Formal and Enriched Coalgebra

Joint work in progress with Dr. Henning Basold

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Outline

Introduction and Motivation
- Coalgebras
- Enrichment

Formal coalgebra
- Formal coalgebra in 2-categories
- Formal coalgebra in double categories

Wrapping Up

Introduction and Motivation

Contributions

Main goal: A general, formal theory of coalgebras that can be instantiated to the desired settings

Contributions:

1. Generalise categories of coalgebras for an endofunctor as internal objects in a double category
2. We show that coalgebra objects are given functorially
3. We prove a general result about the creation of weighted colimits
4. Instantiate our construction to the enriched setting

A Taste of the Scope of Coalgebra

Theory of observable systems and recursion

- Automata and labelled transition systems
- Markov chains with discrete and continuous time
- Differential equations, dynamical systems and hybrid systems
- Session types for concurrent systems
- Infinitary rewriting
- Program semantics (e.g., reactive and concurrent systems)

Provides abstract framework and reusable results

- definition and proof principles (coinduction, coinductive predicates, distributive laws)
- modal logic for system behaviour (coalgebraic logic)
- system composition (e.g., colimits and tensor products)
- foundation for long-term behaviour (e.g., stationary states and stability)
- syntactic representations of infinite behaviour (regular expression, cyclic proofs and recursive types)

Coalgebras

General Setup: the category Coalg(F)

- A base category $\mathcal{C}$ — underlying "states" of system
- A functor $F\colon \mathcal{C} \to \mathcal{C}$ — possible observations on system
- A coalgebra is a morphism $c\colon X \to FX$ in $\mathcal{C}$
- A homomorphism $c \to d$ is a morphism $f\colon X \to Y$ with commuting square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle d} \\ FX & \xrightarrow[Ff]{} & FY \end{array}$$

Comparing coalgebras for two functors $F\colon \mathcal{C} \to \mathcal{C}$ and $G\colon \mathcal{D} \to \mathcal{D}$

- Suppose $K\colon \mathcal{C} \to \mathcal{D}$ functor and $\alpha\colon K \circ F \to G \circ K$ natural transformation called distributive law
- This induces a functor $\mathrm{CoAlg}(\alpha)\colon \mathrm{CoAlg}(F) \to \mathrm{CoAlg}(G)$

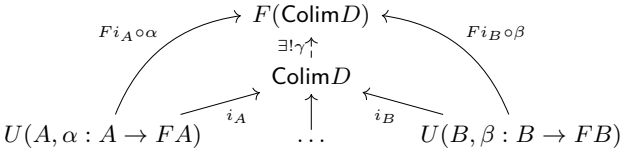
Creation of colimits

Taking colimits of coalgebras lets us combine systems. We can compute these in the base category:

Theorem (Creation of colimits)

The forgetful functor $U : \mathsf{Coalg}(F : \mathcal{C} \to \mathcal{C}) \to \mathcal{C} :: (X, \alpha) \to X$ ***creates colimits****: if for a diagram* $D : \mathbb{I} \to \mathsf{Coalg}(F)$ *there is a colimit* K *for* $U \circ D : \mathbb{I} \to \mathcal{C}$*, then there is a colimit* K' *for* D *such that* $U(K') = K$*.*

Idea: equip a colimit in $\mathcal{C}$ with coalgebra structure to get a colimit in $\mathsf{Coalg}(F : \mathcal{C} \to \mathcal{C})$:



- $i_A, i_B, \ldots$ are coalgebra homomorphisms precisely because the diagram commutes
- Universal property essentially amounts to F-coalgebra homomorphisms being morphisms in $\mathcal{C}$

Adding structure

- Coalgebras model behaviour, and morphisms preserve and reflect behaviour.
- We often want to consider coalgebras with additional structure:
  - Order enrichment[1]
  - Quantitative behaviour via quantaloids[2]
  - Topology and order in coalgebraic modal logic[3]
- How does enrichment interact with coalgebraic behaviour (bisimulation, logic, etc.)?

[1] **BKV15:PositiveFragmentsCoalgebraic**.
[2] **BKV19:ExtendingSetFunctors**.
[3] **BdV22:CoalgebraicGeometricLogic**.

Enriched category theory

Definition

Let $\mathcal{V} = (\mathcal{V}, \otimes, *)$ be a monoidal category. A $\mathcal{V}$-**category** $\mathcal{C}$ consists of:

- Collection of objects $A, B, C, \ldots$
- Hom-objects $\mathcal{C}(A, B) \in \mathrm{ob}\mathcal{V}$,
- Composition morphisms $\circ_{A,B,C} : \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \to \mathcal{C}(A, C)$
- Identity elements $j_A : * \to \mathcal{C}(A, A)$

Examples: locally small category is $\mathbf{Set}$-enriched, 2-category is $\mathbf{Cat}$-enriched, preorder is $\mathbf{2}$-enriched, $\mathbf{Pos}$ can be seen as self-enriched, etc.

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The double category $\mathcal{V}$-$\mathbf{Cat}$

- $\mathcal{V}$-functors $F : \mathcal{C} \to \mathcal{D}$ are maps $F : \text{ob}\mathcal{C} \to \text{ob}\mathcal{D}$ and maps $F_{AB} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$ for $A, B \in \text{ob}\mathcal{C}$
- $\mathcal{V}$-natural transformations
- $\mathcal{V}$-profunctors $R : \mathcal{C} \nrightarrow \mathcal{D}$ are functors $R : \mathcal{D}^{op} \otimes \mathcal{C} \to \mathcal{V}$

Weighted colimits

- In the enriched setting, we have *weighted colimits*, generalizing (conical) colimits.
- Colimits can be only for certain diagrams, rather than all colimits of a certain shape.

Definition

The colimit of a $\mathcal{V}$-functor $D : A \to M$ weighted by a profunctor $W : B \otimes A^{op} \to \mathcal{V}$ is a $\mathcal{V}$-functor $W \star D : B \to M$ with a $\mathcal{V}$-natural isomorphism

$$M((W \star D)-, -) \cong \mathsf{Hom}_{A^{op}}(W(-,-), M(D-,-)).$$

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Everything is profunctors:

- $M((W \star D)-, -) : \ B \otimes M^{op} \to \mathcal{V}$
- $M(D-, -) : \ A \otimes M^{op} \to \mathcal{V}$
- $W : B \otimes A^{op} \to \mathcal{V}$
- $\mathsf{Hom}_{A^{op}} : \ A \otimes A^{op} \to \mathcal{V}$

Why internal category theory?

- Additional structure affects coalgebraic behaviour
- We want to avoid redoing proofs: so we develop an abstract theory of coalgebra that encompasses all of our cases, by instantiating to them.
- We want to do category theory about category theory!
- So we characterize categories of coalgebras as universal objects, where everything happens in the same category.

Formal coalgebra

Results that we would like

Coalgebra objects

- Coalg(F) characterized internally, as special limits.

Functoriality

- For an endofunctor $F : \mathcal{C} \to \mathcal{C}$, we would like the assignment $F \to \mathrm{Coalg}(F)$ to be functorial.
- Many known results are instances of 2-functoriality:
  - transport of adjunctions
  - monoidal structure on coalgebras
  - adequacy of coalgebraic modal logic

Creation of colimits

- General theorem on creation of *weighted* colimits

Formal Coalgebra in 2-Categories

- Coalgebra objects charactarized as inserters[4]: weighted 2-limits of $\bullet \rightrightarrows \bullet$ of the form $A \overset{id_A}{\underset{f}{\rightrightarrows}} A$

[4] **HermidaJacobs97:IndCoindFib**; **BKV19:ExtendingSetFunctors**.

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- For $f : A \to A$ in $\mathcal{C}$, a triple $(C, p : C \to A, \pi : p \implies fp)$ (+UMP).
  - Instantiated to **Cat**:
    - $f : \mathbb{A} \to \mathbb{A}$ is a functor, C is Coalg(F)
    - p : Coalg$(F) \to \mathbb{A}$ projects a coalgebra (X, α) to its carrier X in $\mathbb{A}$
    - $\pi : p \implies Fp$ is a natural transformation with $(\pi_{(X,\alpha)} : X \to FX) = \alpha$.

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- **Pros:**
  - Coalgebra objects are given functorially
  - There is a projection that creates weighted colimits, *provided our category is cartesian closed*.

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Solution

- Move into Fibrant double categories

[4] **HermidaJacobs97:IndCoindFib**; **BKV19:ExtendingSetFunctors**.

Coalgebra objects in double categories

Definition (Coalgebra objects)

A *coalgebra object* of a morphism $f\colon A \to A$ in a double category $\mathcal{C}$ is a triple (C, p, π) consisting of a object C, a morphism $p\colon C \to A$ and a cell $\pi\colon \left[\begin{smallmatrix} U_A & \overset{p}{} & U_B \\ & {\scriptstyle fp} & \end{smallmatrix}\right]$ that is universal

- On morphisms: for all triples $(B, u\colon B \to A, \beta\colon u \Rightarrow fu)$ there is a unique $u^\flat\colon B \to C$ with $pu^\flat = u$ and $\pi u^\flat = \beta$, and
- On cells: for all $(B, u\colon B \to A, \beta\colon u \Rightarrow fu)$, $(B', v\colon B' \to A, \gamma\colon v \Rightarrow fv)$, $R : B \nrightarrow B'$ and $\rho\colon \left[\begin{smallmatrix} R & \overset{u}{} & U_A \\ & {\scriptstyle v} & \end{smallmatrix}\right]$ with $\frac{\beta}{\rho \mid U_F} = \frac{\rho}{\gamma}$, there is a unique cell $\rho^\flat\colon \left[\begin{smallmatrix} R & \overset{u^\flat}{} & U_C \\ & {\scriptstyle v^\flat} & \end{smallmatrix}\right]$ such that $p\rho^\flat = \rho$.

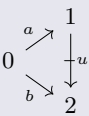
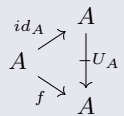
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Equivalently:

Coalgebra objects are weighted limits of shape $\mathbb{I} =$  of the form 

Coalgebra objects, functorially

- Define double category $\mathcal{C}^{\circlearrowleft}$ of
  - endomorphisms (A, f),
  - distributive laws $(h, \delta)\colon f \to g$,
  - endocells $(R, \gamma)\colon f \nrightarrow g$ and
  - distributive law morphisms $\alpha : \begin{bmatrix} & (h_1,\delta_1) & \\ (R,\gamma) & & (S,\rho) \\ & (h_2,\delta_2) & \end{bmatrix}$

$$\begin{array}{c} A \\ {\scriptstyle f}\downarrow \\ A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & A \xrightarrow{h} B \\ {\scriptstyle U_A}\downarrow & \delta & \downarrow{\scriptstyle U_B} \\ A & \xrightarrow{h} & B \xrightarrow{g} B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & A \\ {\scriptstyle R}\downarrow & \gamma & \downarrow{\scriptstyle R} \\ B & \xrightarrow[g]{} & B \end{array} \qquad \begin{array}{ccc} A_1 & \xrightarrow{h_1} & B_1 \\ {\scriptstyle R}\downarrow & \alpha & \downarrow{\scriptstyle S} \\ A_2 & \xrightarrow[h_2]{} & B_2 \end{array}$$

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- forgetful double functor $U\colon \mathcal{C}^{\circlearrowleft} \to \mathcal{C}$ with $U(A, f) = A$
- inclusion double functor $I\colon \mathcal{C} \to \mathcal{C}^{\circlearrowleft}$ with $IA = (A, \mathrm{id}_A)$

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Theorem (Functorial construction of coalgebra objects)

If $\mathcal{C}$ has enough double limits, then I has a right adjoint $\mathrm{CoAlg}\colon \mathcal{C}^{\circlearrowleft} \to \mathcal{C}$ and a $p\colon \mathrm{CoAlg} \to U$.

The double natural transformation p

- $p = U\varepsilon$, where $\varepsilon : I\mathsf{CoAlg} \to \mathsf{Id}$ is the counit of the adjunction $I \dashv \mathsf{CoAlg}$
- Component-wise, p is a projection with $p_{(C,F)} : \mathsf{Coalg}(F) \to C$.
- Applying p to a diagram gives you a new diagram in $U(\mathcal{C}^{\circlearrowleft})$
- p will create our weighted colimits!

Fibrant Double Categories for weighted colimits

Double category $\mathcal{C}$ is fibrant[5] if every niche

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & & \downarrow S \\ A & \xrightarrow[g]{} & B \end{array}$$

can be universally filled

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle S(f,g)}\downarrow & \overline{S}(f,g) & \downarrow S \\ A & \xrightarrow[g]{} & B \end{array}$$

Special case: companion

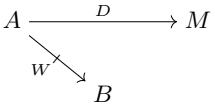
$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ {\scriptstyle A(f,1)}\downarrow & \overline{A}(f,1) & \downarrow U_A \\ A & = & A \end{array}$$

- Give a way to compose diagrams with promorphisms, to characterize weighted colimits.
- Examples: $\mathbf{Cat}$, $\mathcal{V}$-$\mathbf{Cat}$, $\mathbf{Set}$. In $\mathbf{Cat}$, $A(F,1)$ is given by $A(F,1)(b,a) = A(Fb,a)$

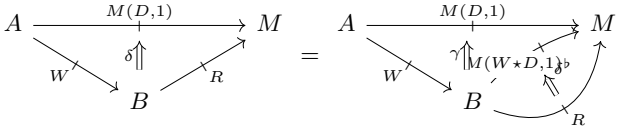
[5] **Shulman08:FramedBicategoriesMonoidal**.

Weighted Colimits in Fibrant Double Categories

Colimit of a diagram D weighted by W



is right Kan extension $(W \star D, \gamma)$ with $W \star D \colon B \to M$ and unique factorisation property[6]



Theorem

The projection $p \colon \mathrm{CoAlg} \to U$ *creates colimits:* $W \star (p_{(M,f)} D)$ *induces* $W \star D$ *for* $D \colon B \to \mathrm{CoAlg}(M, f)$.

[6] **Verity92:EnrichedCategoriesInternal**.

Instantiating to $\mathcal{V}$-Cat: enriched coalgebra objects

Theorem

If $(\mathcal{V}, \times, *)$ is a closed SMC with equalisers, $\mathcal{M}$ a $\mathcal{V}$-category and $F : \mathcal{M} \to \mathcal{M}$ a $\mathcal{V}$-functor, then there is a coalgebra object $\mathrm{CoAlg}(F)$ *for F in the double category $\mathcal{V}$-$\mathbf{Cat}$.*

$$|\mathcal{C}| \;=\; \coprod_{A \in |M|} \mathcal{V}(*, \underline{\mathcal{M}}(A, FA))$$

- We define the Hom-object as equalizer in $\mathcal{V}$:

$$\underline{\mathcal{C}}(a,b) \xrightarrow{\;p_{a,b}\;} \underline{\mathcal{M}}(|a|,|b|) \underset{a^* \circ F_{|a|,|b|}}{\overset{b_*}{\rightrightarrows}} \underline{\mathcal{M}}(|a|, F|b|)$$

- The resulting $\mathcal{V}$-category satisfies the universal property of coalgebra objects.
- From this, we get that $\mathrm{CoAlg}(F)$ is given functorially, and has a projection p that creates weighted colimits.

Wrapping Up

Wrapping up

Summary:

- Brought the 2-categorical characterization up to double categories
- Showed that coalgebra objects are given functorially, and come with a natural projection that creates weighted colimits
- Instantiated to $\mathcal{V}$-$\mathbf{Cat}$

Next goals:

1. Instantiate to double category of fibrations.
2. How does modal logic unfold in enriched fibrations?
3. General theory of coalgebraic modal logic:
   - Dual adjunctions framework for coalgebraic logic
   - Adequacy result: behavioural equivalence is captured by logical equivalence

Thank you for your attention!