Representability of Generalized Multicategories

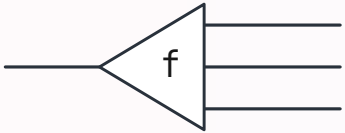
Matt Earnshaw
University of Tartu, Estonia

work in progress with Nathanael Arkor

Third Virtual Workshop on Double Categories
10th November 2026

Multicategories and generalized multicategories

A multicategory X has:



$$X : X_0 \nrightarrow \coprod_{n \in \mathbb{N}} X_0^n$$

hom sets, $f \in X(A_1, \ldots, A_n; B)$

A generalized multicategory $\mathcal{X}$ has:



$$\mathcal{X} : \mathcal{X}_0 \nrightarrow T\mathcal{X}_0$$

$\ldots$

Examples: symmetric multicategories, functors over a base, virtual double categories, topological spaces

Burroni, 1971 – *T-catégories (catégories dans un triple)*

Hermida, 2001 – *From Coherent Structure to Universal Properties*

Leinster, 2004 – *Higher operads, higher categories*

Cruttwell and Shulman, 2010 – *A Unified Framework for Generalized Multicategories*

Representability

A representable multicategory is one in which multimorphisms factor uniquely through unary morphisms,

$$\frac{A_1, ..., A_n \longrightarrow B}{\otimes(A_1, ..., A_n) \longrightarrow B}$$

They are equivalent to monoidal categories, but $\otimes$ acquires a *universal property*.

Hermida, 2000 – *Representable Multicategories*

The double categorical framework for generalized multicategories is well suited to the study of representability.

Cruttwell and Shulman, 2010 – *A Unified Framework for Generalized Multicategories*

Henceforth, (T, η, μ) will be a normal monad on a virtual equipment.

A generalized multicategory is a loose T-coalgebra. *Representability* characterizes those loose T-coalgebras which arise from (tight) T-algebras.

Representability for some T-multicategories

T	T-**multicategory**	**Pseudo** T**-algebra** $\simeq$ **Representable** T**-multicategory**
Free monoidal category monad	Multicategory	Monoidal category
Free symmetric monoidal category monad	Symmetric multicategory	Symmetric monoidal category
Free presheaf monad on $\mathbb{D}\mathrm{ist}/\mathrm{ob}(S)$	Functor over S	Pseudofunctor $S^{\mathrm{op}} \to \mathrm{Cat}$
$\mathbb{M}\mathrm{od}$(ultrafilter)	Topological space	Ordered compact Hausdorff space
Prof(free category)	Virtual double category	Pseudo double category

Hermida, 2001 – *From Coherent Structure to Universal Properties*

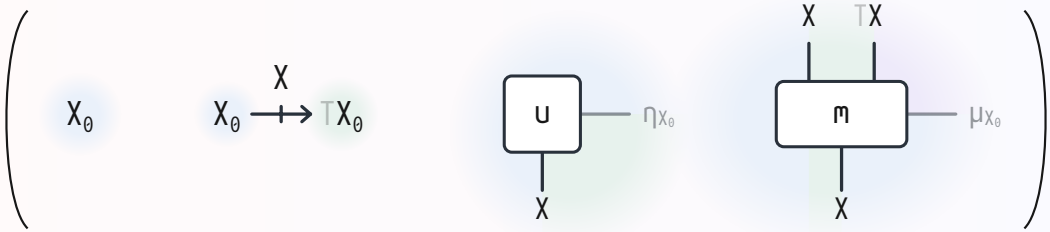
Cruttwell and Shulman, 2010 – *A Unified Framework for Generalized Multicategories*

T-multicategories

unit cell u is *cartesian*

Definition. A T-multicategory is a normalized loose monad in $\mathbb{L}\mathrm{Kl}(\mathbb{X}, T)$.

Cruttwell and Shulman, 2010 – *A Unified Framework for Generalized Multicategories*

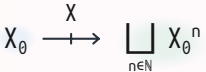


Definition. $T\text{-}\mathbb{M}\mathrm{ult} := \mathbb{M}\mathrm{od}_n(\mathbb{L}\mathrm{Kl}(\mathbb{X}, T))$

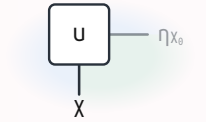
? *Why are loose T-coalgebras the same as loose monads in loose Kleisli?*

For $T : \mathbb{D}\mathrm{ist} \longrightarrow \mathbb{D}\mathrm{ist}$ the free monoidal category monad...

a category X_0



a profunctor, giving hom-sets $X((A_1, \ldots, A_n); B)$



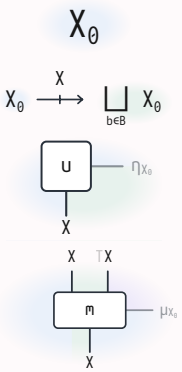
$u_x : X_0(A; B) \longrightarrow X((A); B)$

$m_{(\vec{A}_1,\ldots,\vec{A}_n),C} : X(\vec{B}; C) \times \prod_{i=1}^{n} X(\vec{A}_i; B_i) \longrightarrow X(\vec{A}_1 \cdots \vec{A}_n; C).$

and normalized means $X_0(A; B) \cong X((A); B)$.

> T-multicategories are multicategories.

For a monoid B, and $T = \coprod_{b \in B}(-) : \mathbb{D}\mathrm{ist} \longrightarrow \mathbb{D}\mathrm{ist}$...



a category X_0

hom-sets $X_b(A; B)$ for $b \in B$ and $A, B \in X_0$
("morphisms over b")

$u_{x,y} : X_0(A; B) \longrightarrow X_e(A; B)$,

$X_c(B; C) \times X_b(A; B) \xrightarrow{m_{A,C}} X_{cb}(A; C)$,
$(g, f) \longmapsto g \circ f$

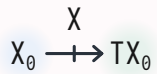
and normalized means that $u_{x,y}$ is an isomorphism.

> T-multicategories are functors over the monoid B.

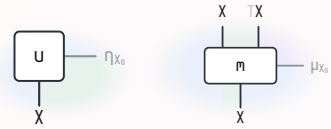
For $T = \mathbb{M}\mathrm{od}(\text{free category}) : \mathbb{M}\mathrm{od}(\mathbb{S}\mathrm{pan}(\text{Graph})) \longrightarrow \mathbb{M}\mathrm{od}(\mathbb{S}\mathrm{pan}(\text{Graph}))$...



internal category with graph of objects and tight morphisms X_{00}, and loose morphisms and unary cells X_{01}



internal distributor
$X_0(s\vec{p}; sq) \leftarrow X_1((\overset{p_1}{\nrightarrow}, \ldots \overset{p_n}{\nrightarrow}); \overset{q}{\nrightarrow}) \longrightarrow X_0(t\vec{p}; tq)$

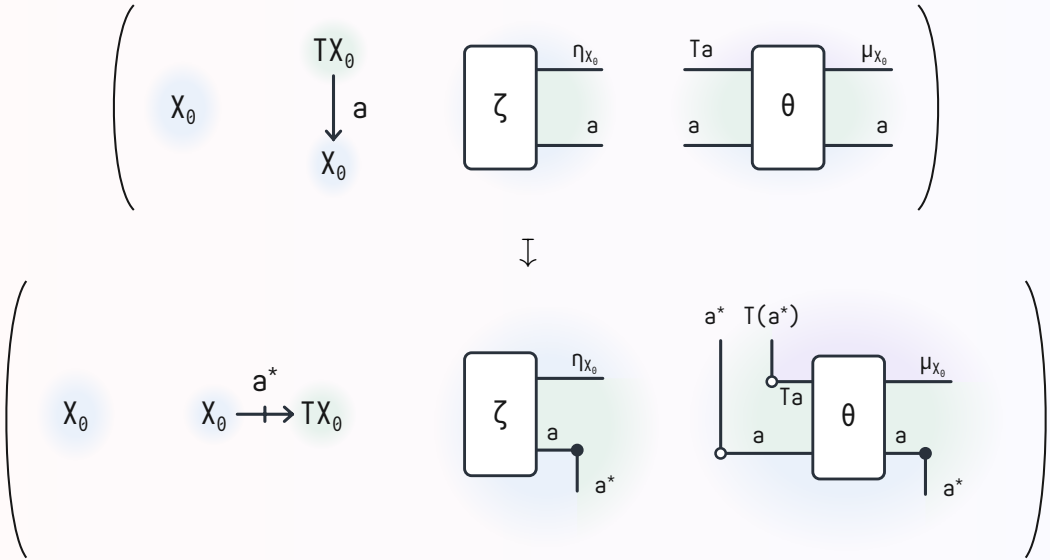


inclusion of unary cells and VDC composition.

and normalized means that unary cells agree $X_{01}((p); q) \cong X_1((p); q)$.

> T-multicategories are virtual double categories.

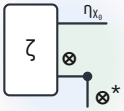
Underlying T-multicategory of a pseudo T-algebra



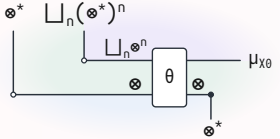
Example: underlying multicategory of a monoidal category

X_0 — the underlying category of $(X_0, \otimes : TX_0 \longrightarrow X_0, \zeta, \theta)$

$X_0 \xrightarrow{\otimes^*} \coprod_{n\in\mathbb{N}} X_0{}^n$ — $\otimes^*(A_1, ..., A_n; B) = X_0(\otimes(A_1, ..., A_n); B)$,



$u_{A,B} : X_0(A; B) \longrightarrow X_0(\bigotimes(A); B)$,
$f \longmapsto f \circ \zeta_A$, where $\zeta_A : \bigotimes(A) \xrightarrow{\sim} A$



$$X_0(\bigotimes \vec{A}, C) \times \prod_i X_0(\bigotimes \vec{A}_i, B_i) \xrightarrow{m_{\vec{A},C}} X_0(\bigotimes(\vec{A}_1 \cdots \vec{A}_n), C)$$
$$(g, (f_i)_i) \longmapsto g \circ (\bigotimes_i f_i) \circ \theta,$$

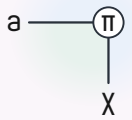
Representability (part one)

Underlying multicategories are *representable*. This means firstly that...

> 1. For every list of objects $A_1, ..., A_n$, there is an object $a(A_1, ..., A_n)$ and multimorphism $\pi_{\vec{A}} : A_1, ..., A_n \longrightarrow a(A_1, ..., A_n)$ such that composition induces a natural bijection
>
> $$\mathbb{X}(a(A_1, ..., A_n); B) \cong \mathbb{X}(A_1, ..., A_n; B).$$
>
> Hermida, 2000 – *Representable Multicategories*

In the virtual equipment on which T acts, we instead say:

> 1. There exists a tight morphism $a : TX_0 \longrightarrow X_0$ and an cocartesian cell π witnessing $X : X_0 \nrightarrow TX_0$ as the conjoint of a.



Representability (part two)

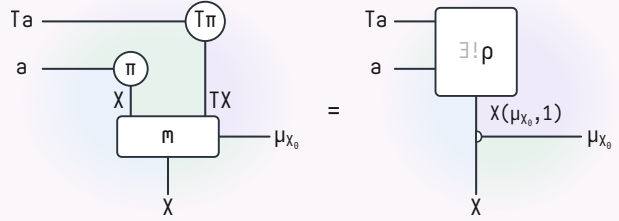
and secondly that...

2. For every list of lists $\vec{x}_1, \ldots, \vec{x}_n$, the composite $\pi_{(a\vec{x}_1,\ldots,a\vec{x}_n)} \circ (\pi_{\vec{x}_1}, \ldots, \pi_{\vec{x}_n})$ is again universal: precomposition with it induces a bijection (natural in y)

$$\mathbb{X}_0\big(a(a\vec{x}_1, \ldots, a\vec{x}_n); y\big) \xrightarrow{\sim} \mathbb{X}\big((\vec{x}_1 \cdots \vec{x}_n); y\big),$$

In the virtual equipment on which T acts, we instead say:

2. The unique cell ρ obtained by factoring the cell on the left is cocartesian.



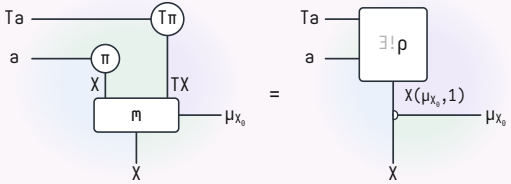
Representable T-multicategories

Definition. A T-multicategory $(X_0, X : X_0 \nrightarrow TX_0, X_u, X_m)$ is *representable* if

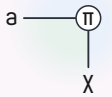
1. There exists a tight morphism $a : TX_0 \longrightarrow X_0$ and a cocartesian cell π witnessing $X : X_0 \nrightarrow TX_0$ as the conjoint of a.



2. and the unique cell ρ obtained by factoring the cell on the left is cocartesian

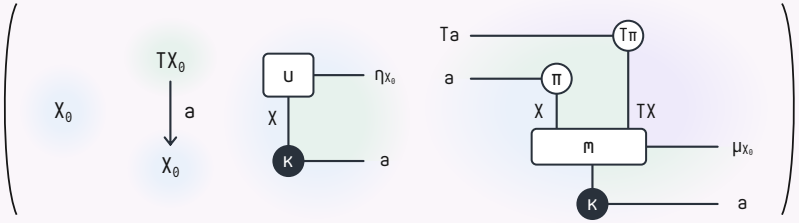


Representable T-multicategories are those underlying pseudo T-algebras

Theorem. Given a T-multicategory X and representation 

,

The following is a pseudo-T-algebra whose underlying T-multicategory is isomorphic to X



Corollary. There is an equivalence of VDCs, $\mathbb{P}\text{s-T-Alg}_{\text{lax}} \simeq \mathbb{R}\text{ep}$.

Free T-algebra on a T-multicategory

Theorem. Taking underlying T-multicategories extends to a fully faithful functor of VDCs

$$U : \mathbb{X}^{\mathcal{T}} \longrightarrow \mathbb{M}\mathrm{od}(\mathbb{L}\mathrm{Kl}(\mathbb{X}, T)).$$

We show that this has a left adjoint by building it from manageable pieces:

1. a left adjoint to the forgetful $\mathbb{X}^{\mathcal{T}} \longrightarrow \mathbb{L}\mathrm{Kl}(\mathbb{X}, T)$: *a VDC analogue of the Kl-EM functor.*
2. a left adjoint to $\mathbb{X} \longrightarrow \mathbb{M}\mathrm{od}(\mathbb{X})$: *collapse of loose monads.*

Loose analogue of the Kl-EM functor

Definition. A monad T on a virtual equipment has *loose Kleisli extensions* if for every loose morphism $N : X \nrightarrow TY$, the composite $TN \odot \mu_{Y*}$ exists.

Theorem. For a monad T with loose Kleisli extensions (and loosely strong multiplication), there is a VDC functor $E : \mathbb{L}\mathrm{Kl}(\mathbb{X}, T) \longrightarrow \mathbb{X}^T$

$$\begin{array}{rcl} X & \longmapsto & (TX, \mu_X) \\ f : X \longrightarrow Y & \longmapsto & Tf \\ N : X \nrightarrow TY & \longmapsto & TN \odot \mu_{Y*} : TX \nrightarrow TY \end{array}$$

and this is left adjoint to the forgetful $V : \mathbb{X}^T \longrightarrow \mathbb{L}\mathrm{Kl}(\mathbb{X}, T)$.

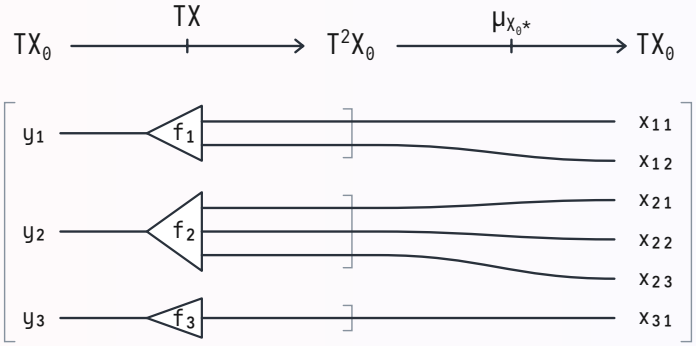
(?) *Why does E act tightly like $X \longrightarrow X^T$ but loosely like $X_T \longrightarrow X^T$?*

A loose monad in T-algebras

Taking $\mathbb{M}\mathrm{od}$, we obtain a VDC functor

$$\mathbb{M}\mathrm{od}(E) : \mathbb{M}\mathrm{od}(\mathbb{L}\mathrm{Kl}(\mathbb{X}, T)) \longrightarrow \mathbb{M}\mathrm{od}(\mathbb{X}^T),$$

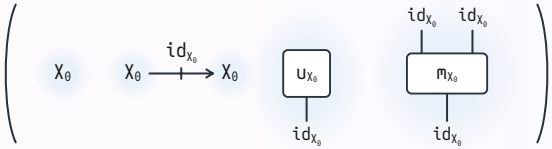
sending a T-multicategory to the free algebra (TX_0, μ_{X_0}) on its carrier, equipped, in particular, with a loose endomorphism $(TX_0, \mu_{X_0}) \xrightarrow{TX \odot \mu_{X_0 *}} (TX_0, \mu_{X_0})$,



Collapsing the loose monad to a T-algebra

For $\mathbb{X}$ a normal virtual double category, we have an inclusion into $\mathbb{M}\mathrm{od}(\mathbb{X})$

$$(-)(1,1) : \mathbb{X} \longrightarrow \mathbb{M}\mathrm{od}(\mathbb{X}) : X_0 \mapsto$$



Definition. *Collapse* is a strict left adjoint retraction

$$\mathbb{X} \underset{(-)(1,1)}{\overset{\ll - \gg}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbb{M}\mathrm{od}(\mathbb{X})$$

with cartesian unit components.

Schultz, 2015 – *Regular and exact (virtual) double categories*
Arkor and McDermott, 2025 – *The nerve theorem for relative monads*

Collapse in $\mathbb{D}\mathrm{ist}$: Kleisli category of a promonad

$$\begin{aligned} \ll - \gg : \mathbb{M}\mathrm{od}(\mathbb{D}\mathrm{ist}) &\longrightarrow \mathbb{D}\mathrm{ist} \\ (\mathcal{C}, \mathcal{T} : \mathcal{C} \nrightarrow \mathcal{C}, u, m) &\longmapsto \ll \mathcal{T} \gg \end{aligned}$$

where $\ll \mathcal{T} \gg$ is the category with

objects	those of $\mathbb{C}$
morphisms	$\ll \mathcal{T} \gg (x; y) = \mathcal{T}(x; y)$
identities	$u_{x,x}(1_x)$
composition	$(g, f) \longmapsto m(g, f)$

Unit of $\ll - \gg \vdash (-)(1, 1)$ gives the inclusion $\mathcal{C} \longrightarrow \ll \mathcal{T} \gg$

Free T-algebra on a T-multicategory

We obtain a composite adjunction in **VDbl**:

$$\mathbb{X}^T \underset{(-)(1,1)}{\overset{\ll - \gg}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbb{M}\mathrm{od}(\mathbb{X}^T) \underset{\mathbb{M}\mathrm{od}(V)}{\overset{\mathbb{M}\mathrm{od}(E)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbb{M}\mathrm{od}(\mathbb{L}\mathrm{Kl}(\mathbb{X}, T))$$

with *fully faithful* unit,

A tight morphism f is *fully faithful* if $\mathtt{f}$——$\mathtt{f}$ is cartesian.

and *cofully faithful* counit.

A tight morphism f is *cofully faithful* if $\mathtt{f}$——$\mathtt{f}$ is cocartesian.

Properties of the adjunction

$$\mathbb{X}^T \underset{(-)(1,1)}{\overset{\ll - \gg}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbb{M}\mathrm{od}(\mathbb{X}^T) \underset{\mathbb{M}\mathrm{od}(V)}{\overset{\mathbb{M}\mathrm{od}(E)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbb{M}\mathrm{od}(\mathbb{L}\mathrm{Kl}(\mathbb{X}, T))$$

The composite adjunction, $F \dashv U$, should have the properties of being:

Lax-idempotent representability is *property-like structure*.

Monadic $\mathbb{X}^T \simeq \mathbb{M}\mathrm{od}_n(\mathbb{L}\mathrm{Kl}(\mathbb{X}, T))^{UF}$

"*monoidal categories are multicategories with UF-algebra structure.*"

Comonadic* $\mathbb{M}\mathrm{od}_n(\mathbb{L}\mathrm{Kl}(\mathbb{X}, T)) \simeq (\mathbb{X}^T)^{FU}$

"*multicategories are monoidal categories with FU-coalgebra structure.*"

Consequence: T-multicategory X is representable iff $\eta_X : X \to UFX$ has a left adjoint.