THE (VIRTUAL) DOUBLE CATEGORY OF RESTRICTION DISTRIBUTORS

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joint work *in progress* with N. Arkor, Y. Chamoun, B. Clarke, V. Jelínek and H. Nasu
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SECTION 1

Restriction categories

Restriction categories provide a framework where one can have a *calculus of partially defined maps* which ordinary categories cannot capture.

Example: Set_p the category of sets and partial functions. Given a partial function $f : X \rightharpoonup Y$, we can *describe how far off f is from being total* using the map

$$\overline{f} : X \to X : \overline{f}(x) = \begin{cases} x, & \text{if } f(x) \text{ is defined,} \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Idea: *use such maps as the primitive notion* for partiality.

Definition [Cockett, Lack 02]

A ***restriction category*** is a category $\mathcal{A}$ equipped with a ***restriction structure***, that is assignments

$$\overline{(-)} : \mathcal{A}(X,Y) \to \mathcal{A}(X,X) : f \mapsto \overline{f}$$

for every $X, Y \in \mathcal{A}_0$ so that

$$f\overline{f} = f, \quad \overline{f}\,\overline{g} = \overline{g}\,\overline{f}, \quad \overline{f\overline{g}} = \overline{f}\,\overline{g} \quad \text{and} \quad f\overline{gf} = \overline{g}f$$

whenever these make sense. Arrows of the form $\overline{f}$ satisfy $\overline{f}\,\overline{f} = \overline{f}$ and are thus called ***restriction idempotents***. An arrow $f \in \mathcal{A}(X,Y)$ will be called ***total*** if $\overline{f} = 1_X$.

Example (sets and partial functions)

The category Set_p of sets and partial functions is a restriction category if we define $\overline{f}$ as before. Total arrows are the total functions.

Example (partial recursive functions)

The category Rec whose objects are natural numbers and arrows $n \to m$ are partial recursive functions $\mathbb{N}^n \to \mathbb{N}^m$ is a restriction category if we define $\overline{f}$ as above. Its total arrows are the total recursive functions.

Example (meet-semilattices)

The category $\mathcal{S} = \mathsf{StabLat}^{\mathrm{op}}$ whose objects are *meet-semilattices* (posets with binary meets $x \wedge y$ and a top element $\top$) and arrows $X \to Y$ are *stable maps* $Y \to X$ (functions preserving binary meets) is a restriction category; to each arrow $f : X \to Y$ given by a stable map $Y \to X$ we define $\overline{f}(x) = x \wedge f(\top_Y)$. Total are the stable maps that also preserve the top element.

Any restriction category $\mathcal{A}$ is a *locally posetal 2-category* or Pos-*enriched* in a natural way. For any $X, Y \in \mathcal{A}$ and $f, g : X \rightrightarrows Y$ define

$$f \leqslant g \iff f = g\overline{f}.$$

Intuitively, *f is below g when the two of them agree wherever f is defined*.

Suppose $\mathcal{A}, \mathcal{B}$ are two restriction categories. A ***restriction functor*** is a functor that *respects the restriction structure* in the sense that $F\overline{f} = \overline{Ff}$ for all $f \in \mathcal{A}_1$. Any restriction functor $F : \mathcal{A} \to \mathcal{B}$ is a *2-functor*; it respects the ordering since this is defined equationally.

Example (the fundamental restriction functor)

For every restriction category $\mathcal{A}$ there is a restriction functor $\mathcal{O} : \mathcal{A} \to \mathcal{S}$ called its ***fundamental functor***. It sends an object X to the meet-semilattice $\mathcal{O}(X) = \{\overline{f} : X \to X \mid f \in \mathcal{A}_1, \mathrm{dom} f = X\}$.

If $F, G : \mathcal{A} \to \mathcal{B}$ are two restriction functors, a ***restriction natural transformation*** is a natural transformation $\varphi : F \Rightarrow G$ *whose components are total arrows*.

Restriction categories, restriction functors and restriction natural transformations form the *2-category* rCat. From the above we get a *forgetful functor*

$$\Gamma : \mathrm{rCat} \to \text{2-Cat}.$$

SECTION 2

Restriction categories as enriched categories

Suppose $\mathcal{V}$ is a *monoidal category*. A ***$\mathcal{V}$-category*** $\mathcal{C}$ consists of:

- a collection $\mathcal{C}_0$ of *objects* of $\mathcal{C}$.
- for any $X, Y \in \mathcal{C}_0$, a *hom-object* $\mathcal{C}(X,Y) \in \mathcal{V}_0$.
- for each $X \in \mathcal{C}_0$, an *arrow* $1_X : I \to \mathcal{C}(X,X)$ in $\mathcal{V}$ providing the identities.
- *arrows* $\circ_{XYZ} : \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$ in $\mathcal{V}$ giving the composition for all $X, Y, Z \in \mathcal{C}_0$.

enriching in a monoidal category $\equiv$ enriching in a one-object bicategory $\xrightarrow{\text{generalize to}}$ enriching in a general bicategory

Let $\mathcal{W}$ be a *bicategory*. A ***$\mathcal{W}$-category*** $\mathcal{C}$ consists of:

- a collection $\mathcal{C}_0$ of *objects* of $\mathcal{C}$.
- an *assignment* $|-| : \mathcal{C}_0 \to \mathcal{W}_0$ called the ***extent***.
- for any $X, Y \in \mathcal{C}_0$ a *1-cell* $\mathcal{C}(X,Y) : |X| \leftarrow |Y|$ in $\mathcal{W}$.
- for each $X \in \mathcal{C}_0$, a *2-cell* $i_X : 1_{|X|} \Rightarrow \mathcal{C}(X,X)$ in $\mathcal{W}$ providing the identities.
- *2-cells* $\circ_{XYZ} : \mathcal{C}(Y,Z) \odot \mathcal{C}(X,Y) \Rightarrow \mathcal{C}(X,Z)$ in $\mathcal{W}$ giving the composition for all $X, Y, Z \in \mathcal{C}_0$.

Let $\mathbb{D}$ be a double category. ***$\mathbb{D}$-categories*** are the categories *enriched in the underlying horizontal bicategory*.
A ***$\mathbb{D}$-functor*** $F : \mathcal{C} \to \mathcal{D}$ between $\mathbb{D}$-categories consists of a function $\mathcal{C}_0 \to \mathcal{D}_0$, tight arrows $F_X : |X| \to |FX|$ for every object $X \in \mathcal{C}$ and cells

$$\begin{array}{ccc} |X| & \xleftarrow[\mathcal{C}(X,Y)]{} & |Y| \\ {\scriptstyle F_X}\downarrow & F_{XY} & \downarrow{\scriptstyle F_Y} \\ |FX| & \xleftarrow[\mathcal{D}(FX,FY)]{} & |FY| \end{array}$$

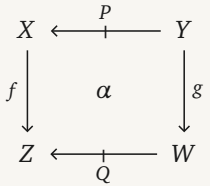
satisfying the usual axioms. A ***$\mathbb{D}$-natural transformation*** $\alpha : F \Rightarrow G$ is given by a family of cells

$$\begin{array}{ccc} |X| & \xleftarrow[1_{|X|}]{} & |X| \\ {\scriptstyle F_X}\downarrow & \alpha_X & \downarrow{\scriptstyle G_X} \\ |FX| & \xleftarrow[\mathcal{D}(FX,GX)]{} & |GX| \end{array}$$

satisfying the usual naturality axioms.

Define the double category $\mathbb{S}$ as:

- Its objects are the objects of $\Gamma\mathcal{S}$ which are *meet-semilattices*.
- Tight arrows are discrete fibrations in $\Gamma\mathcal{S}$ which are *stable maps preserving the top element*.
- Loose arrows $X \nrightarrow Y$ are *presheaves* on $\Gamma\mathcal{S}(X, Y)$; equivalently, posets with a discrete fibration to $\Gamma\mathcal{S}(X, Y)$.
- Cells



  are *monotone maps* that commute with f, g.
- Composition of loose arrows is given by Day convolution.

Theorem [Cockett, Garner]

The 2-category rCat is 2-equivalent to the 2-category of $\mathbb{S}$-enriched categories.

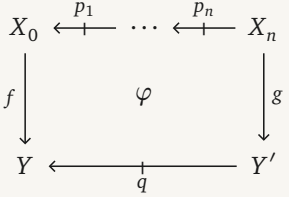
Remark. Restriction categories are one of the few naturally occurring (non trivial) examples of categories enriched in a double category.

SECTION 3

The virtual double category of restriction distributors

Recall that a ***virtual double category*** $\mathbb{V}$ consists of:

- *objects* $X, Y, Z, \ldots$;
- *tight arrows* between objects, denoted $X \to Y$;
- *loose arrows* between objects, denoted $X \nrightarrow Y$;
- *cells* of the form



$\mathbb{V}$ is called ***normal*** when it has loose identities.

There is a *normal virtual double category* $\mathbb{R}\mathrm{DIST}$ whose

- objects are *restriction categories* $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \ldots$
- tight arrows are *restriction functors* $F : \mathcal{A} \to \mathcal{B}$
- loose arrows are ***restriction distributors*** $P : \mathcal{A} \nrightarrow \mathcal{B}$ [DeWolf]; that is *ordinary distributors equipped with assignments*

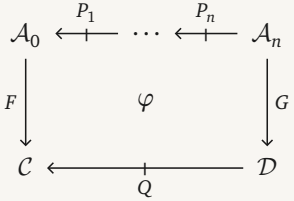
$$\overline{(-)} : P(a,b) \to \mathcal{O}(a)$$

for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$ so that

$$p\overline{p} = p, \quad \overline{p\overline{f}} = \overline{p}\,\overline{f}, \quad p\overline{fp} = \overline{f}\,p \quad \text{and} \quad f\overline{pf} = \overline{p}f$$

for every heteromorphism p and arrows f in $\mathcal{A}$ or $\mathcal{B}$ whenever the above make sense.

- cells



are *restriction natural transformations* (natural in a_0, a_n and dinatural in the other variables such that $\overline{\varphi(p_1, \ldots, p_n)} = F\left(\overline{\overline{p_n} \cdots p_1}\right)$).

EXTENDING THE EQUIVALENCE OF [COCKETT, GARNER]

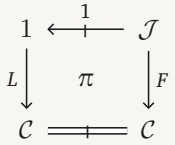
Recall [Cockett, Garner]: $\mathrm{rCat} \simeq \mathbb{S}\text{-}\mathrm{Cat}$, an equivalence of *2-categories*.

Theorem (Arkor, Chamoun, Clarke, Jelínek, Nasu, T.)
There is an equivalence of virtual double categories

$$\mathbb{R}\mathrm{DIST} \simeq \mathbb{S}\text{-}\mathrm{DIST}.$$

On tight arrows and nullary cells we recover the theorem of Cockett and Garner.

In a virtual double category, limits are *weighted by loose arrows*. A cone with total components is exactly a cell of $\mathbb{R}$DIST



L is the ***conical weighted limit*** of F when π is universal among such cells.

Let $F : \mathcal{J} \to \mathcal{C}$ be a restriction functor. A cone $\pi_j : L \to Fj$ with *total* components is a ***restriction limit*** of F if for every object X, every *restriction idempotent* e on X and every cone $\gamma_j : X \to Fj$ whose legs all satisfy $\overline{\gamma_j} = e$, there is a unique morphism of cones $m : X \to L$ with $\overline{m} = e$.

Proposition (Arkor, Chamoun, Clarke, Jelínek, Nasu, T.)

*F has a restriction limit if and only if it has a conical weighted limit in $\mathbb{R}$*DIST*, and the two coincide.*

Remark. For $\mathcal{J}$ empty this is the restriction terminal object of [Cockett, Lack 07]; for $\mathcal{J}$ free on a finite nonempty category it agrees with their restriction limits.

We expect the same ideas to apply to variants of restriction categories:

1. **Join restriction categories**: every family of parallel maps that is *compatible*, $f\overline{g} = g\overline{f}$, has a join, and joins are stable under composition.
2. **Range restriction categories**: every $f : A \to B$ also has a *range* $\hat{f} : B \to B$, a restriction idempotent playing the role of its image.
3. **Support categories**: weaken the fourth restriction axiom.
4. **Meet restriction categories**: parallel maps have meets $f \wedge g$, preserved by composition on one side only.

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- Darien DeWolf, *Restriction Category Perspectives of Partial Computation and Geometry*, Ph.D. thesis, Dalhousie University, Halifax, Nova Scotia, 2017.

Thank you for your attention!