $\mathbf{Z}_c(*,*) =$ (loose unit on c).

Observation

$$\text{An } \mathbb{X}\text{-functor } \mathbf{Z}_c \longrightarrow \mathbf{A} \quad \longleftrightsquigarrow \quad \text{A pair } (x \in \mathbf{A}, \ \ c \xrightarrow{u} |x| \text{ in } \mathbb{X})$$

$\therefore$ By looking at arrows from the $\mathbb{X}$-categories $\mathbf{Z}_c$, we should be able to reconstruct $\mathbf{A}$.

For example, the hom-loose arrows $\mathbf{A}(x,y)$ can be recovered by the restriction:



in $\mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$.

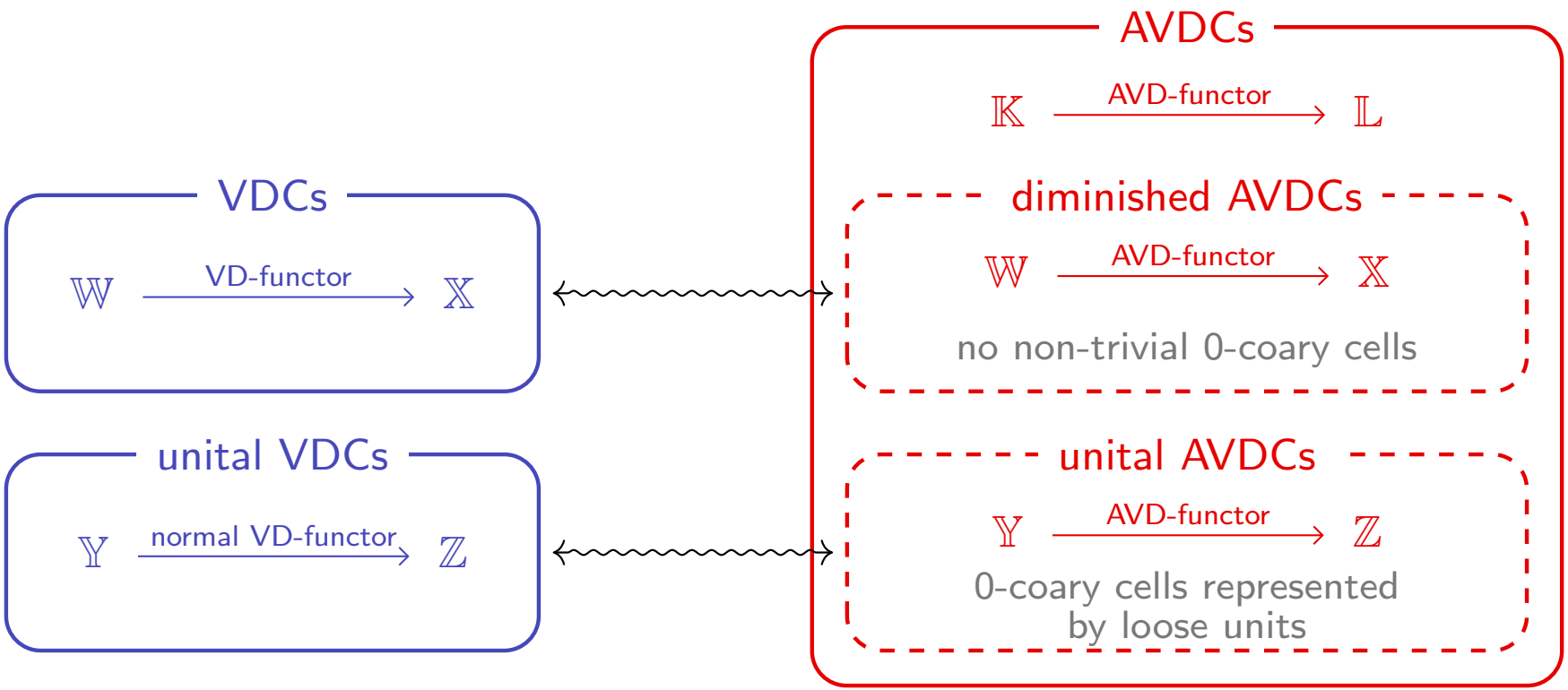
Additionally, every $\mathbb{X}$-category $\mathbf{A}$ looks like a "gluing" of the $\mathbf{Z}_{|x|}$ for $x \in \mathbf{A}$.

We will make this precise as a versatile colimit.

A technical remark: VDC vs. AVDC

Remark

We use *augmented* virtual double categories (AVDC), allowing 0-coary cells, for a "loose-unit-free" treatment of cocones.



A unital VDC has both interpretations; the choice depends on which functors we consider.

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Collages of a profunctor: 1/2

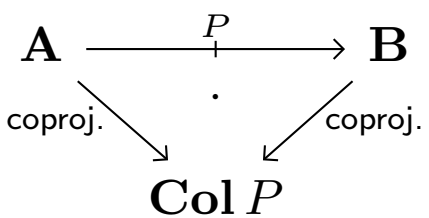
$\mathscr{V}$: a monoidal category with $\varnothing$ (The initial object preserved by $\otimes$)
$\mathbf{A} \xrightarrow{P} \mathbf{B}$: a $\mathscr{V}$-profunctor

Definition

A collage (or *cograph*) of P is the $\mathscr{V}$-category $\mathbf{Col}\, P$:

$$\mathrm{Ob}(\mathbf{Col}\, P) \coloneqq \mathrm{Ob}\mathbf{A} + \mathrm{Ob}\mathbf{B};$$

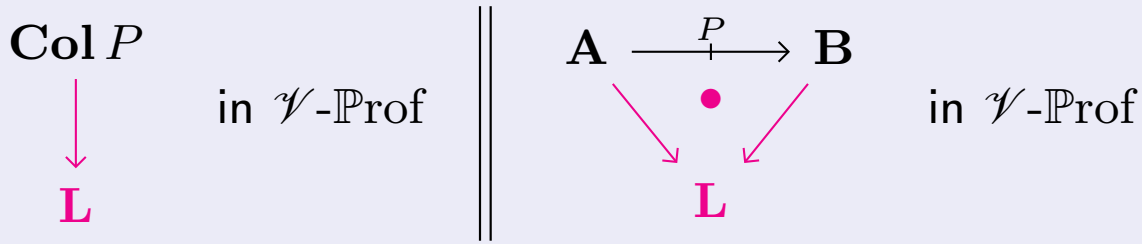
$$\mathbf{Col}\, P(x,y) \coloneqq \begin{cases} \mathbf{A}(x,y) & \text{if } x,y \in \mathbf{A} \\ \mathbf{B}(x,y) & \text{if } x,y \in \mathbf{B} \\ P(x,y) & \text{if } x \in \mathbf{A}, y \in \mathbf{B} \\ \varnothing & \text{if } x \in \mathbf{B}, y \in \mathbf{A} \end{cases}$$



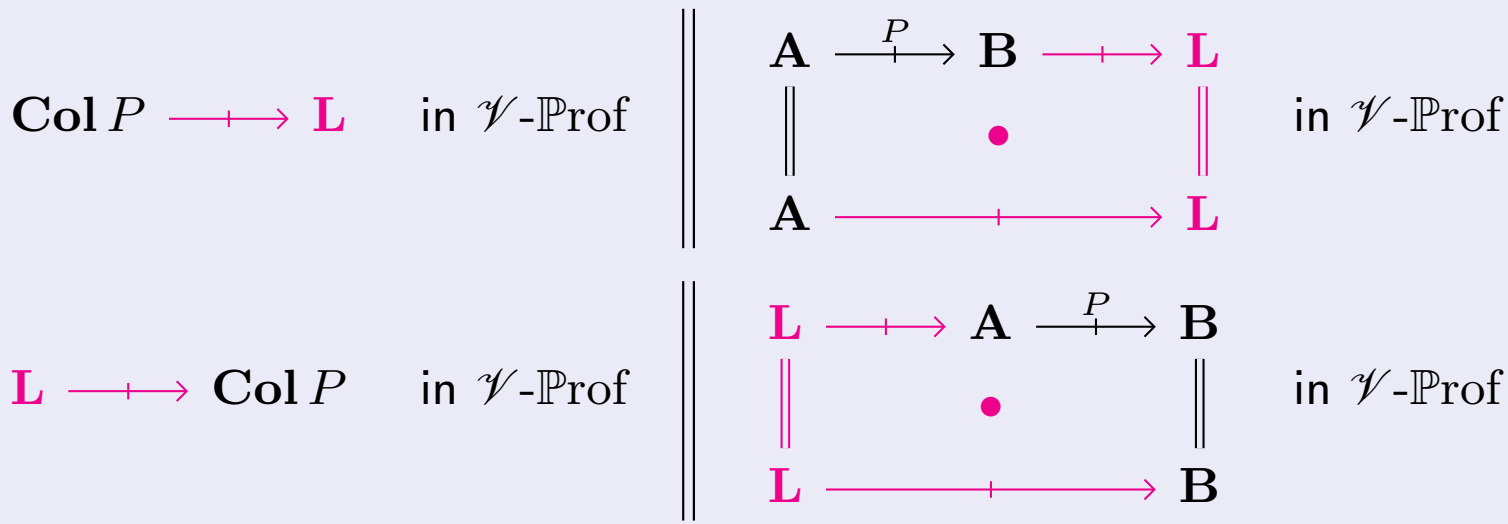
in $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

Collages of a profunctor: 2/2

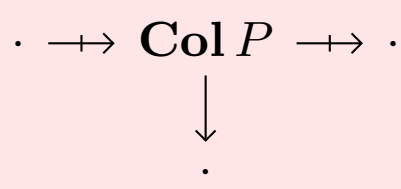
Universal property (tightwise)



Universal property (loosewise)



Col P has universal properties in the three directions:



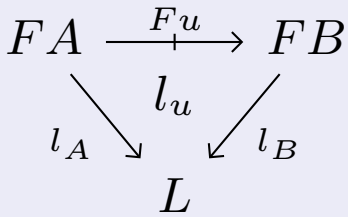
Tight cocones and modules

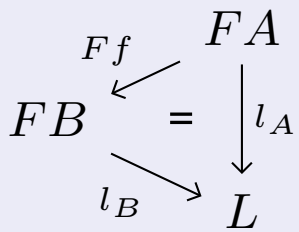
$\mathbb{K} \xrightarrow{F} \mathbb{L}$: an AVD-functor.

Definition (tight cocone)

A tight cocone l (over F) consists of:

- a vertex $L \in \mathbb{L}$;
- tight arrows $FA \xrightarrow{l_A} L$ $(A \in \mathbb{K})$;
- cells l_u for $A \xrightarrow{u} B$ in $\mathbb{K}$.





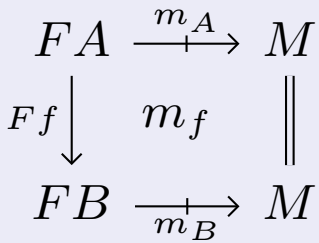
The cells l_u are compatible with all cells of $\mathbb{K}$.

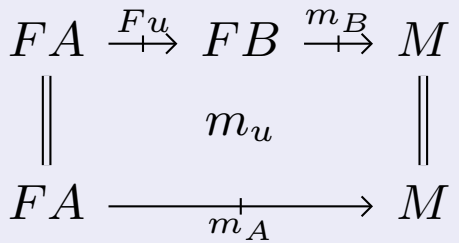
For tight $A \xrightarrow{f} B$ in $\mathbb{K}$: $(Ff)\fatsemi l_B = l_A$. ⚠ No comparison cell l_f.

Definition (modules)

A left module m (over F) consists of:

- a vertex $M \in \mathbb{L}$;
- loose arrows $FA \xrightarrow{m_A} M$ $(A \in \mathbb{K})$;
- cartesian cells m_f for $A \xrightarrow{f} B$ in $\mathbb{K}$;
- cells m_u for $A \xrightarrow{u} B$ in $\mathbb{K}$.





All data respect identities, composition, and cells in $\mathbb{K}$.

Right modules $\cdots$ loosewise dual.

Cocones and modules for a collage

Example

General concept	Collage example
AVDC $\mathbb{L}$	$\mathcal{V}$-$\mathbb{P}\mathrm{rof}$
Shape $\mathbb{K}$	$\{\ 0 \xrightarrow{!} 1\ \}$ with only identity tight arrows.
Diagram $\mathbb{K} \xrightarrow{F} \mathbb{L}$	$\mathbf{A} \xrightarrow{P} \mathbf{B}$ in $\mathcal{V}$-$\mathbb{P}\mathrm{rof}$
Tight cocone $F \Rightarrow L$	$\mathbf{A} \xrightarrow{P} \mathbf{B}$, $l_!$, l_0, l_1, $\mathbf{L}$ in $\mathcal{V}$-$\mathbb{P}\mathrm{rof}$
Left module $F \xRightarrow{m} M$	$\mathbf{A} \xrightarrow{P} \mathbf{B} \xrightarrow{m_1} \mathbf{M}$, $m_!$, $\mathbf{A} \xrightarrow[m_0]{} \mathbf{M}$ in $\mathcal{V}$-$\mathbb{P}\mathrm{rof}$
Right module $N \xRightarrow{n} F$	$\mathbf{N} \xrightarrow{n_0} \mathbf{A} \xrightarrow{P} \mathbf{B}$, $n_!$, $\mathbf{N} \xrightarrow[n_1]{} \mathbf{B}$ in $\mathcal{V}$-$\mathbb{P}\mathrm{rof}$

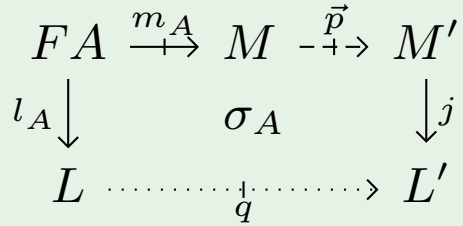
Modulations and their boundaries

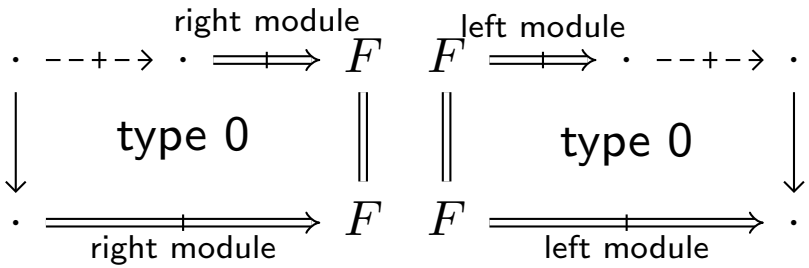
Definition (informal)

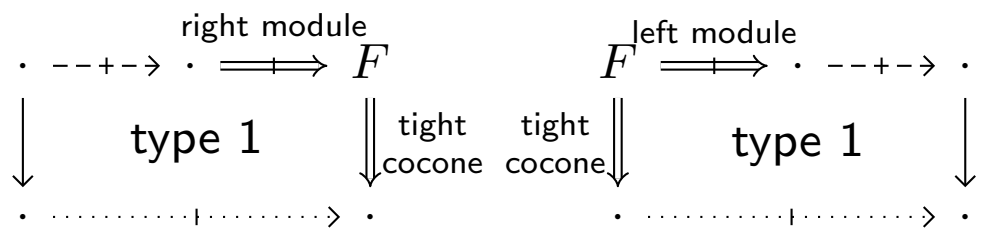
Modulations are "2-cells" between tight cocones and left/right modules.

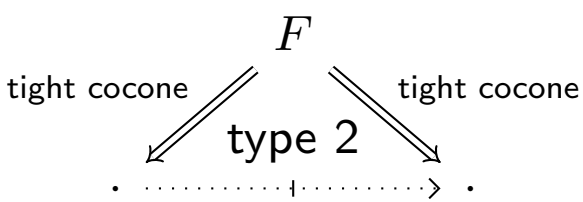
Example (type 1 with a left module)

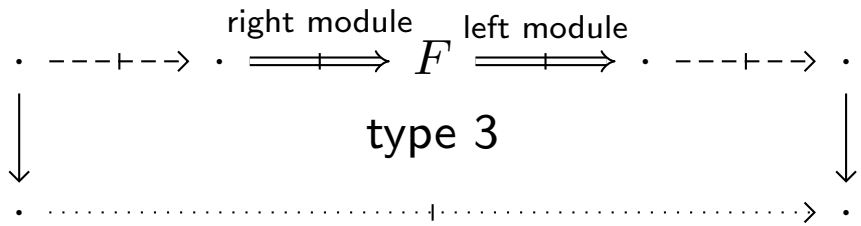
l, m: a tight cocone and a left module over F.
A modulation σ consists of cells σ_A $(A \in \mathbb{K})$ with fixed boundary $\vec{p}, j, q$, compatible with l_u, m_f, m_u.











Double arrows: cocones/modules. Dashed: loose paths. Dotted: loose paths of length ≤ 1.

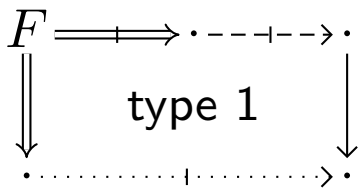
$F \xRightarrow{\xi} \Xi$: a tight cocone.

(T) $F \xRightarrow{\text{tight cocone}} L$ bijective $\Xi \longrightarrow L$ in $\mathbb{L}$

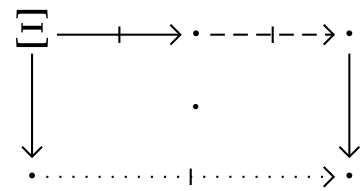
(L-ℓ) $F \xRightarrow{\text{left module}} L$ ess.surj. $\Xi \nrightarrow L$ in $\mathbb{L}$

(L-r) $L \xRightarrow{\text{right module}} F$ ess.surj. $L \nrightarrow \Xi$ in $\mathbb{L}$

(M1-ℓ)

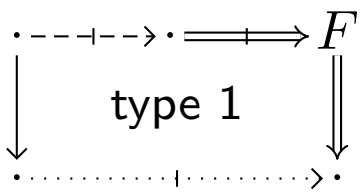


bijective

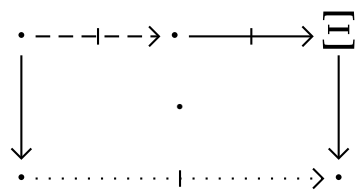


in $\mathbb{L}$

(M1-r)

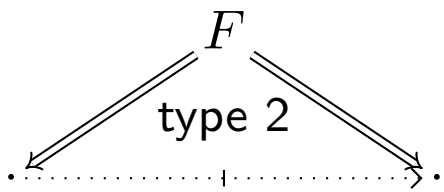


bijective

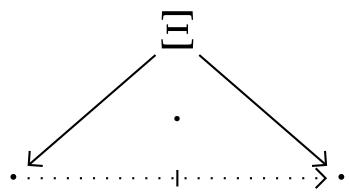


in $\mathbb{L}$

(M2)

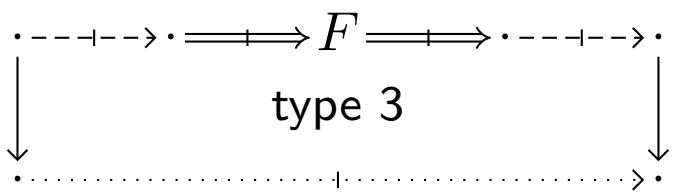


bijective

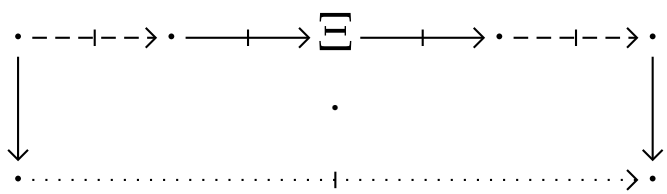


in $\mathbb{L}$

(M3)



bijective



in $\mathbb{L}$

Definition

A versatile colimit is a tight cocone ξ satisfying all these conditions.

cf. If loose composites exist, ξ satisfies (T) and (M2) $\iff$ it is a *strict double colimit* in the sense of Grandis–Paré [GP99].

Collages in general: 1/2

S: a set

Notation

$\mathbb{I}^\flat\mathrm{S}$ $\cdots$ the VDC described by:

- $\mathrm{Ob}(\mathbb{I}^\flat\mathrm{S}) := \mathrm{S}$; only identity tight arrows;
- For $i, j \in \mathrm{S}$, there is a unique loose arrow $i \overset{!_{ij}}{\longrightarrow} j$ in $\mathbb{I}^\flat\mathrm{S}$;
- For $n \geq 0$ and $i_0, \ldots, i_n \in \mathrm{S}$, there is a unique cell

$$\begin{array}{ccccccc} i_0 & \overset{!_{i_0 i_1}}{\longrightarrow} & i_1 & \overset{!_{i_1 i_2}}{\longrightarrow} & \cdots & \overset{!_{i_{n-1} i_n}}{\longrightarrow} & i_n \\ \| & & & !_{i_0 \ldots i_n} & & & \| \\ i_0 & & \longrightarrow & & & & i_n \\ & & {\scriptstyle !_{i_0 i_n}} & & & & \end{array}$$

A VD-functor $\mathbb{I}^\flat\mathrm{S} \longrightarrow \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ is equivalent to the following data:

- $\mathbf{A}_i$: $\mathbb{X}$-categories $(i \in \mathrm{S})$
- $\begin{array}{ccc} & \mathbf{A}_i & \\ & \overset{\eta_i}{\Vert\!\Vert} & \\ \mathbf{A}_i & \overset{P_{ii}}{\longrightarrow} & \mathbf{A}_i \end{array}$ $(i \in \mathrm{S})$
- $\mathbf{A}_i \overset{P_{ij}}{\longrightarrow} \mathbf{A}_j$: $\mathbb{X}$-profunctors $(i, j \in \mathrm{S})$
- $\begin{array}{ccccc} \mathbf{A}_i & \overset{P_{ij}}{\longrightarrow} & \mathbf{A}_j & \overset{P_{jk}}{\longrightarrow} & \mathbf{A}_k \\ \| & & \mu_{ijk} & & \| \\ \mathbf{A}_i & & \longrightarrow & & \mathbf{A}_k \\ & & {\scriptstyle P_{ik}} & & \end{array}$ $(i, j, k \in \mathrm{S})$ (+Axioms)

Collages in general: 2/2

Definition

Versatile collages are versatile colimits of shape $\mathbb{I}^\flat \mathrm{S}$ for some set S.

Example

$P \coloneqq (\mathbf{A}_i, P_{ij}, \eta_i, \mu_{ijk})_{i,j,k \in \mathrm{S}}$: a VD-functor $\mathbb{I}^\flat \mathrm{S} \xrightarrow{P} \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$
The $\mathbb{X}$-category $\mathbf{Col}\, P$:

$$\mathrm{Ob}(\mathbf{Col}\, P) \coloneqq \coprod_{i \in \mathrm{S}} \mathrm{Ob}\mathbf{A}_i;$$

$$\mathbf{Col}\, P(x, y) \coloneqq P_{ij}(x, y) \quad (\text{where } x \in \mathbf{A}_i, y \in \mathbf{A}_j).$$

This gives a versatile collage of P.

$$\begin{array}{ccc} \mathbf{A}_i & \xrightarrow{P_{ij}} & \mathbf{A}_j \\ & {\scriptstyle \xi_i} \searrow \quad \swarrow {\scriptstyle \xi_j} & \\ & \mathbf{Col}\, P & \end{array} \qquad \text{in } \mathbb{X}\text{-}\mathbb{P}\mathrm{rof} \qquad (i, j \in \mathrm{S})$$

Additional condition simplifies the axioms

Theorem [Kaw25]

Suppose $\mathbb{L}$ has:

- loose units/composites,
- restrictions,
- extensions/lifts of loose arrows.

Then, ξ is a versatile colimit $\iff$ for every $L \in \mathbb{L}$,

$\mathbf{Hom}_{\mathbb{L}}(\frac{\Xi}{L}) \longrightarrow \{\text{tight cocones } F \Rightarrow L\}$ is an isomorphism of categories,

$\mathbf{Hom}_{\mathbb{L}}(\Xi, L) \longrightarrow \{\text{left modules } F \nRightarrow L\}$ is an equivalence of categories,

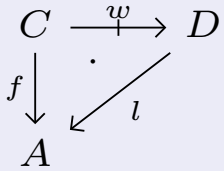
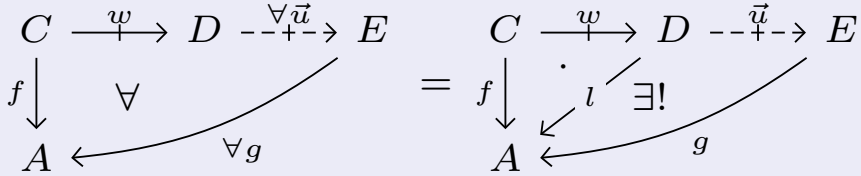
$\mathbf{Hom}_{\mathbb{L}}(L, \Xi) \longrightarrow \{\text{right modules } L \nRightarrow F\}$ is an equivalence of categories.

Remark

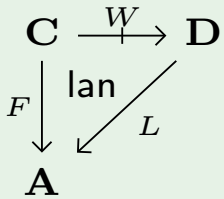
This allows us to compare our versatile colimits with Wood's notion of colimit [Woo85]. We will discuss this comparison later.

An application to Kan extensions

Definition [Kou24]

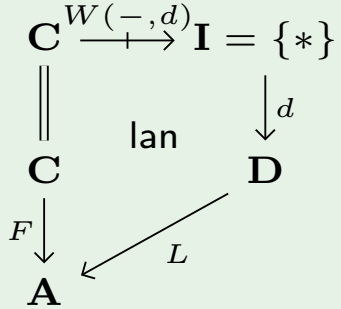
A cell  in $\mathbb{L}$ is a left-Kan-extension cell $\overset{\text{def}}{\Leftrightarrow}$ 

Example (Weighted colimits)

 in $\mathcal{V}$-$\mathbb{P}\mathrm{rof}$.

For $d \in \mathbf{D}$, $L(d) \cong \int^{c \in \mathbf{C}} W(c,d) \cdot F(c)$.

Here, L is obtained by gluing the "local" lan-cells:

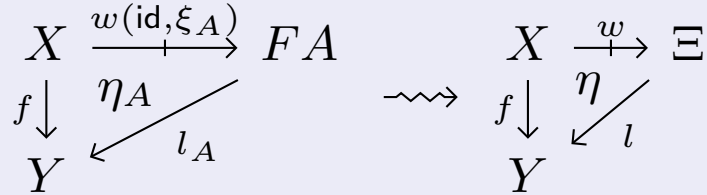
 in $\mathcal{V}$-$\mathbb{P}\mathrm{rof}$.

This observation can be formalized by using versatile colimits.

Theorem

$\mathbb{K} \xrightarrow{F} \mathbb{L}$, ξ: a versatile colimit with vertex Ξ; $\mathbb{K}$ has only identity tight arrows.

For $X \xrightarrow{f} Y$ and $X \xrightarrow{w} \Xi$, lan-cells η_A for all $A \in \mathbb{K}$ glue to a lan-cell η, with $\xi_A \fatsemi l = l_A$.



1 Background

2 Versatile colimits

3 Characterization

4 Bicolimits and future directions

The characterization theorem

Theorem [Kaw25]

The following are equivalent for an AVDC $\mathbb{L}$:

1. $\mathbb{L}$ is ~~equivalent~~ admissibly equivalent to $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$ for some iso-fibrant unital AVDC $\mathbb{X}$.
2. $\mathbb{L}$ has versatile collages and an iso-fibrant full sub-AVDC $\mathbb{X} \subseteq \mathbb{L}$ such that:
   - every object of $\mathbb{X}$ is ~~"atomic"~~ collage-atomic in $\mathbb{L}$;
   - every object of $\mathbb{L}$ is a versatile collage of a diagram in $\mathbb{X}$.

Definition

An AVDC is iso-fibrant $\overset{\text{def}}{\Leftrightarrow}$ All restrictions of a loose arrow along tight isos exist.

$$\begin{array}{ccc} X' & \longrightarrow\!\!\!\!\!+ & Y' \\ {\scriptstyle \mathbb{R}}\downarrow & \text{cart} & \downarrow{\scriptstyle \mathbb{R}} \\ X & \longrightarrow\!\!\!\!\!+ & Y \end{array}$$

We next explain *admissible equivalences* and *collage-atomicity*. ↪

Admissible equivalences

Observation

Ordinary equivalences in the 2-category $\mathcal{AVDC}$ do not behave well:

$$\left\{ \begin{array}{rc} & Y \\ & \cong\downarrow k \\ X & \overset{u}{\nrightarrow} Z \end{array} \right\} \xrightarrow[\simeq]{\Phi} \left\{ 0 \overset{!}{\mapsto} 1 \right\}, \quad \Phi X = 0, \quad \Phi Y = \Phi Z = 1.$$

However, $\mathbf{Hom}(X, Y) = \varnothing \not\cong \mathbf{1} = \mathbf{Hom}(\Phi X, \Phi Y)$. $\quad \because$ missing restriction along k.

Definition

- A tight iso h is admissible $\overset{\text{def}}{\Leftrightarrow}$ Restrictions of any loose arrow along h or h^{-1} exist.
- An AVD-natural iso is admissible $\overset{\text{def}}{\Leftrightarrow}$ Every component is admissible.
- An AVD-equivalence is admissible $\overset{\text{def}}{\Leftrightarrow}$ Both unit and counit are admissible.

Proposition

Admissible equivalences define an equivalence relation on AVDCs.

⚠ This is non-trivial because AVDCs, AVD-functors, and admissible AVD-nat.trans. do not form a 2-category.

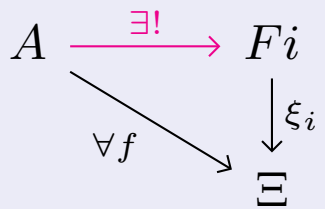
Theorem [Kaw25]

Versatile colimits are preserved by any admissible equivalence.

Collage-atomicity and characterization

Definition

$A \in \mathbb{L}$ is collage-atomic $\overset{\text{def}}{\Leftrightarrow}$ Every tight arrow into a versatile collage uniquely factors through a unique coprojection:



$\exists!\ i \in \mathrm{S}.$

Proposition

Suppose $\mathbb{X}$ has loose units.

$$\mathbf{A} \in \mathbb{X}\text{-}\mathbb{P}\mathrm{rof} \text{ is collage-atomic } \iff \mathbf{A} \underset{\text{tight iso}}{\cong} \mathbf{Z}_c \text{ for some } c \in \mathbb{X}.$$

Theorem (revisited) [Kaw25]

The following are equivalent for an AVDC $\mathbb{L}$:

1. $\mathbb{L}$ is admissibly equivalent to $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$ for some *iso-fibrant* unital AVDC $\mathbb{X}$.
2. $\mathbb{L}$ has versatile collages and an *iso-fibrant* full sub-AVDC $\mathbb{X}$ such that:
   - every object of $\mathbb{X}$ is collage-atomic in $\mathbb{L}$;
   - every object of $\mathbb{L}$ is a versatile collage of a diagram in $\mathbb{X}$.

Variants: matrices and modules

Remark

We also have analogous characterization theorems:

Construction from $\mathbb{X}$	Corresponding versatile colimits	Diagram
$\mathbb{M}\mathrm{od}(\mathbb{X})$	Versatile collapses	A monoid
$\mathbb{X}$-$\mathbb{M}\mathrm{at}$	VD-versatile coproducts	A family of objects

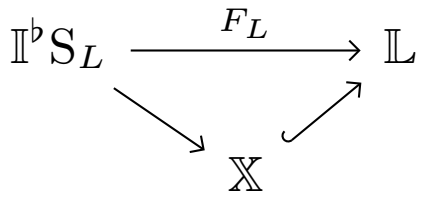
Definition (informal)

VD-versatile colimits satisfy the same universal properties for tight and loose arrows. But the cell universal properties are required only for 1-coary boundaries.

We will not discuss VD-versatility in detail in this talk because of its technicality. . .

Why does the characterization work?

Fix $\mathbb{X} \subseteq \mathbb{L}$ as in the characterization. For each $L \in \mathbb{L}$, choose a versatile collage



$$F_L \overset{\xi^L}{\Longrightarrow} L.$$

Construction

$$\mathbb{X}\text{-}\mathbb{P}\mathrm{rof} \underset{N}{\overset{C}{\rightleftarrows}} \mathbb{L}$$

C: take the versatile collages in $\mathbb{L}$.

N: reconstruct each $\mathbb{X}$-category from $L \in \mathbb{L}$ using $(\mathrm{S}_L, F_L, \xi^L)$.

To show $N \circ C \cong \mathrm{Id}$, we must recover the original $\mathbb{X}$-enrichment from $L \in \mathbb{L}$.

1. The collage-atomicity recovers **(colored) objects**.
2. Why can we also recover **hom-loose arrows**? $\rightsquigarrow$ *"Effectiveness"* of vers. collages. $\hookrightarrow$

Strongness of versatile collages: 1/2

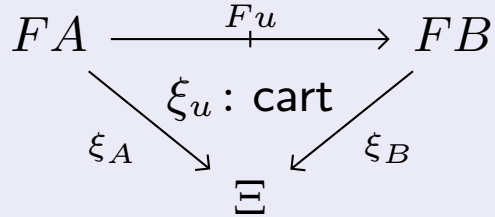
Recall

In $\mathbf{Set}$, any equivalence relation R with quotient $X \overset{q}{\twoheadrightarrow} Q$ is effective, i.e., $R \cong X \times_Q X$.

$\rightsquigarrow$ Q does not collapse R too much.

Definition

A tight cocone ξ is strong if ξ_u is cartesian for every u.



Strongness (effectiveness) now follows from the universal property:

Theorem (Strongness) [Kaw25]

$\mathbb{K}$: one loose arrow per object pair; one cell per 1-coary boundary; at most one per 0-coary boundary.

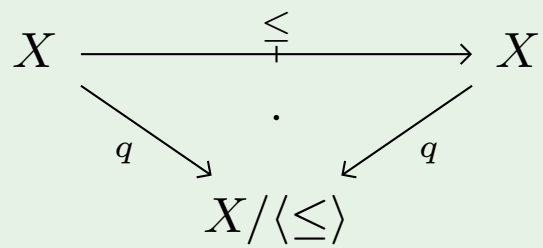
$\rightsquigarrow$ Versatile colimits of shape $\mathbb{K}$ are strong. (e.g. versatile collages.)

Rem. The proof was first given by Hayato Nasu for pseudo double categories!

Strongness of versatile collages: 2/2

Non-example

A monoid in $\mathbb{R}\mathrm{el}$ is a preorder $(X, \leq)$. Its tight cocones identify comparable elements, so their universal quotient is



where $\langle\leq\rangle$ is the smallest equivalence relation containing $\leq$.
This cannot be a versatile colimit because it is not strong!

1 Background

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Versatile bicolimits

Definition

Versatile bicolimits are like versatile colimits, but their tightwise universal property holds only up to equivalence.

~~Recall~~ Theorem

Suppose $\mathbb{L}$ has:

- loose units/composites,
- restrictions,
- extensions/lifts of loose arrows.

Then, ξ is a versatile ~~colimit~~ (bicolimit) $\iff$ for every $L \in \mathbb{L}$,

$$\mathbf{Hom}_{\mathbb{L}}\left(\begin{smallmatrix}\Xi\\ L\end{smallmatrix}\right) \longrightarrow \{\text{tight cocones } F \Rightarrow L\} \qquad \text{is an ~~isomorphism~~ (equivalence) of categories,}$$

$$\mathbf{Hom}_{\mathbb{L}}(\Xi, L) \longrightarrow \{\text{left modules } F \nRightarrow L\} \qquad \text{is an equivalence of categories,}$$

$$\mathbf{Hom}_{\mathbb{L}}(L, \Xi) \longrightarrow \{\text{right modules } L \nRightarrow F\} \qquad \text{is an equivalence of categories.}$$

Comparison with Wood's proarrow equipments

Fact [Shu08]

A *proarrow equipment* whose domain is a 2-category $\coloneqq$ a pseudo double category + restrictions

$\coloneqq$ an AVDC + loose composites / restrictions

Theorem [Kaw25]

For a proarrow equipment $\mathcal{K} \to \mathcal{M}$ s.t. $\mathcal{K}$: a 2-category, $\mathcal{M}$: *biclosed*;

Wood's axiom 4 $\iff$ has all finite versatile bicoproducts,

Wood's axiom 5 $\iff$ has all versatile bicollapses.

Does the theorem mean $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$ is a free cocompletion?

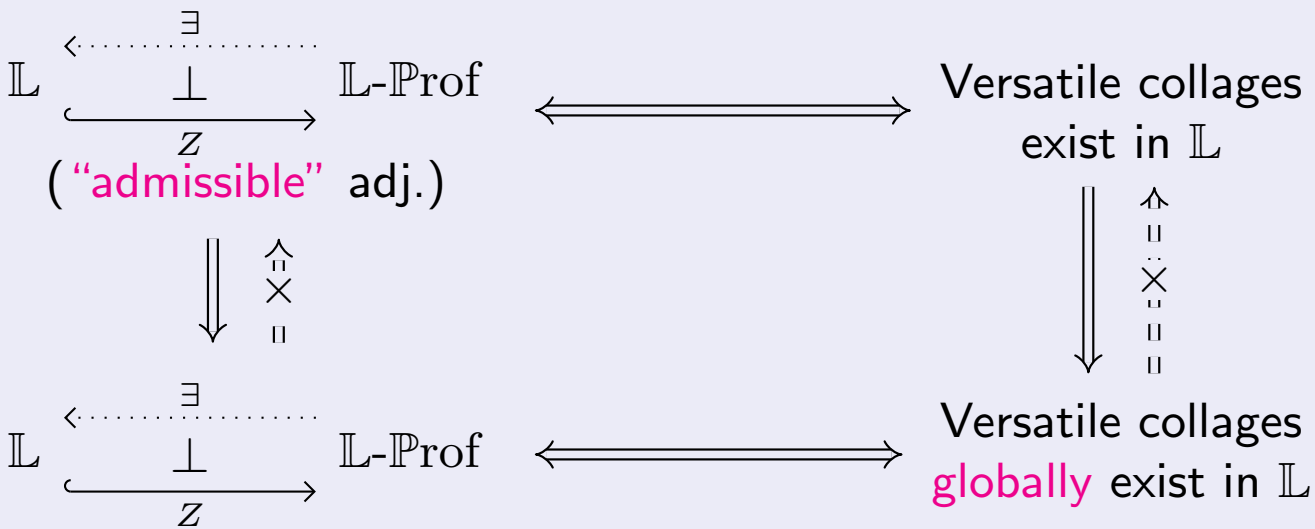
Question

Is $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$ a *free cocompletion* of $\mathbb{X}$ under versatile collages?

$\rightsquigarrow$ Conjecturally **YES**, but the admissibility needs to be considered.

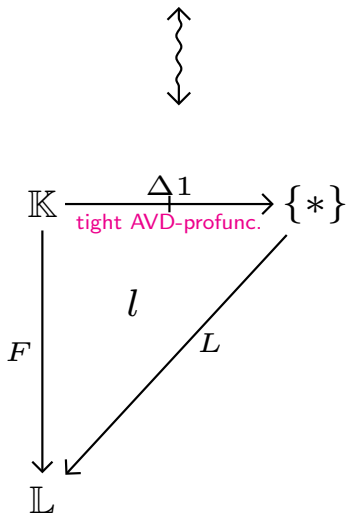
Conjecture

$\mathbb{L}$: has loose units.



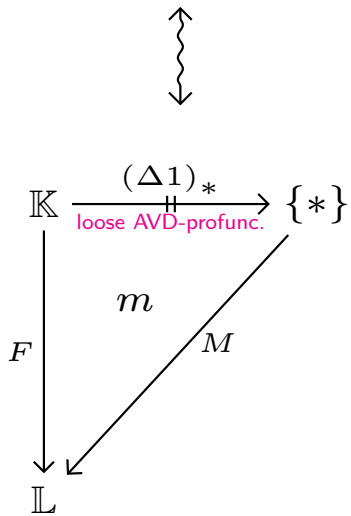
Towards a formal theory of versatile colimits

A tight cocone

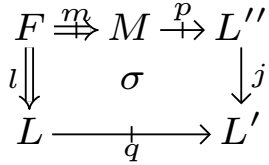


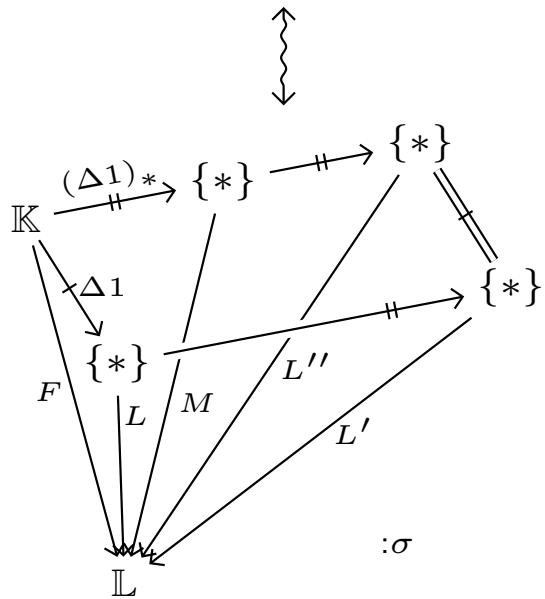
A left module



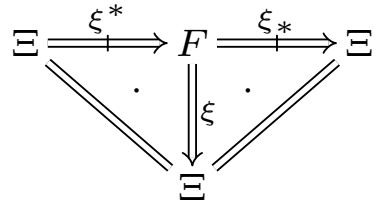


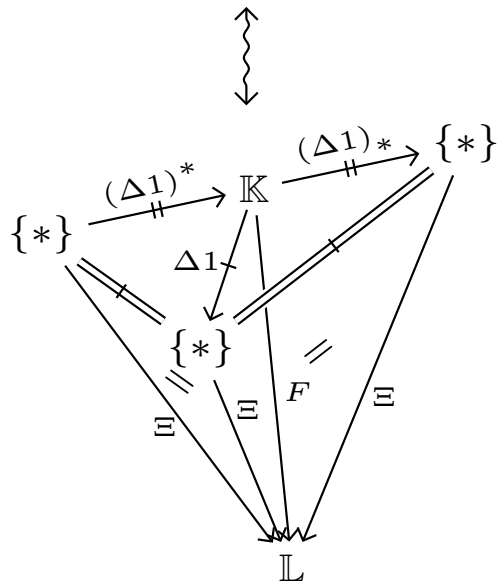
A modulation





A versatile colimit





Kan-extending 3-cell?

in the *augmented virtual triple category(?)*
of AVDCs, AVD-functors, tight AVD-profunctors, and loose AVD-profunctors.

May this talk be versatile for you!

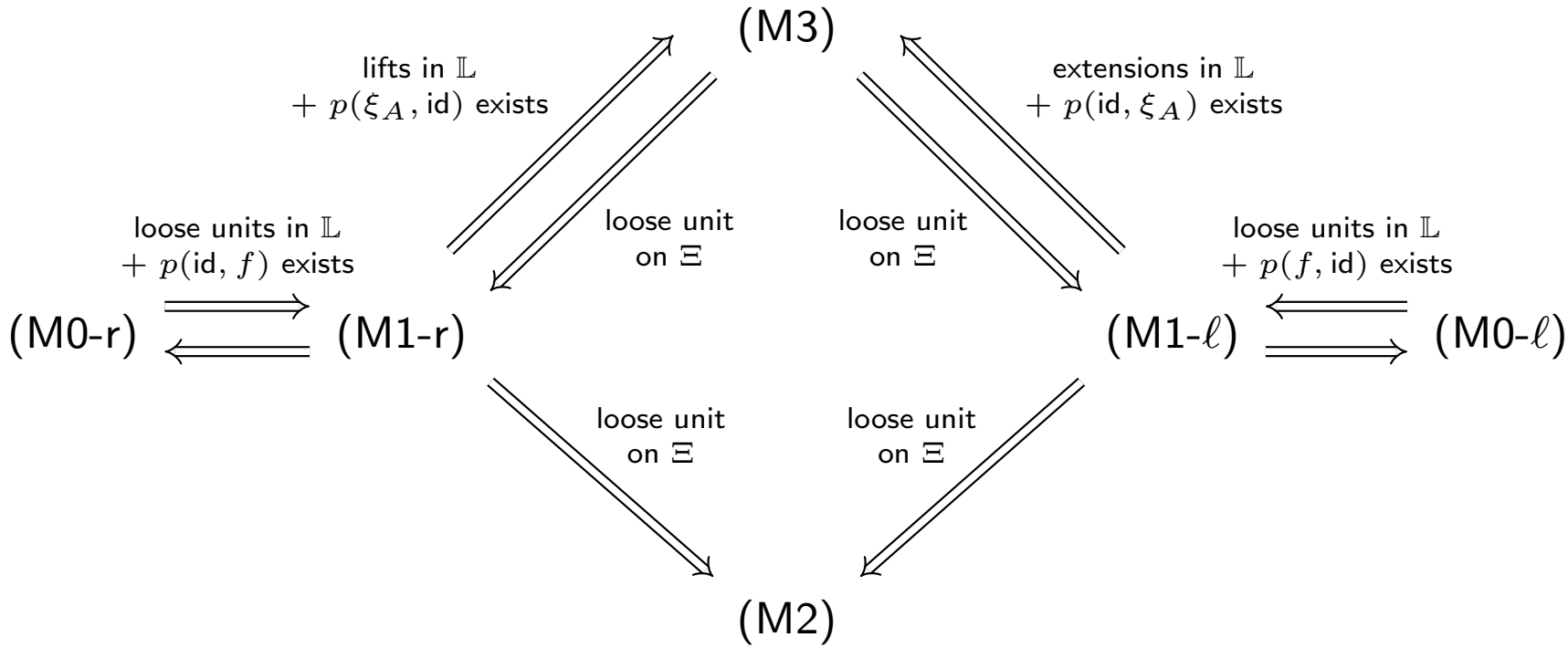
Paper

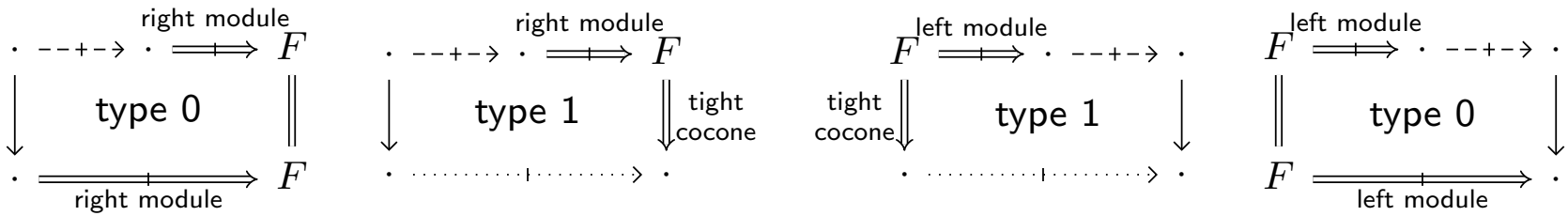
Double categories of profunctors

arXiv:2504.11099

Implications between the cell axioms

$\mathbb{K} \xrightarrow{F} \mathbb{L}$, ξ: a tight cocone with vertex Ξ.





Dashed: loose paths. Dotted: loose paths of length ≤ 1.

References

[GP99] M. Grandis and R. Pare. "Limits in double categories". In: *Cahiers Topologie Géom. Différentielle Catég.* 40.3 (1999), pp. 162–220.

[Kaw25] Y. Kawase. *Double categories of profunctors*. 2025. arXiv: 2504.11099.

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[Shu08] M. Shulman. "Framed bicategories and monoidal fibrations". In: *Theory Appl. Categ.* 20 (2008), Paper No. 18, 650–738.

[Woo85] R. J. Wood. "Proarrows. II". In: *Cahiers Topologie Géom. Différentielle Catég.* 26.2 (1985), pp. 135–168.