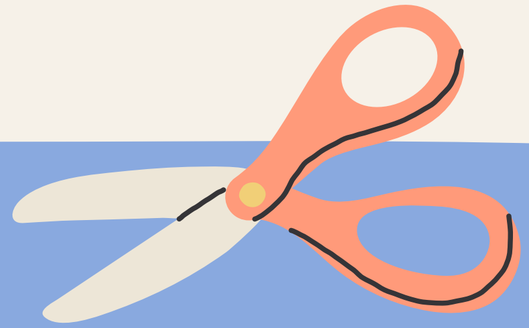
Third Virtual Double Categories Workshop

November 2026



On the cut-and-paste K-theory of manifolds

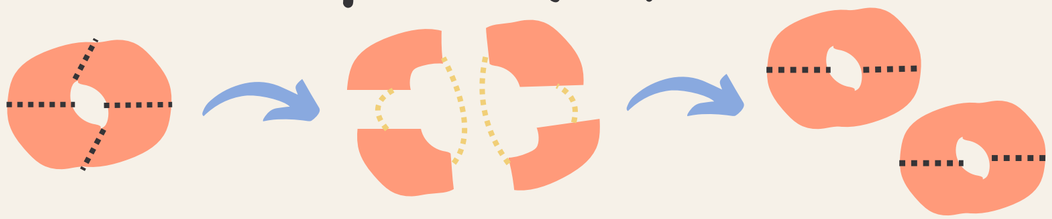
Maxine Calle

University of Bonn

jt. w/ Maru Sarazola

Cut-and-paste manifolds

A "cut-and-paste" (SK) move



Two manifolds are **SK-equivalent** if they are connected by a finite sequence of SK-moves

SK-groups

computed by Karras-Kreck-Neumann-Ossa (1970s)

$$SK_d := \mathbb{Z}\left[\begin{matrix}\text{d-manifolds}\\ \text{w/ boundary}\end{matrix}\right]/\sim$$

where ~ is generated by:

- diffeomorphism,
- $[M \sqcup N] = [M] + [N]$,
- SK-equivalence

Example (d=2)





$SK_2 \cong \mathbb{Z}^2$ (Euler characteristic, boundary components)

Recent idea

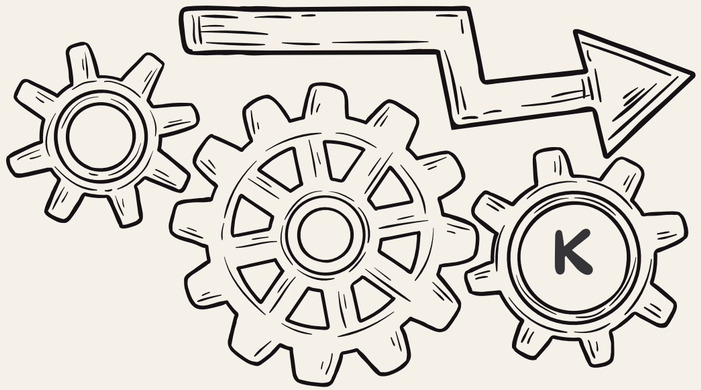
Study SK-equivalences themselves using perspective of higher algebraic K-theory!

Zakharevich '16, Hoekzema-Merling-Murray-Rovi-Semikina '22

Flavors of algebraic K-theory

Categorical structure

with notion of "decomposition"



Space (spectrum)

points = objects
paths = decompositions

1. **Finitely generated projective R-modules** w/ short exact sequences

2. **Finite pointed CW complexes** w/ $A \subseteq X \to X/A$ and weak homotopy equivalences

3. **Finite sets** w/ disjoint union

4. **Smooth d-dimensional manifolds** w/ "cut-and-paste" equivalences

Quillen
Q-construction
exact cats

Waldhausen
S-construction
Waldhausen cats

Segal
group completion
symm. mon. cats

???

Axiom

The outputs of K-theory are interesting

All of these examples can be built from double categories!

Squares K-theory

Nice double category

"decompositions" = squares

$K^{\square}(C) := \Omega|NC|$

where $|NC|$ is the "classifying space"

Space (spectrum)

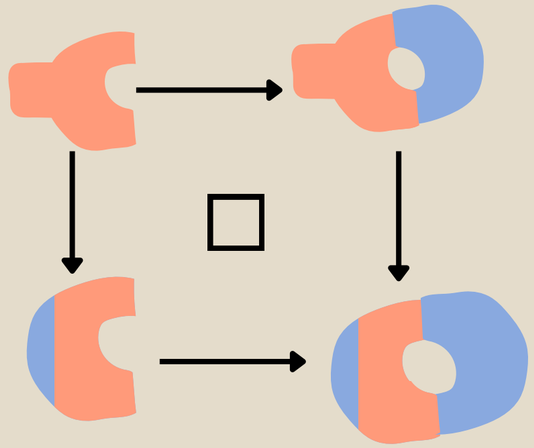
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3. **Finite sets** w/ disjoint union
4. **Smooth d-dimensional manifolds** w/ "cut-and-paste" equivalences

Defn. A **squares category** is a flat double category C with a "doubly initial" object 0. The **squares K-theory** of C is $K^{\square}(C)=\Omega|NC|$.

4. Let $\mathbf{M_d}$ denote the squares category with:

- Obj = d-manifolds with boundary
- $\mathrm{Hom}^h = \mathrm{Hom}^v$ = "SK-embeddings"
- Squares = pushouts
- 0 = empty set



Squares K-theory

Nice double category

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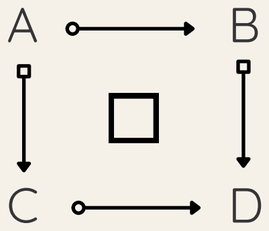
Space (spectrum)

points = objects
paths = decompositions

Thm (HMMRS). The space $K^{\square}(M_d)$ is an infinite loop space (spectrum) and

$$\pi_0 K^{\square}(M_d) \cong SK_d \cong \mathbb{Z}[Ob(M_d)] /\sim$$

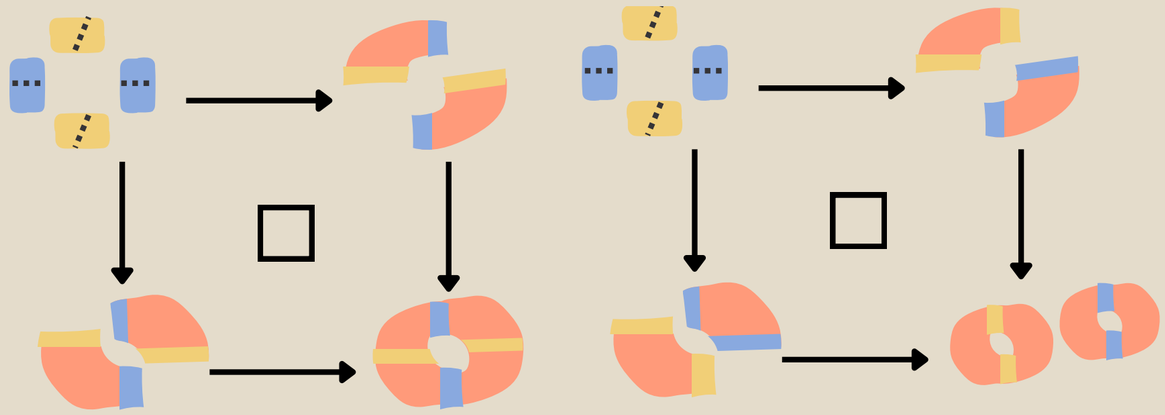
where [D] + [A] ~ [B] + [C] whenever



This result holds more generally (Campbell-Kuijper-Merling-Zakharevich, '25)

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Squares K-theory

Nice double category

"decompositions"
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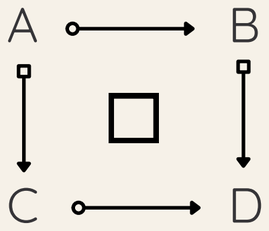
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$\pi_n K^{\square}(C) \cong$???

develop new techniques

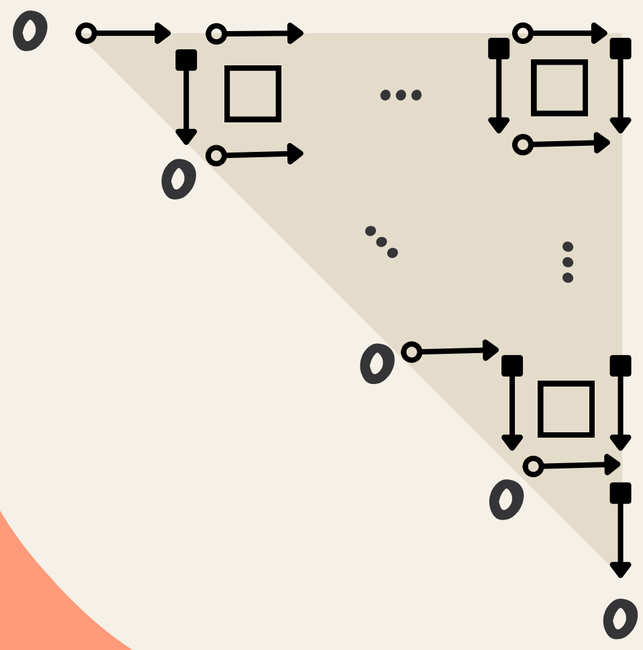
develop new models

The S-construction

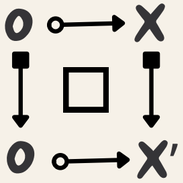
(Waldhausen, Bergner-Osorno-Ozornova-Rovelli-Scheimbauer, C.-Sarazola)

Let C be a flat double category with a distinguished object 0 which is horizontally initial and vertically terminal. An object of S_nC looks like:

(nxn)-staircases



The morphisms are vertical natural transformations, pointwise valued in $X \to X'$ such that



simplicial category SC

If C is a **proto-Waldhausen** squares category, then $|C| \simeq |NSC|$, so $K^{\square}(C) \simeq \Omega|NSC|$

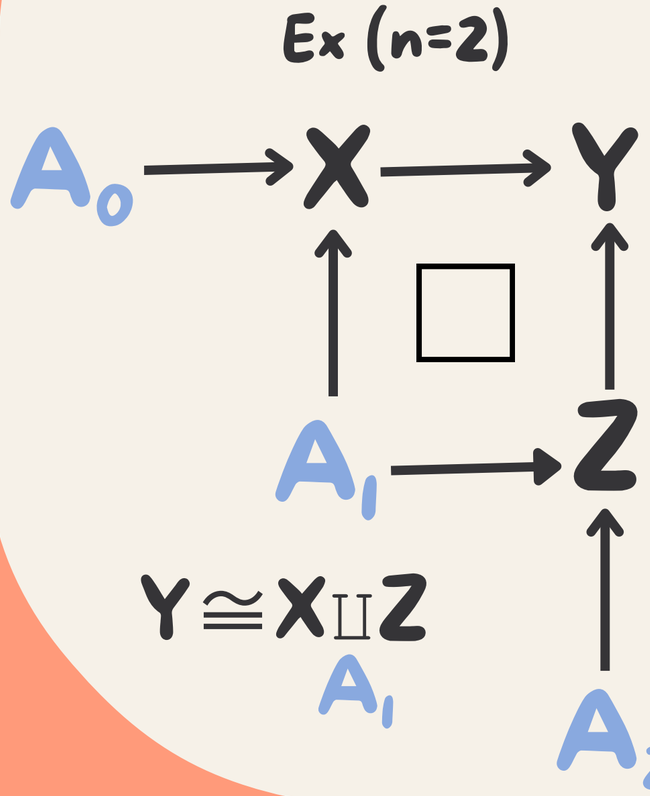
Thm Waldhausen-Thomason, C.-Sarazola

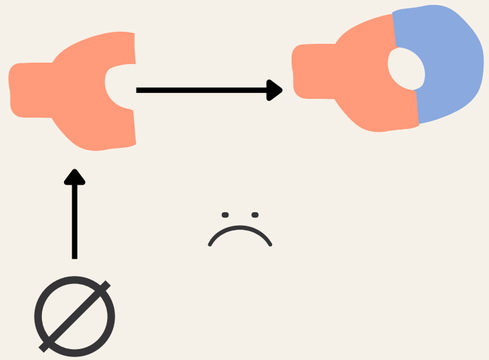
proto-Waldhausen: roughly means squares ≈ pushouts AND "pushouts always exist"

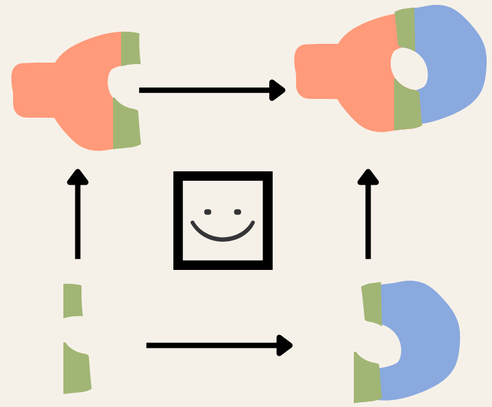
for SK-manifolds: even though squares ≈ pushouts, not proto-Waldhausen :(

The augmented S-construction

Question: Given an SK-embedding N → M, how can we complete it to a square?







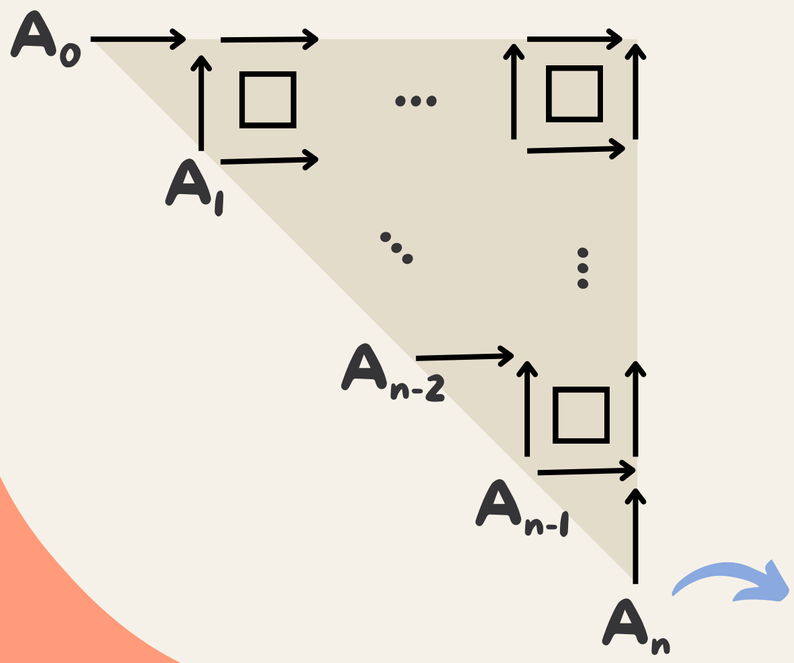
Each $A_i = \Sigma_i \times [0,1]$ cylinder
plus condition on the cospans

The augmented S-construction

Define $K(M_d) := \Omega|S^{aug}M_d|$

An object of $S^{aug}M_d$ looks like:

(nxn)-staircases



Each $A_i = \Sigma_i \times [0,1]$ cylinder
plus condition on the cospans

Thm
C.-Sarazola

The space $K(M_d)$ is amenable to known techniques in higher algebraic K-theory:

- it's an infinite loop space (spectrum) and $\pi_0 K(M_d) \cong \pi_0 K^{\square}(M_d) \cong SK_d$
- description of $\pi_1 K(M_d)$ as H_1(simplicial set)
- $K(M_d) \simeq \Omega|fCob_d|$ for a weak 2-category of "cobordisms with free boundary"

Is $\pi_n K(M_d) \cong \pi_n K^{\square}(M_d)$ for all n?

Takeaways

Results about classifying spaces of double categories could have interesting consequences in squares K-theory

Some double categories (like SK-manifolds) have other models for the squares K-theory space where we can apply known tools

The K-theory of SK-manifolds encodes SK-manifold invariants, with interesting connections to cobordisms

Thanks!