A loose Yoneda theory for equipments

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1-Categorical Yoneda

[Yoneda '54] For a 1-category $\mathcal{C}$, object $a \in \mathcal{C}$, and functor $F\colon \mathcal{C} \to \mathbf{Set}$,

$$[\mathcal{C}, \mathbf{Set}](\mathcal{C}(a, -), F) \cong Fa$$
$$\theta \mapsto \theta_a(\mathrm{id}_a)$$

Applications:

- *forward:* tame large amount of data, *e.g. morphisms of polynomial functors*
- *backward:* express properties natively, *e.g. simplices of simplicial sets*

1-Categorical Yoneda, implicitly

[Yoneda '54] For a 1-category $\mathcal{C}$, object $a \in \mathcal{C}$, and functor $F\colon \mathcal{C} \to \mathbf{Set}$,

$$[\mathcal{C}, \mathbf{Set}](\mathcal{C}(a, -), F) \cong Fa$$
$$\theta \mapsto \theta_a(\mathrm{id}_a)$$

- $\mathcal{C}(a, -)\colon \mathcal{C} \to \mathbf{Set}$ is a functor: a ***representable functor***
- $a \mapsto \mathcal{C}(a, -)$ is a functor $\mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathbf{Set}]$: the ***Yoneda embedding***
- naturality w.r.t. $f\colon a \to b$ in $\mathcal{C}$ and $\tau\colon F \Rightarrow G$ in $[\mathcal{C}, \mathbf{Set}]$:

$$\begin{array}{ccc} [\mathcal{C}, \mathbf{Set}](\mathcal{C}(a, -), F) & \xrightarrow{\sim} & Fa \\ {\scriptstyle -\circ\mathcal{C}(f,-)}\downarrow & & \downarrow{\scriptstyle Ff} \\ [\mathcal{C}, \mathbf{Set}](\mathcal{C}(b, -), F) & \xrightarrow{\sim} & Fb \end{array} \qquad \begin{array}{ccc} [\mathcal{C}, \mathbf{Set}](\mathcal{C}(a, -), F) & \xrightarrow{\sim} & Fa \\ {\scriptstyle \tau\circ-}\downarrow & & \downarrow{\scriptstyle \tau_a} \\ [\mathcal{C}, \mathbf{Set}](\mathcal{C}(a, -), G) & \xrightarrow{\sim} & Ga \end{array}$$

1-Categorical Yoneda embedding

For a 1-category $\mathcal{C}$, object $a \in \mathcal{C}$, and object $b \in \mathcal{C}$,

$$\begin{aligned} [\mathcal{C}, \mathbf{Set}](\mathcal{C}(a,-), \mathcal{C}(b,-)) &\cong \mathcal{C}(b,a) \\ \theta &\mapsto \theta_a(\mathsf{id}_a) \\ \mathcal{C}(g,-) &\leftarrow\!\shortmid g \end{aligned}$$

making the Yoneda embedding $\mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathbf{Set}]$ fully faithful.

Applications:

- identification/uniqueness via universal property
- with **density theorem**, colimit completion

Bicategorical Yoneda

Idea: Replace hom-sets with hom-categories, *with unitality & associativity of composition up to isomorphism & functoriality up to isomorphism*

[Street '80] For a bicategory $\underline{\mathbf{B}}$, object $a \in \underline{\mathbf{B}}$, and pseudofunctor $F\colon \underline{\mathbf{B}} \to \underline{\mathbf{Cat}}$,

$$[\underline{\mathbf{B}}, \underline{\mathbf{Cat}}](\underline{\mathbf{B}}(a, -), F) \simeq Fa$$
$$\theta \mapsto \theta_a(\mathrm{id}_a)$$

- $\underline{\mathbf{B}}(a, -)\colon \underline{\mathbf{B}} \to \underline{\mathbf{Cat}}$ is a pseudofunctor: a ***representable pseudofunctor***
- $a \mapsto \underline{\mathbf{B}}(a, -)$ is a pseudofunctor $\underline{\mathbf{B}}^{\mathrm{op}} \to [\underline{\mathbf{B}}, \underline{\mathbf{Cat}}]$: the ***Yoneda embedding***
- pseudonaturality in $a \in \underline{\mathbf{B}}$ & $F \in [\underline{\mathbf{B}}, \underline{\mathbf{Cat}}]$

Bicategorical Yoneda embedding

For a bicategory $\underline{\mathbf{B}}$, object $a \in \underline{\mathbf{B}}$, and object $b \in \underline{\mathbf{B}}$,

$$\begin{aligned} [\underline{\mathbf{B}}, \underline{\mathbf{Cat}}](\underline{\mathbf{B}}(a,-), \underline{\mathbf{B}}(b,-)) &\simeq \underline{\mathbf{B}}(b,a) \\ \theta &\mapsto \theta_a(\mathrm{id}_a) \\ \underline{\mathbf{B}}(g,-) &\leftarrow\!\shortmid g \end{aligned}$$

making the Yoneda embedding $\underline{\mathbf{B}}^{\mathrm{op}} \to [\underline{\mathbf{B}}, \underline{\mathbf{Cat}}]$:

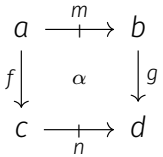
- essentially full on 1-cells
- fully faithful on 2-cells

Application: strictification of bicategories into 2-categories, because $\underline{\mathbf{Cat}}$ is strict

Double categories: tight & loose arrows

In a ***double category***:

- **Tight** arrow composition satisfies unitality & associativity *strictly*
- **Loose** arrow composition satisfies unitality & associativity *up to isomorphism*



Examples.

- $\mathbb{S}\mathrm{pan}(\mathcal{C})$, when $\mathcal{C}$ has pullbacks: **morphisms** & **spans**
- $\mathbb{P}\mathrm{rof}$: **functors** & **profunctors**

Tight Yoneda theory for double categories

Let $\mathbb{D}$ be a double category & $a \in \mathbb{D}$ an object. We write $\mathbb{S}\mathrm{et} \coloneqq \mathbb{S}\mathrm{pan}(\mathbf{Set})$.

[Paré '11] Yoneda theory for **tight** representable *lax* functors:

$$
\begin{aligned}
&\mathbb{D}_{\to}(a,-)\colon \mathbb{D} \to \mathbb{S}\mathrm{et} \\
&\qquad (x \in \mathbb{D}) \mapsto \{\textbf{tight arrows } a \to x \text{ in } \mathbb{D}\} \\
&\qquad (x \xrightarrow{f} y) \mapsto (\mathbb{D}_{\to}(a,x) \xrightarrow{f\circ -} \mathbb{D}_{\to}(a,y)) \\
&\qquad (x \overset{m}{\nrightarrow} y) \mapsto (\mathbb{D}_{\to}(a,x) \xleftarrow{\mathsf{dom}} \{\text{cells } \mathsf{id}_a \Rightarrow m \text{ in } \mathbb{D}\} \xrightarrow{\mathsf{cod}} \mathbb{D}_{\to}(a,y))
\end{aligned}
$$

Yoneda embedding full & faithful on tight arrows, *not* full on loose arrows

Loose Yoneda theory for double categories?

Let $\mathbb{D}$ be a double category & $a \in \mathbb{D}$ an object.

Loose representables?

- $(x \in \mathbb{D}) \mapsto \mathbb{D}(a, x)$, a *category* with:
  - *objects:* **loose** arrows $a \nrightarrow x$ in $\mathbb{D}$
  - *morphisms:* ***globular*** cells bounded by tight identities $a = a$ & $x = x$
- $(x \overset{m}{\nrightarrow} y) \mapsto (\mathbb{D}(a, x) \xrightarrow{-\odot m} \mathbb{D}(a, y))$, a *functor* (*tight* in $\mathbb{P}\mathrm{rof}$)
- $(x \xrightarrow{f} y) \mapsto (\mathbb{D}(a, f) \colon \mathbb{D}(a, x) \nrightarrow \mathbb{D}(a, y))$, a *profunctor* (*loose* in $\mathbb{P}\mathrm{rof}$)

$$(a \overset{p}{\nrightarrow} x, a \overset{q}{\nrightarrow} y) \mapsto \left\{ \text{cells} \begin{array}{ccc} a & \overset{p}{\nrightarrow} & x \\ \| & & \downarrow f \\ a & \underset{q}{\nrightarrow} & y \end{array} \text{ in } \mathbb{D} \right\}$$

Twisted functors

[LMP '26] For double categories $\mathbb{D}$ and $\mathbb{E}$, a ***twisted functor*** $F\colon \mathbb{D} \nrightarrow \mathbb{E}$ sends:

- objects $x \in \mathbb{D}$ to objects $Fx \in \mathbb{E}$
- loose arrows $x \overset{m}{\nrightarrow} y$ in $\mathbb{D}$ to tight arrows $Fx \xrightarrow{Fm} Fy$ in $\mathbb{E}$
- tight arrows $x \xrightarrow{f} y$ in $\mathbb{D}$ to loose arrows $Fx \overset{Ff}{\nrightarrow} Fy$ in $\mathbb{E}$
- ...

Focus on case where:

- identities are preserved up to isocells
- loose-to-tight composition is preserved up to isocell: $F(m \odot m') \cong Fm' \circ Fm$
- F may be lax on tight-to-loose composition: ***comparison cell*** $F_{f,g}\colon Ff \odot Fg \Rightarrow F(g \circ f)$

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- loose arrows $x \overset{m}{\nrightarrow} y$ in $\mathbb{D}$ to tight arrows $Fx \xrightarrow{Fm} Fy$ in $\mathbb{E}$
- tight arrows $x \xrightarrow{f} y$ in $\mathbb{D}$ to loose arrows $Fx \overset{Ff}{\nrightarrow} Fy$ in $\mathbb{E}$
- ...

[LMP '26] There is a 2-category $\underline{\mathbf{Tw}}(\mathbb{D}, \mathbb{E})$ with:

- twisted functors $\mathbb{D} \nrightarrow \mathbb{E}$ as objects
- tight pseudonatural transformations as 1-cells
- modifications as 2-cells

Twisted functors

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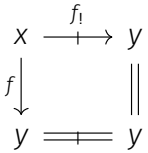
[LNP] There is a double category $\mathbb{T}\mathrm{w}(\mathbb{D}, \mathbb{E})$ with:

- twisted functors $\mathbb{D} \nrightarrow \mathbb{E}$ as objects
- tight (resp. loose) pseudonatural transformations as tight (resp. loose) arrows
- modifications as cells

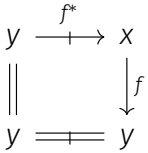
Equipments: companions & conjoints

An ***equipment*** is a **double category** in which every *tight arrow* $f\colon x \to y$ has:

a universal ***companion*** loose arrow $f_!\colon x \nrightarrow y$ with cell



a universal ***conjoint*** loose arrow $f^*\colon y \nrightarrow x$ with cell



Example. In $\mathbb{P}\mathrm{rof}$, the functor $F\colon \mathcal{C} \to \mathcal{D}$ has:

- companion $\mathcal{D}(F(-),=)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$
- conjoint $\mathcal{D}(-,F(=))\colon \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$

Twisted functors between equipments

[LNP] For equipments $\mathbb{D}, \mathbb{E}$, there is an equivalence of 2-categories

$$\underline{\mathsf{Tw}}(\mathbb{D}, \mathbb{E}) \simeq [\underline{\mathsf{Loose}}(\mathbb{D}), \underline{\mathsf{Tight}}(\mathbb{E})]$$

$\underline{\mathsf{Tw}}(\mathbb{D}, \mathbb{E})$, 2-category with:

- twisted functors $\mathbb{D} \nrightarrow \mathbb{E}$ as objects
- tight ps.nat. trans. as 1-cells
- modifications as 2-cells

$[\underline{\mathsf{Loose}}(\mathbb{D}), \underline{\mathsf{Tight}}(\mathbb{E})]$, 2-category with:

- pseudofunctors $\underline{\mathsf{Loose}}(\mathbb{D}) \to \underline{\mathsf{Tight}}(\mathbb{E})$ as objects
- ps.nat. trans. as 1-cells
- modifications as 2-cells

Twisted copresheaves (i.e. functors to $\mathbb{P}\mathrm{rof}$) on equipments

[LNP] For an equipment $\mathbb{D}$, there is an equivalence of 2-categories

$$\underline{\mathsf{Tw}}(\mathbb{D}, \mathbb{P}\mathrm{rof}) \simeq [\underline{\mathsf{Loose}}(\mathbb{D}), \underline{\mathsf{Cat}}]$$

$\underline{\mathsf{Tw}}(\mathbb{D}, \mathbb{P}\mathrm{rof})$, 2-category with:

- twisted functors $\mathbb{D} \nrightarrow \mathbb{P}\mathrm{rof}$ as objects
- tight ps.nat. trans. as 1-cells
- modifications as 2-cells

$[\underline{\mathsf{Loose}}(\mathbb{D}), \underline{\mathsf{Cat}}]$, 2-category with:

- pseudofunctors $\underline{\mathsf{Loose}}(\mathbb{D}) \to \underline{\mathsf{Cat}}$ as objects
- ps.nat. trans. as 1-cells
- modifications as 2-cells

Loose Yoneda for double categories from Yoneda for bicategories

[LNP] For an equipment $\mathbb{D}$, object $a \in \mathbb{D}$, and twisted functor $F\colon \mathbb{D} \nrightarrow \mathbb{P}\mathrm{rof}$,

$$\underline{\mathbf{Tw}}(\mathbb{D}, \mathbb{P}\mathrm{rof})(\mathbb{D}(a,-), F) \simeq Fa$$
$$\theta \mapsto \theta_a(\mathrm{id}_a)$$

- $\mathbb{D}(a,-)\colon \mathbb{D} \nrightarrow \mathbb{P}\mathrm{rof}$ is a twisted functor: a ***twisted representable*** [LMP '26]
- $a \mapsto \mathbb{D}(a,-)$ is a twisted functor $\mathbb{D}^{\mathrm{co}} \nrightarrow \mathbb{T}\mathrm{w}(\mathbb{D}, \mathbb{P}\mathrm{rof})$: the ***twisted Yoneda embedding***, faithful & essentially full in loose-to-tight direction
  - $\mathbb{D}^{\mathrm{co}}$ is $\mathbb{D}$ with all *loose* arrows flipped
- pseudonaturality in $a \in \mathbb{D}$ w.r.t. loose arrows & in $F \in \mathbb{T}\mathrm{w}(\mathbb{D}, \mathbb{P}\mathrm{rof})$ w.r.t. tight natural transformations

Loose Yoneda for double categories

[LNP] For an equipment $\mathbb{D}$, object $a \in \mathbb{D}$, and twisted functor $F\colon \mathbb{D} \nrightarrow \mathbb{P}\mathrm{rof}$,

$$\underline{\mathbf{Tw}}(\mathbb{D}, \mathbb{P}\mathrm{rof})(\mathbb{D}(a,-), F) \simeq Fa$$
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- $a \mapsto \mathbb{D}(a,-)$ is a twisted functor $\mathbb{D}^{\mathrm{co}} \nrightarrow \mathbb{T}\mathrm{w}(\mathbb{D}, \mathbb{P}\mathrm{rof})$: the ***twisted Yoneda embedding***, faithful & essentially full in loose-to-tight direction
  - $\mathbb{D}^{\mathrm{co}}$ is $\mathbb{D}$ with all *loose* arrows flipped
- pseudonaturality in $a \in \mathbb{D}$ w.r.t. loose ***& tight*** arrows & in $F \in \mathbb{T}\mathrm{w}(\mathbb{D}, \mathbb{P}\mathrm{rof})$ w.r.t. tight ***& loose*** natural transformations

Loose Yoneda for double categories

[LNP] For an equipment $\mathbb{D}$, object $a \in \mathbb{D}$, and twisted functor $F\colon \mathbb{D} \nrightarrow \mathbb{P}\mathrm{rof}$,

$$\underline{\mathrm{Tw}}(\mathbb{D}, \mathbb{P}\mathrm{rof})(\mathbb{D}(a, -), F) \simeq Fa$$
$$\theta \mapsto \theta_a(\mathrm{id}_a)$$

- pseudonaturality in $a \in \mathbb{D}$ & in $F \in \mathbb{T}\mathrm{w}(\mathbb{D}, \mathbb{P}\mathrm{rof})$ w.r.t. ***tight*** natural transformations: **bicategorical Yoneda lemma** & **theory of equipments**
- pseudonaturality in $F \in \mathbb{T}\mathrm{w}(\mathbb{D}, \mathbb{P}\mathrm{rof})$ w.r.t. ***loose*** natural transformations: **tight Yoneda lemma for double categories**
- cell component at ***loose*** natural transformation is an equivalence (of profunctors): **new**

All three needed for ***twisted Yoneda embedding***.

Future directions

- Understanding universal properties w.r.t. loose arrows in equipments via loose Yoneda
  - Density theorem for colimits of twisted double diagrams
- Understanding twisted functors from equipments to $\mathbb{P}\mathrm{rof}$ as pseudofunctors to $\underline{\mathbf{Cat}}$, i.e. fibered/indexed categories
- Formal category theory: compact double categories, dualization