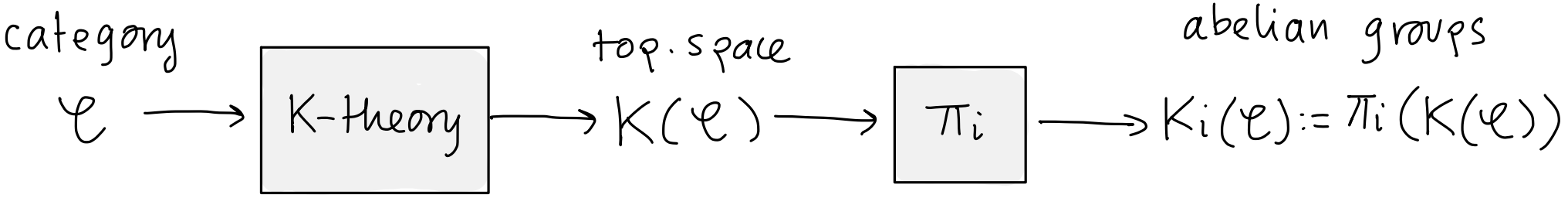
K-THEORY OF MONOIDS VIA DOUBLE CATEGORIES

joint w/ Brandon Shapiro and Inna Zakharevich

arXiv/2611.05265

WHAT IS ALGEBRAIC K-THEORY ABOUT?



IDEA: This is some universal procedure built out of ways to decompose the objects in $\mathcal{C}$ into smaller pieces.

EX: G finite group, $\mathcal{C} = G\text{-sets}$: obj = finite pointed G-sets
mor = pointed G-equivariant maps

Here we have short exact sequences

$$X \hookrightarrow Y \twoheadrightarrow Y/X$$

that decompose Y as $Y \cong X \amalg Y/X$.

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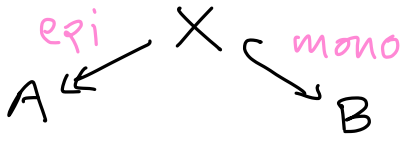
How do we construct the space $K(\mathcal{C})$?

STEP 1

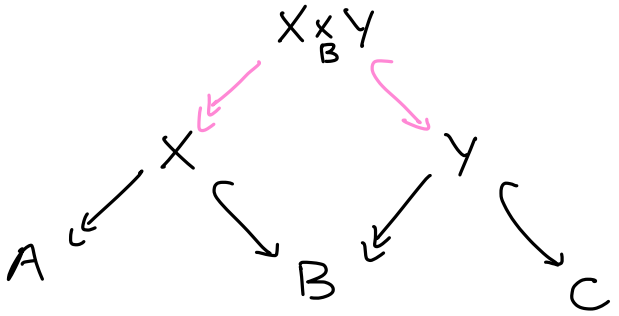
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obj = obj $\mathcal{C}$

mor = (iso classes of) spans



composition: pullback



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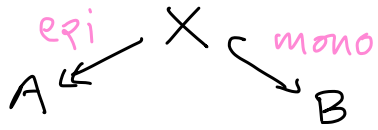
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STEP 1

Define a cat $Q\mathcal{C}$:

obj = obj $\mathcal{C}$

mor = (iso classes of) spans



STEP 2

Get simplicial set

$N Q\mathcal{C}$

STEP 3

Get top. space

$\Omega |N Q\mathcal{C}|$

$\|$

$K(\mathcal{C})$

$$K_0(\mathcal{C}) = \pi_0(K(\mathcal{C})) = \mathbb{Z}[\text{obj } \mathcal{C}] \Big/ [Y] = [X] + [Y/X] \text{ for each } X \hookrightarrow Y \twoheadrightarrow Y/X$$

A FORMULA FOR THE K-THEORY OF G-SETS

G finite abelian group, $K(G\text{-Sets}) \simeq \coprod_{H \leq G} \Sigma^\infty_+ B(G/H)$

Where is this formula coming from?

1. All short exact sequences split:

$X \hookrightarrow Y \twoheadrightarrow Y/X \rightsquigarrow Y \cong X \amalg Y/X$

↳ G-iso!

Suffices to understand: how G-sets decompose as coproducts

2. Every G-set is the coproduct of its orbits: $X \cong \coprod O_G(x)$

- Orbit-Stabilizer thm: $O_G(x) \cong G/\mathrm{Stab}(x)$

$K(G\text{-Sets})$

$\simeq (G\text{-sets}^{iso}, \amalg)^{gp}$

$\simeq \left(\prod_{H \leq G} \langle G/H, \amalg \rangle^{iso}\right)^{gp}$

equiv. of cats

$\Rightarrow \prod_{H \leq G} \langle G/H, \amalg \rangle^{iso} \longrightarrow G\text{-sets}^{iso}$

$(X_i) \longmapsto \coprod X_i$

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$$
\begin{aligned}
K(G\text{-Sets}) &\simeq (G\text{-sets}^{\mathrm{iso}}, \amalg)^{gp} \\
&\simeq \Big(\prod_{H \leq G} \langle G/H, \amalg \rangle^{\mathrm{iso}}\Big)^{gp} \\
&\simeq \coprod_{H \leq G} \Sigma^\infty_+ B \underbrace{\mathrm{Aut}(G/H)}_{\cong G/H}
\end{aligned}
$$

CHANGING THE GROUP G BY A MONOID A

Finite abelian monoid A, we have (pointed) A-sets

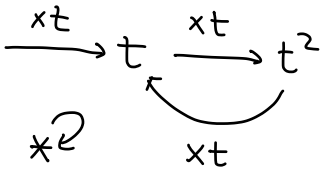
unit $1 \in A$
assoc.
no inverses

GOAL: compute $K(A\text{-sets})$.

Try to follow previous strategy? Problems galore!!

PROBLEM #1: orbits are kinda weird.

$A = \{1, t, t^2 \mid t^3 = t\}$



$O_A(t) = \{t, t^2\} \subsetneq O_A(1) = A$

- No longer true that orbits partition an A-set
- Very hard (if at all possible??) to classify orbits.

THE PROBLEMS W/ ORBITS CAN BE SOLVED!

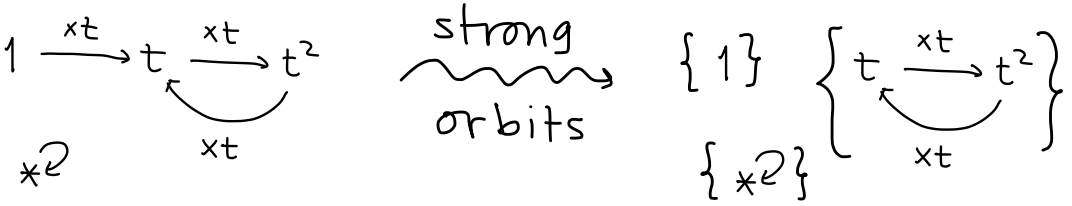
By redefining what a meaningful "orbit" is.

Defn For an A-set X and $x \in X$, the strong orbit of x is

$$SO_A(x) := \{a.x \mid \exists\, b \in A \text{ with } ba.x = x\} = R_x.x$$

↳ the "x-reversible" elements in A

[Weissmann]



Strong orbits are still a little odd: not invariant under A-action

Orbit-Stabilizer Thm [S-Shapiro-Zakharevich]

For any invariant strong orbit,

$$SO_G(x) = R_x.x \cong R_x/\mathrm{Stab}(x)$$

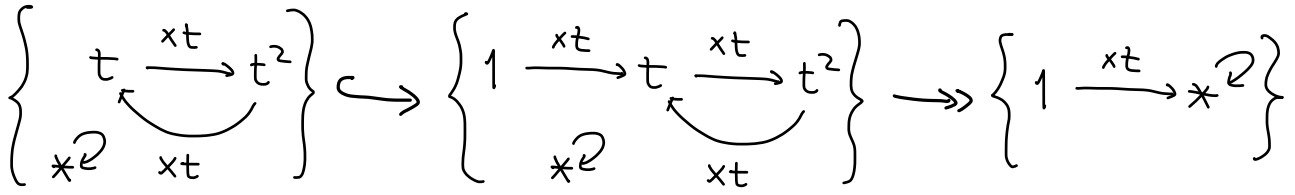
G-sets: for any orbit,

$$O_G(x) = G.x \cong G/\mathrm{Stab}(x)$$

BACK TO THE GENERAL STRATEGY

PROBLEM #2: short exact sequences no longer split

$A = \{1, t, t^2 \mid t^3 = t\}$



This cannot be fixed!

Very bad news for our formula: we relied on this for the crucial simplification in the first step

$$K(G\text{-Sets}) \simeq (G\text{-sets}^{iso}, \amalg)^{gp} \simeq \Big(\prod_{H \leq G} \langle G/H, \amalg \rangle^{iso}\Big)^{gp} \simeq \coprod_{H \leq G} \Sigma^\infty_+ B\underbrace{Aut(G/H)}_{\cong G/H}$$

How do we simplify $K(A\text{-sets})$?

double categories:



K-THEORY VIA DOUBLE CATEGORIES

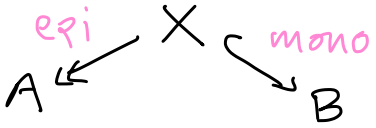
Recall: to do K-theory we started w/ a cat $\mathcal{C}$, and did

STEP 1

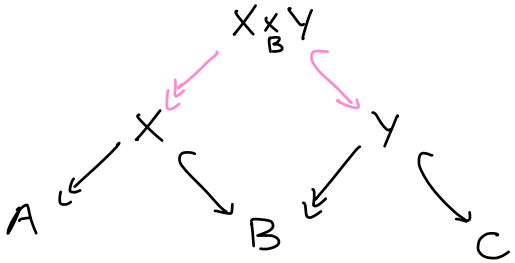
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composition: pullback



STEP 2

Get simplicial set

$NQ\mathcal{C}$

STEP 3

Get top. space

$\Omega|NQ\mathcal{C}| = K(\mathcal{C})$

K-THEORY VIA DOUBLE CATEGORIES

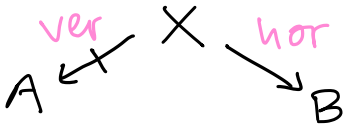
Start w/ a double cat $\mathbb{C}$, and

STEP 1

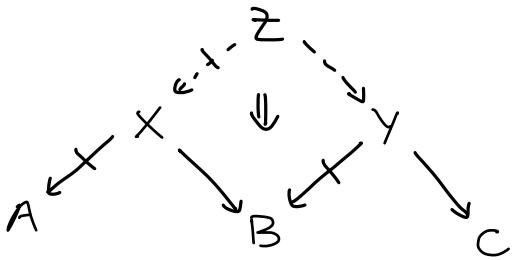
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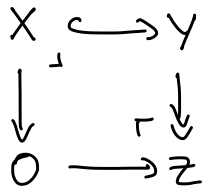
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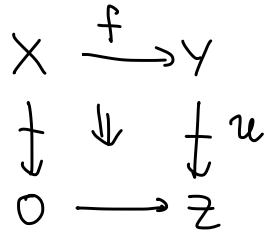
short exact sequences $X \hookrightarrow Y \twoheadrightarrow Z$

$\Updownarrow$

bicartesian square



Now becomes: a square in $\mathbb{C}$



K-THEORY VIA DOUBLE CATEGORIES

One needs certain conditions/axioms on $\mathbb{C}$ for this to work

"CGW-categories" [Campbell–Zakharevich]

What do we gain?

previously only available for abelian cats!

NEW COMPUTATIONAL TOOLS:

Devissage Thm [Campbell–Zakharevich] $\mathbb{A} \subseteq \mathbb{C}$ CGW-subcategory s.t. every obj. $C \in \mathbb{C}$ admits

$$C = C_0 \xrightarrow{u_0} C_1 \xrightarrow{u_1} C_2 \longrightarrow \cdots \longrightarrow C_{n-1} \xrightarrow{u_{n-1}} C_n = 0$$

with

$$\begin{array}{ccc} A_i & \longrightarrow & C_i \\ \downarrow & \Downarrow & \downarrow u_i \\ 0 & \longrightarrow & C_{i+1} \end{array}, \quad A_i \in \mathbb{A}$$

$$\Rightarrow K(\mathbb{A}) \simeq K(\mathbb{C}).$$

Devissage Thm [Campbell-Zakharevich] $\mathbb{A} \subseteq \mathbb{C}$ full CGW-subcategory s.t. every obj. $C \in \mathbb{C}$ admits $C = C_0 \overset{u_0}{\twoheadrightarrow} C_1 \overset{u_1}{\twoheadrightarrow} C_2 \twoheadrightarrow \cdots \twoheadrightarrow C_{n-1} \overset{u_{n-1}}{\twoheadrightarrow} C_n = 0$ with $\begin{array}{ccc} A_i & \longrightarrow & C_i \\ \downarrow & \Downarrow & \downarrow u_i \\ 0 & \longrightarrow & C_{i+1} \end{array}$, $A_i \in \mathbb{A}$

$\Rightarrow K(\mathbb{A}) \simeq K(\mathbb{C})$.

In our setting:

$\mathbb{C} = A\text{-sets}$, i.e. the double cat w/

obj = pointed A-sets

$\longrightarrow$ = inclusions $X \hookrightarrow Y$

$\downarrow$ = quotient maps $\begin{array}{c} Y \\ \downarrow \\ Y/X \end{array}$

$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \downarrow & \Downarrow & \downarrow \\ \cdot & \longrightarrow & \cdot \end{array}$ = bicartesian.

$\mathbb{A} = A\text{-sets}^{inv} \subseteq A\text{-sets}$

the double subcat on the A-sets for which every strong orbit is invariant.

Prop Every A-set X has at least one invariant strong orbit $X_I \subseteq X$

Then, for $X \in A\text{-sets}$, do

$X = X_0 \twoheadrightarrow X_0/(X_0)_I = X_1 \twoheadrightarrow X_1/(X_1)_I = X_2 \twoheadrightarrow \cdots \twoheadrightarrow \{*\}$

with $\begin{array}{ccc} (X_k)_I & \longrightarrow & X_k \\ \downarrow & \Downarrow & \downarrow \\ \{*\} & \longrightarrow & X_k/(X_k)_I \end{array}$, $(X_k)_I \in \mathbb{A}$

$\Rightarrow$ $K(A\text{-sets}) \simeq K(A\text{-sets}^{inv})$ and we've reduced our problem to understanding invariant strong orbits.

But we do understand those!

Orbit-Stabilizer Thm [S-Shapiro-Zakharevich]

For any invariant strong orbit, $SO_G(x) \cong R_x/Stab(x)$

Thm [S-S-Z] There is an equivalence of double cats

$$\prod_{S \leq R \leq A} \langle R/S, \amalg \rangle \longrightarrow A\text{-sets}^{inv}$$
$$(X_i) \longmapsto \coprod X_i$$

Thm [S-S-Z] There is an equivalence of double cats

$$\prod_{S \leq R \leq A} \langle R/S, \amalg \rangle \longrightarrow A\text{-sets}^{inv}$$
$$(X_i) \longmapsto \coprod X_i$$

KEY REMARK: this **only** works because we have double cats.

All that inclusions ($\hookrightarrow$) and quotients ($\twoheadrightarrow$) can do is

- map an inv. strong orbit to an isomorphic one,
- collapse an entire inv. strong orbit to $\{*\}$

$\Rightarrow$ hor/ver maps **do not** mix $\neq$ orbit types.

VS.

In the (1-)cat $A\text{-sets}^{inv}$, we have maps like $G/H \longrightarrow G/K$.

PUTTING IT ALL TOGETHER

$$K(A\text{-}\mathrm{Sets}) \simeq K(A\text{-}\mathrm{Sets}^{\mathrm{inv}})$$ double cat Devissage

$$\simeq K\Big(\prod_{S \leq R \in A} \langle R/S, \amalg \rangle\Big)$$ double cat equiv

$$\simeq \coprod_{S \leq R \in A} K(\langle R/S, \amalg \rangle)$$

$$\simeq \coprod_{S \leq R \in A} \Sigma^\infty_+ B\mathrm{Aut}(R/S)$$

same old K-theory shenanigans

$$K(A\text{-}\mathrm{Sets}) \simeq \coprod_{S \leq R \in A} \Sigma^\infty_+ B(R/S)$$

$\mathrm{Aut}(R/S) \cong R/S$ (easy)

THANKS FOR YOUR TIME!

REFERENCES

[Campbell-Zakharevich] Jonathan Campbell and Inna Zakharevich, "Devissage and Localization for the Grothendieck spectrum of varieties", Advances in Mathematics (2022).

[Sarazola-Shapiro-Zakharevich] Maru Sarazola, Brandon Shapiro and Inna Zakharevich, "K-theory of finite monoid sets", arXiv/2611.05265 (2026).

[Weissmann] Jeremy Weissmann, "A Burnside ring for monoids", Journal of Algebra (2025).