Unbiased Algebra

Claudio Pisani

Virtual Double Categories Workshop
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What is unbiased algebra?

Unbiased algebra is based on the belief that operations on a set X concern families $I \to X$ of elements of X.

The introduction of spurious orders on the index set I, as when one considers I in a skeleton of finite sets, is unnatural, and inevitably leads to awkwardness.

I wish to convey the idea that the unbiased approach to algebra is effective and opens up new perspectives on the subject.

What is unbiased algebra?

More precisely, it is based on four assumptions:

First assumption

Operations form a category,
whose objects are families of elements of a fixed set.

Second assumption

Operations are indexed by functions, compatibly with composition.

Third assumption

Operations can be reindexed along pullbacks,
compatibly with composition and with sums.

What is unbiased algebra?

These hold both for the *abstract* operations in an algebraic theory $\mathbb{M}$ and for the *concrete* operations in an algebra $\mathbb{A}$ for $\mathbb{M}$.

Fourth assumption

Algebraic theories and algebras share the same nature:
an algebra $\mathbb{A}$ for $\mathbb{M}$ is itself an algebraic theory,
along with morphism $\mathbb{A} \to \mathbb{M}$ satisfying the following condition:

Discrete opfibration condition for algebras

Given an operation $\alpha : a \to b$ in $\mathbb{M}$ and an object a' in $\mathbb{A}$ over a,
there is a unique operation $\alpha' : a' \to b'$ in $\mathbb{A}$ over α.

Single sorted theories or operads

Single sorted theories or operads

If the set of sorts X is terminal,
the objects of the theory are simply finite sets
and we have operations $\alpha : I \to J$
with their underlying functions $f : I \to J$.

$$I \xrightarrow{\quad\alpha\quad} J$$

$$I \xrightarrow{\quad f\quad} J$$

Algebras for an operad

If $\mathbb{A}$ is an algebra on X for the operad $\mathbb{M}$,
we then have, for $a : I \to X$, an unique operation in $\mathbb{A}$ over α
which we can then call α as well:

$$a \xrightarrow{\ \alpha\ } b \qquad \mathbb{A}$$

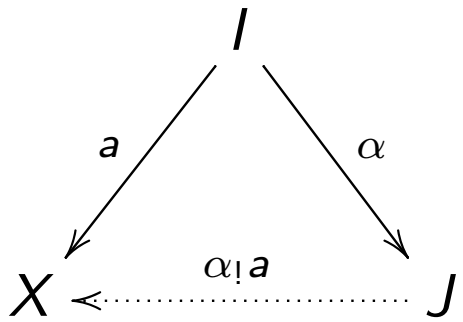
$$I \xrightarrow{\ \alpha\ } J \qquad \mathbb{M}$$

$$I \xrightarrow{\ f\ } J \qquad \mathbf{Set}_{\mathrm{f}}$$

The codomain $b : J \to X$ of α can be considered as the result
of the concrete operation over α, and we can write $b = \alpha_! a$.

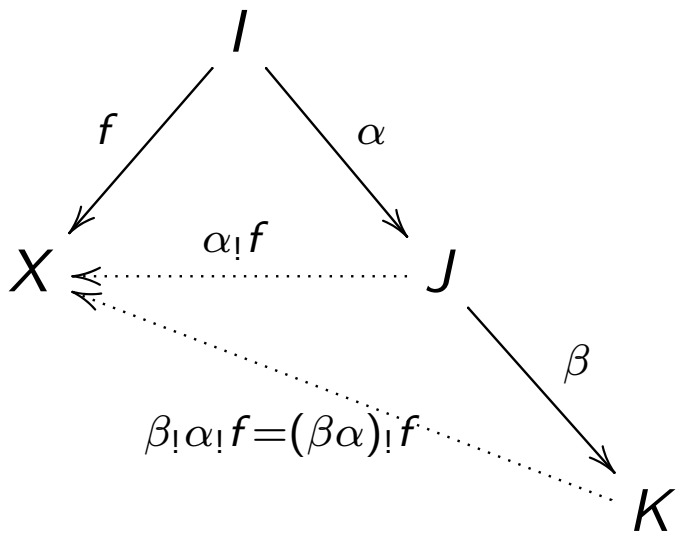
Algebras for an operad, span version

Putting it in an equivalent way, an algebra on X for $\mathbb{M}$ gives a way to "evaluate" spans formed by a family $a : I \to X$ and an operation $\alpha : I \to J$, to get a new family $\alpha_! a : J \to X$:



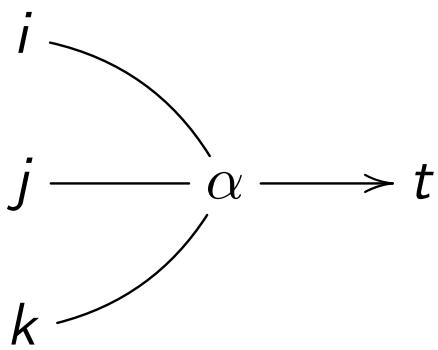
Algebras for an operad, span version

Furthermore, evaluation has to be compatible with compositions and reindexing of operations:



Algebras for an operad, span version

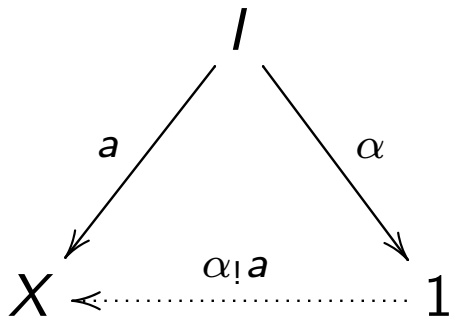
Evaluation is usually considered in the case when
the operation α is "elementary", that is J is a terminal set:



We will see how the reindexing assumption assures that each operation is uniquely a sum of elementary operations.
The great advantage of considering general operations is that they form a category.

Algebras for an operad, span version

Thus, if $\alpha : I \to 1$ is an elementary operation, the evaluation $\alpha_! a$ is an element of X:



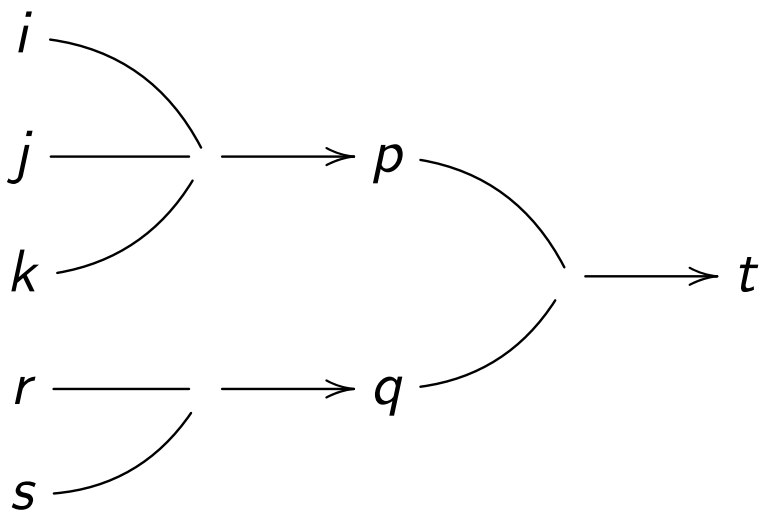
An example

Before presenting the precise definition of algebraic theory,
which involves double categories and fibrations,
let us dive into a simple but significant example
of single sorted theory and its algebras.

An example

Let us say that a function $I \to J$ is a **Tot**-function if each of its fibers is endowed with a total order.

It is easy to see that **Tot**-functions compose in a natural way, namely by the lexicographic sum.



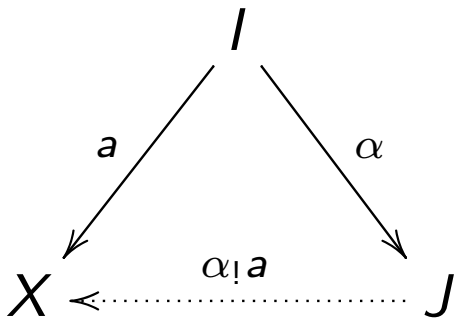
An example

Furthermore, they can be transported along pullbacks,
since these induce bijections on the fibers,
and total orders (as any structure in the sense of Joyal)
are transported along bijections.

Thus, **Tot**-functions $\alpha : I \to J$
are the operation of an operad $\mathbb{T}\mathrm{ot}$.

An example

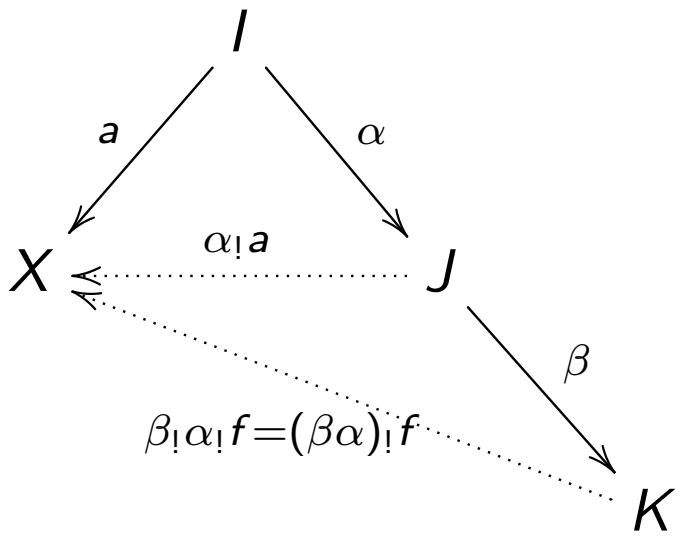
An algebra for $\mathbb{T}\mathrm{ot}$ consists then of a set X and,
for any family $a : I \to X$ and any **Tot**-function $\alpha : I \to J$,
a new family $\alpha_! a : J \to X$.



Thus, in a $\mathbb{T}\mathrm{ot}$-algebra we can evaluate J-families of lists in X
(the span) to get a J-family of elements in X.

An example

Compatibility with compositions of operations amounts, in this case, to the usual condition for the algebras of the monad of lists: we can either evaluate the concatenated list (the big span) or evaluate the list obtained from the evaluations of the inner lists.



An example

Lastly, compatibility with reindexing of operations assures that the evaluation of a family of lists is the family obtained by evaluating the single lists.

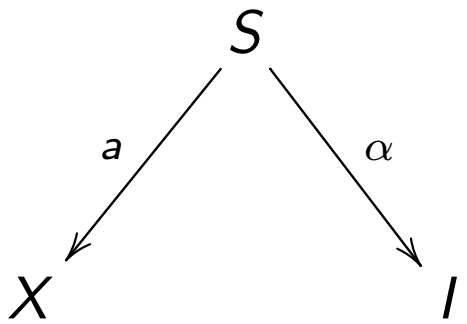
Thus, the algebras for $\mathbb{T}\mathrm{ot}$ are monoids, in their unbiased natural version:

a monoid structure on a set X is a coherent a way to evaluate lists in X.

An example

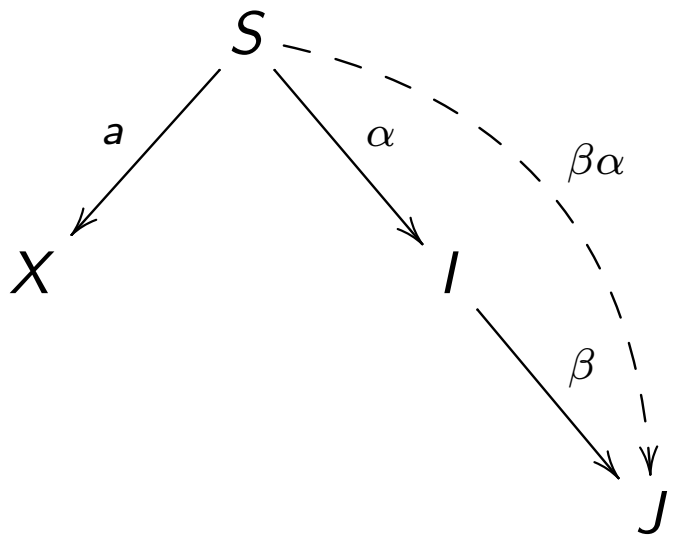
It is clear that we have here a "span version" of the list monad, which is essentially based on the indexing of the set X:

It takes the indexed set X^I, whose elements over I are families $I \to X$, and returns the indexed set $\mathbf{Span}(X, I)$ of equivalence class of spans, whose elements over I are I-families of lists in X.



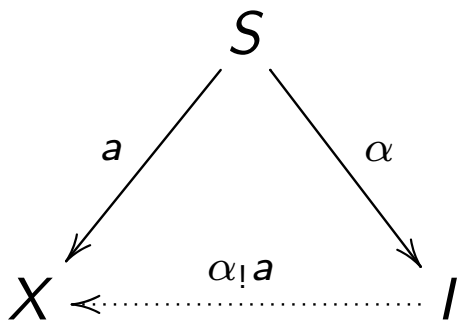
An example

The span form for the family of lists
allows for a smooth definition of the monad structure.
It is given by composition in the operation leg:



An example

The algebras for the monad are the evaluations mentioned above, that is they are the algebras for the associated operad.



This holds for any "multiplicative" structure in place of **Tot**, that is for any operad, as we will see in the sequel.

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Operadic algebraic theories

By the first two assumptions, an X-sorted algebraic theory $\mathbb{M}$ consists, first of all, of a category $\mathbb{M}_\ell$ of "operations"

- whose objects are families $a : I \to X$, $I \in \mathbf{Set}_\mathrm{f}$;
- whose arrows $\alpha : a \to b$ from $a \in X^I$ to $b \in X^J$ have an underlying function $f : I \to J$.

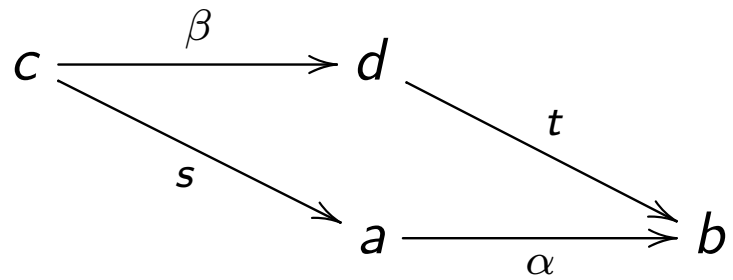
Thus, we have an underlying functor $p : \mathbb{M}_\ell \to \mathbf{Set}_\mathrm{f}$ giving the indexing of objects and arrows.

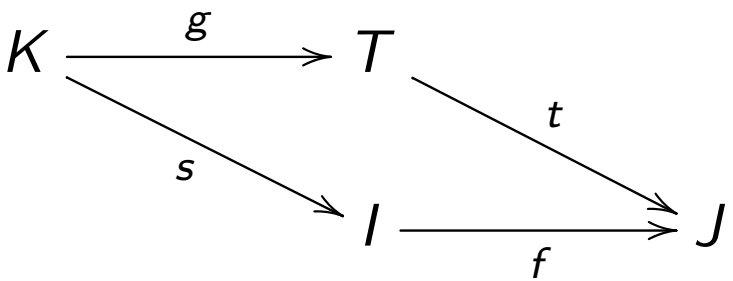
$$a \xrightarrow{\quad\alpha\quad} b \qquad\qquad I \xrightarrow{\quad f = p\alpha \quad} J$$

$$\begin{array}{c} \mathbb{M}_\ell \\ \Big\downarrow{\scriptstyle p} \\ \mathbf{Set}_\mathrm{f} \end{array}$$

Operadic algebraic theories

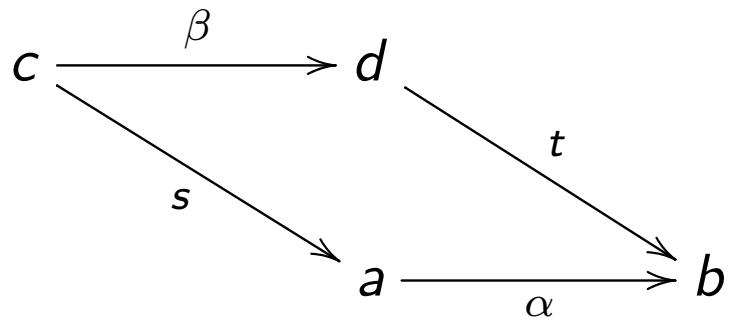
The third assumption can be rephrased by saying that, for any operation $\alpha : a \to b$ over f and any pullback $fs = tg$ in $\mathbf{Set}_{\mathrm{f}}$, there is a specific operation $\beta : c \to d$, where c and d are obtained by reindexing a and b along s and t respectively.

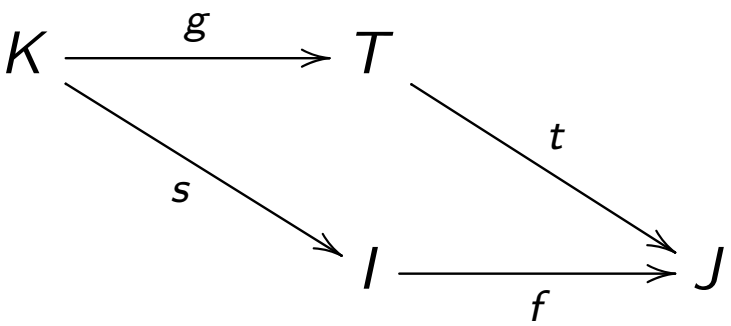


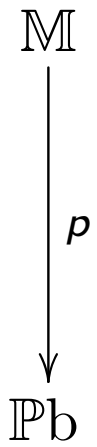


Operadic algebraic theories

Thus, we are led to consider these squares
as the cells of a double category $\mathbb{M}$, with loose part $\mathbb{M}_\ell$,
so that we have a double functor $p : \mathbb{M} \to \mathbb{P}\mathrm{b}$:







Symmetric multicategories

Symmetric multicategory or operadic algebraic theory

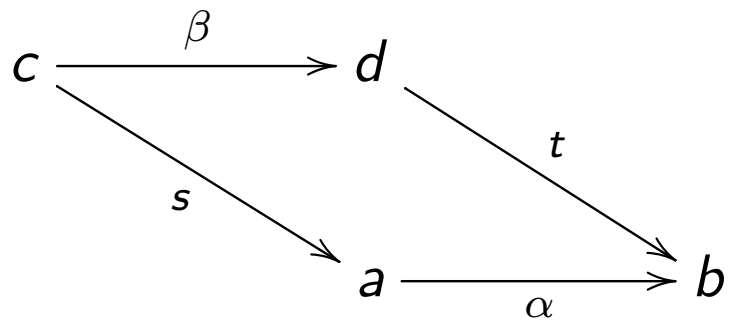
- A double category with finite parallel sums $\mathbb{M}$;
- a sum preserving double functor $p : \mathbb{M} \to \mathbb{P}\mathrm{b}$;
- (*reindexing condition*)
  p is a *discrete fibration* in the tight direction.

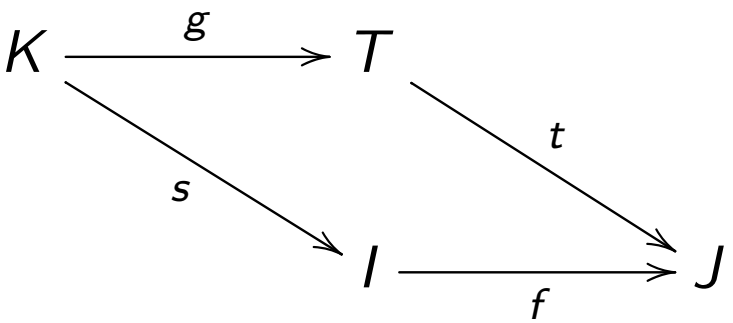
The morphisms are the double functors over $\mathbb{P}\mathrm{b}$.

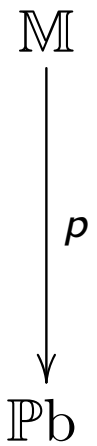
Symmetric multicategories

In $\mathbb{M}$ we have *operations* $\alpha : a \to b$ and *reindexing cells*.

β is obtained by copying, duplicating or deleting some of the components of α.



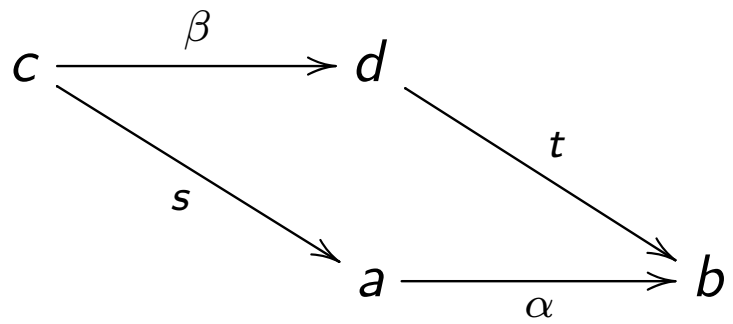


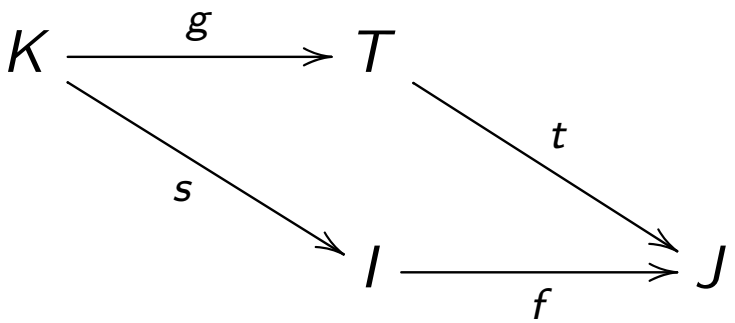


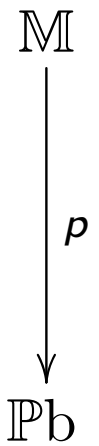
Symmetric multicategories

If t is bijective, β can be thought of as the same as α up to reindexing.

If $T = 1$ is a terminal set, β is one of the components of α.





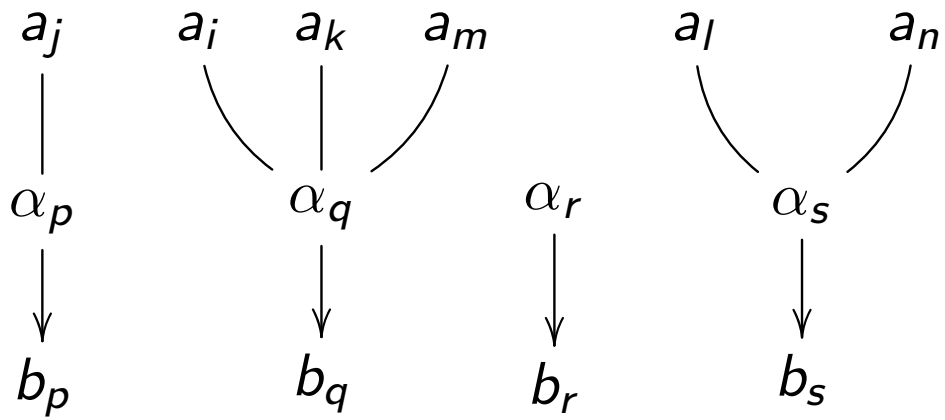


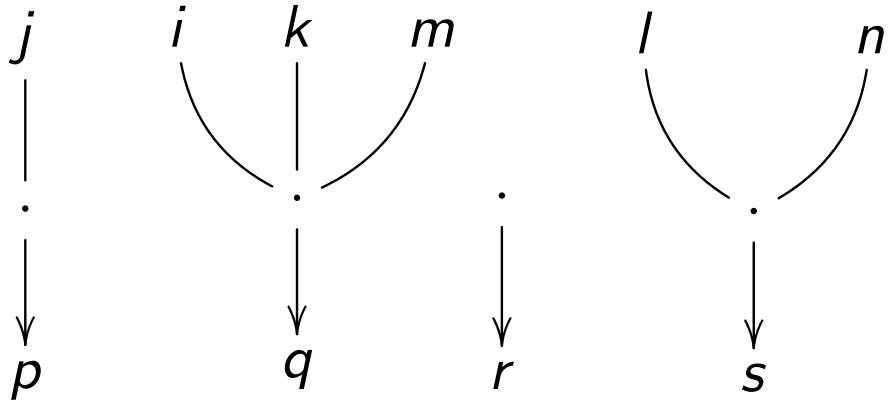
Symmetric multicategories

Good behavior w.r.t. sums and reindexing

- The objects of $\mathbb{M}$ over I are I-families (and sums) of *elementary objects*;
- the operations in $\mathbb{M}$ over $f : I \to J$ are J-families (and sums) of *elementary operations*.

Symmetric multicategories





The unbiased perspective

Idea

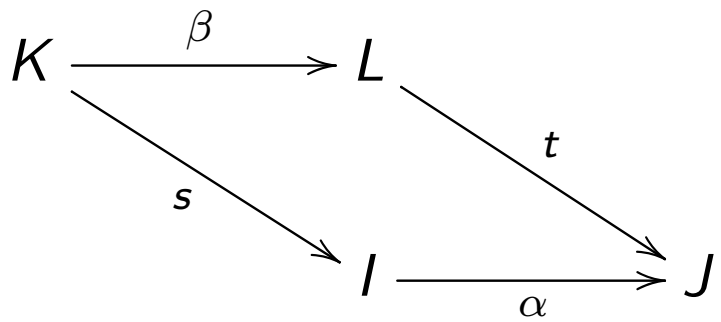
It is unnatural to define symmetric multicategories
by putting together elementary operations with fixed indexing.

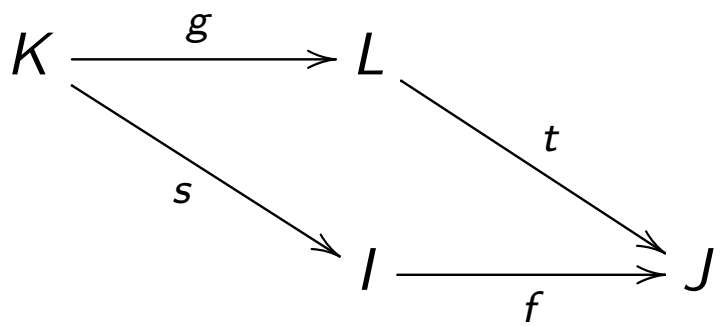
The other way round

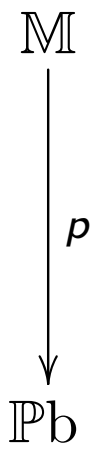
Rather, given an operation we can obtain its components
by the exploiting the reindexing axiom.

Operads

An *operad* has just one elementary object.
Its objects are finite sets.







Operads via species

Given a Joyal's species $\mathcal{S}$,
an $\mathcal{S}$-*function* is a function with an $\mathcal{S}$-structure on each fiber.

$\mathcal{S}$-functions can be reindexed along pullbacks,
since these induce bijections on the fibers.

Multiplicative species

To get an operad $\mathbb{M}_{\mathcal{S}}$ we need also that
$\mathcal{S}$-functions form a category over $\mathbf{Set}_f$.

All operads are obtained in this way.

Examples of operads from multiplicative species

Many basic species are in fact multiplicative.

- The *terminal species* gives the *terminal operad* $\mathbb{P}\mathrm{b}$.
- The *species of total orders* gives the operad $\mathbb{T}\mathrm{ot}$,
  also known as the *the associative operad*.
- The *species characteristic of singletons* gives the *initial operad* $\mathbb{B}\mathrm{ij}$,
  whose operations are bijections.
- The *species of elements* gives the operad $\mathbb{S}\mathrm{ect}$,
  whose operations are functions endowed with a section.

Cocartesian multicategories

Cocartesian operads and multicategories

If R is a monoid, the species $\mathcal{S}(I) = R^I$ of labellings in R gives the *cocartesian* operad $\mathbb{R}_{\triangleright}$.

Note that $1_{\triangleright}$ is the terminal operad $\mathbb{P}\mathrm{b}$.

Cocartesian (a.k.a. "sequential") multicategories

If $\mathcal{C}$ is a category, we have *cocartesian* multicategory $\mathbb{C}_{\triangleright}$, whose elementary operations are families of arrows $f_i : a_i \to b$.

Multicategories over a base

Each operad can serve as a base for defining some kinds of multicategories.

Plain multicategories

Symmetric multicategories over $\mathbb{T}\mathrm{ot}$ are plain multicategories.
Indeed, a morphism $p : \mathbb{M} \to \mathbb{T}\mathrm{ot}$ endows each elementary operation in $\mathbb{M}$ with a total order on its domain.

Unary multicategories

Symmetric multicategories over $\mathbb{B}\mathrm{ij}$ are categories.
Indeed, a morphism $p : \mathbb{M} \to \mathbb{B}\mathrm{ij}$ forces the elementary operations in $\mathbb{M}$ to have an elementary domain (that is, to be unary).

Special multicategories

Symmetric monoidal categories

If the loose part of $p : \mathbb{M} \to \mathbb{P}\mathrm{b}$ is an *opfibration*,
then $\mathbb{M}$ is a *representable* symmetric multicategory.
Opcartasian operations give the unbiased notion *tensor products*.

Plain monoidal categories

Similarly, if the loose part of $p : \mathbb{M} \to \mathbb{T}\mathrm{ot}$ is an opfibration,
then $\mathbb{M}$ is a representable plain multicategory.

Exponentiable multicategories

If the loose part of $p : \mathbb{M} \to \mathbb{P}\mathrm{b}$ is a Conduché fibration,
then $\mathbb{M}$ is an *exponentiable* multicategory.

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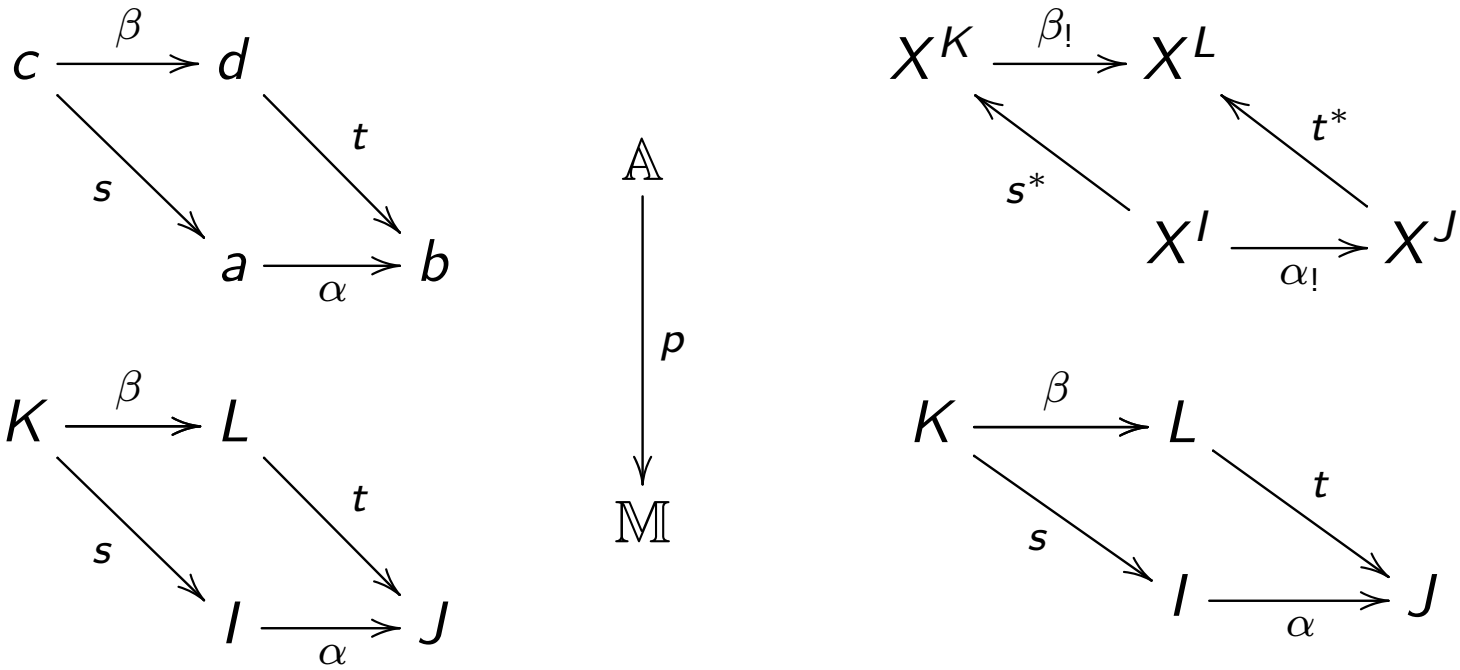
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Algebras for an operad

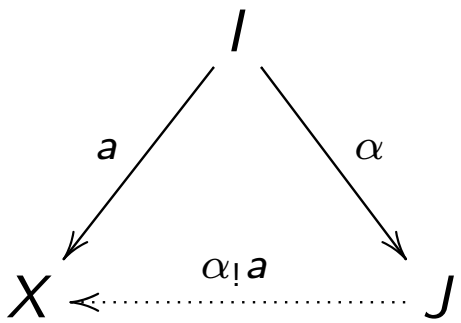
As we have already discussed in the introduction,
an algebra for the operad (or multicategory) $\mathbb{M}$
is a morphism $p : \mathbb{A} \to \mathbb{M}$ whose loose part is a *discrete opfibration*.

Fibered picture on the left, *indexed* picture on the right.



Algebras for an operad via species

If $\mathbb{M}_{\mathcal{S}}$ is the operad associated to the species $\mathcal{S}$, an algebra $\mathbb{A}$ on a set X is a coherent way to evaluate families in X against $\mathcal{S}$-functions.



In particular, when J is terminal, one evaluates a family $a : I \to X$ against a structure S on I to get an element of X.

Algebras for an operad: examples

The algebras for...

- the terminal operad $\mathbb{P}\mathrm{b}$ are *commutative monoids*.
  We evaluate simply families in X.
- the associative operad $\mathbb{T}\mathrm{ot}$ are *plain monoids*.
  We evaluate totally ordered families (a.k.a. lists) in X.
- the initial operad $\mathbb{B}\mathrm{ij}$ are *sets*.
- the unary multicategory $\mathbb{C}$ are the *presheaves* on $\mathcal{C}$.
- the cocartesian operad $\mathbb{R}_{\triangleright}$ are commutative monoids
  on which the monoid R acts by homomorphisms.
  We evaluate R-labelled families in X.

Algebras for an operad: another view

The algebras (over a set X) for $\mathbb{M}$ are the morphisms $\mathbb{M} \to \mathbb{E}\mathrm{nd}\,(\mathrm{X})$.

The endomorphism operad

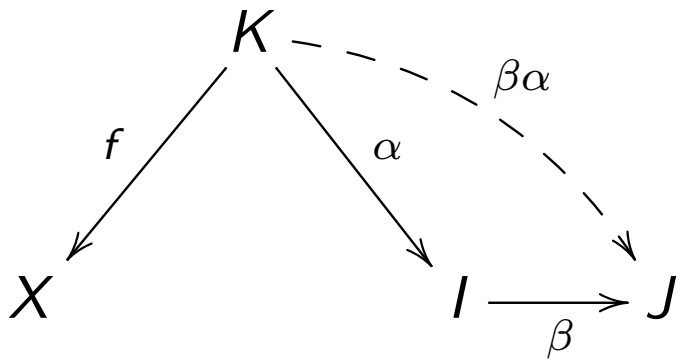
Is the operad associated to the species of multivariate functions on X.
The structures on I are the functions $X^I \to X$.

Free algebras

As seen in the introduction, the algebras for $\mathbb{M}$
are the algebras for the monad $\mathbf{Span}_{\mathbb{M}}$
so that the free algebras are formed by the spans themselves.

The free algebra $\mathbf{Span}_{\mathbb{M}}(X)$ on a set X for the operad $\mathbb{M}$ has

- isomorphism classes of spans as objects over I;
- evaluation along an operation $\beta : I \to J$ is given by composition:



Relatively free algebras

A morphism of operads $F : \mathbb{N} \to \mathbb{M}$ induces, by pulling back, a morphism from $\mathbb{M}$-algebras to $\mathbb{N}$-algebras.
Its left adjoint gives, for an $\mathbb{N}$-algebra X, the relatively free $\mathbb{M}$-algebra $\mathbf{Span}^F_{\mathbb{M}}(X)$ obtained by quotienting $\mathbf{Span}_{\mathbb{M}}(X)$ via $\mathbb{N}$-operations $\gamma : K \to L$.

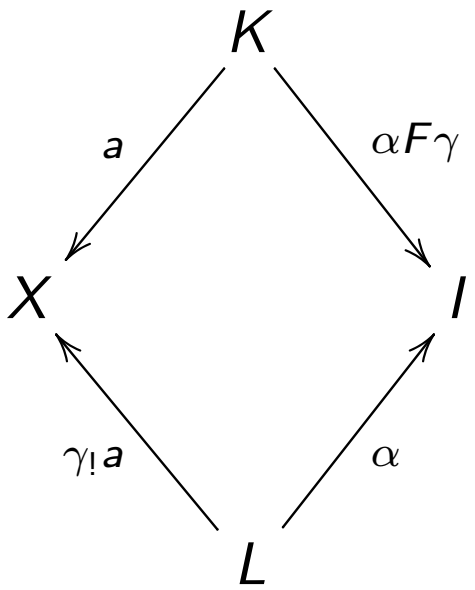


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Cartesian multicategories

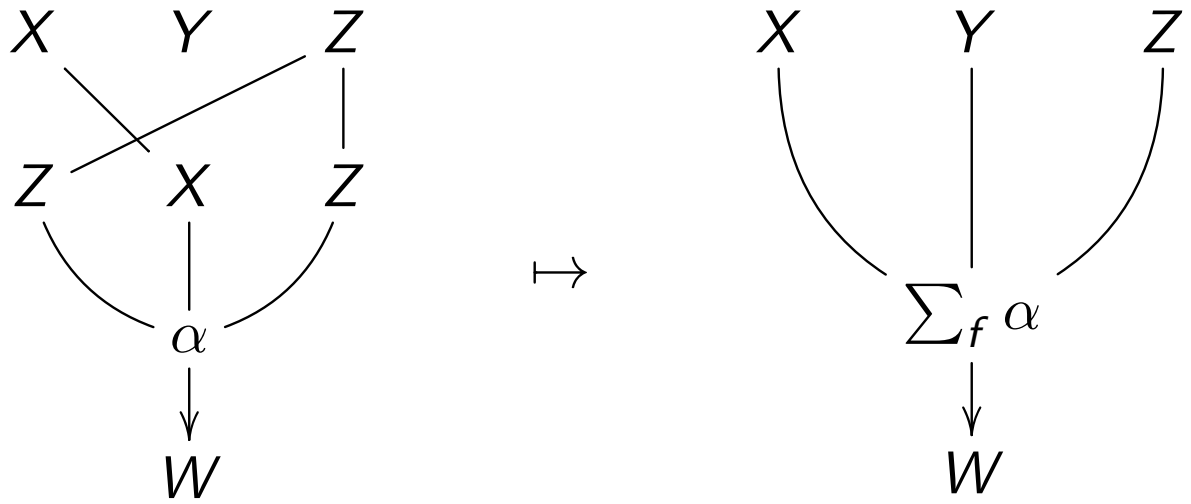
We have presented an unbiased approach to symmetric multicategories, seeing them as a way to encode algebraic theories.

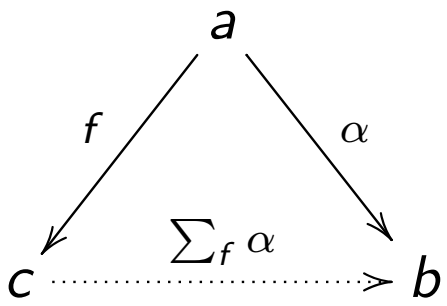
Actually, these do not capture all sorts of algebraic theories; roughly, they do not allow duplication or deletion of variables.

For general algebraic theories, we need cartesian multicategories, which offer an alternative to Lawvere theories in categorical algebra.

Cartesian multicategories

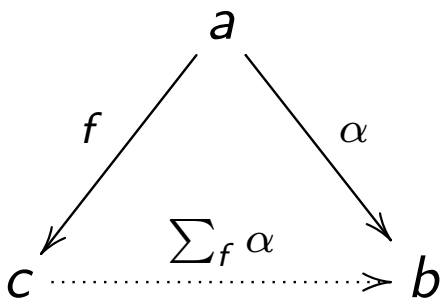
Duplication or deletion of variables $\sum_f \alpha(x, y, z) = \alpha(z, x, z)$:





Cartesian multicategories

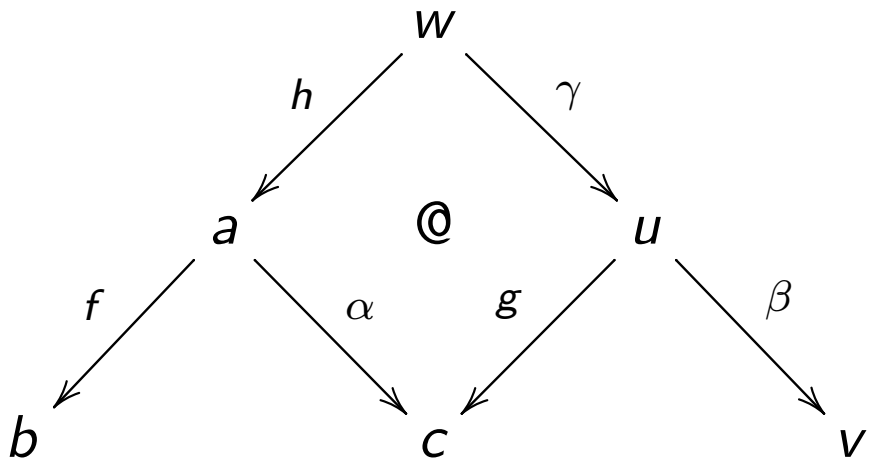
Thus, in a cartesian multicategory $\mathbb{M}$ we can evaluate spans formed by a loose arrow $\alpha : a \to b$ (an operation) and a tight arrow $f : a \to c$ (the reindexing of the family c along the function f), obtaining a new operation $\sum_f \alpha : c \to b$ (the "sum" of α along f):



If the operation $\alpha : a \to b$ is not elementary, we must insist that we evaluate *enhanced* spans, that is, spans over commutative triangles in $\mathbf{Set}_f$.

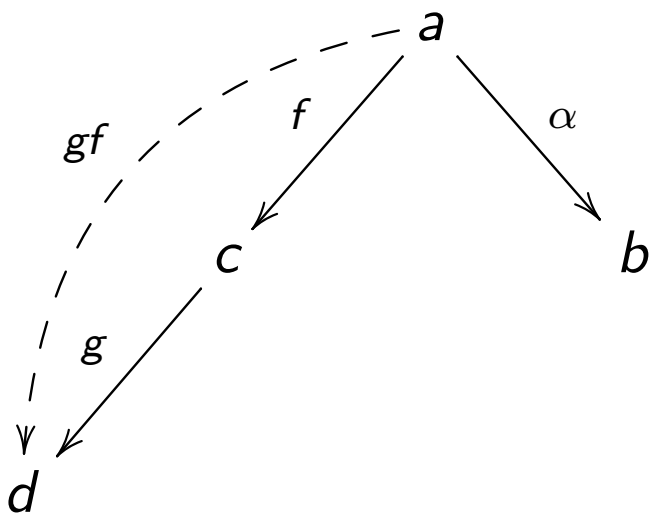
Cartesian multicategories as algebras for a monad

We define cartesian multicategories as the algebras
for a monad **cart** on symmetric multicategories.
The operations in **cart** $\mathbb{M}$ are isomorphism classes of enhanced spans in $\mathbb{M}$;
they are composed like spans, using cells in $\mathbb{M}$ in place of pullbacks:



Cartesian multicategories as algebras for a monad

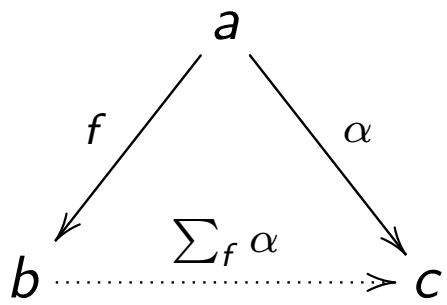
The monad multiplication $\mathbf{cart}\,\mathbf{cart}\,\mathbb{M} \to \mathbf{cart}\,\mathbb{M}$ is obtained by composing the tight legs:



Cartesian multicategories as algebras for a monad

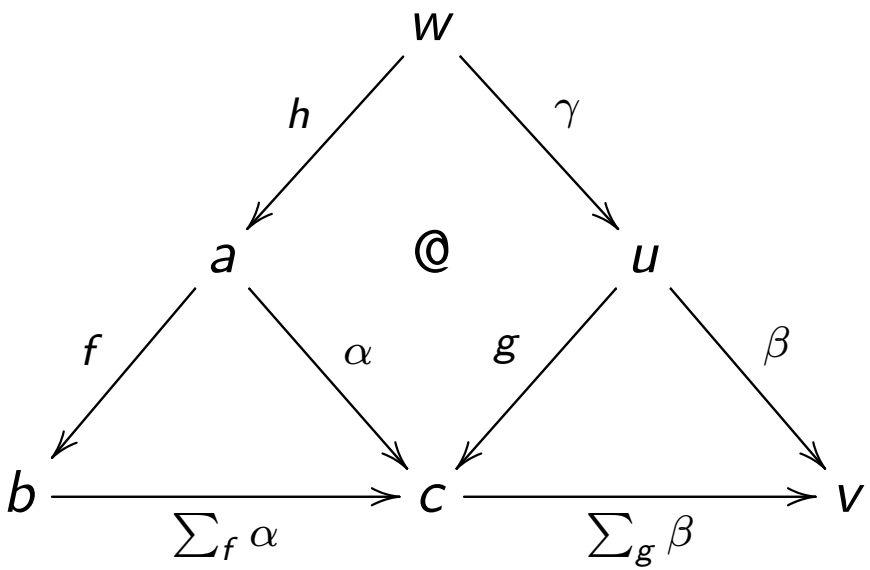
An algebra for the monad **cart** is a morphism in **sMlt**

$\sum : \mathbf{cart}\,\mathbb{M} \to \mathbb{M}$ which takes an operation in $\mathbf{cart}\,\mathbb{M}$, that is a span (f, α) in $\mathbb{M}$, to an operation $\sum_f \alpha$ in $\mathbb{M}$:



The distributive law

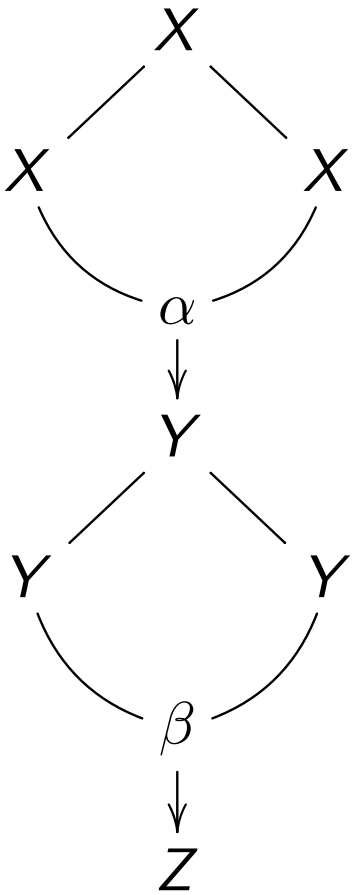
Since $\sum : \mathbf{cart}\,\mathbb{M} \to \mathbb{M}$ is a morphism of multicategories, it preserves composition of operations:



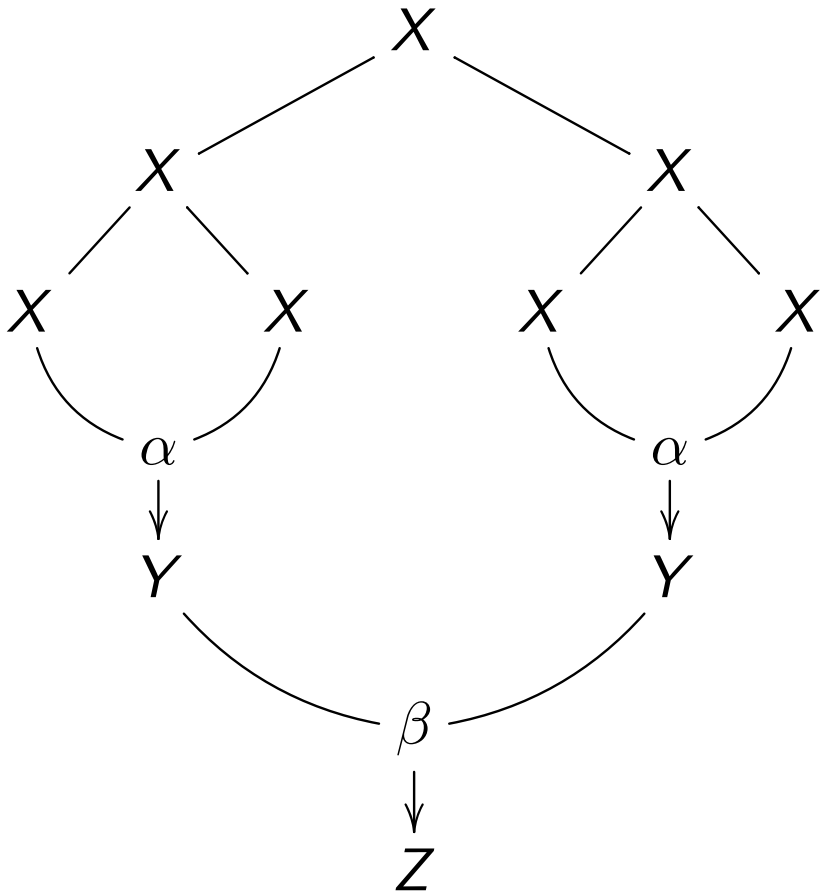
$$\sum_g \beta \sum_f \alpha = \sum_{fh} \beta\gamma$$

The distributive law

The condition can be exemplified by the fact that diagrams below are evaluated to the same operation:



$$\sum_g \beta \sum_f \alpha = \sum_{fh} \beta\gamma$$



The distributive law

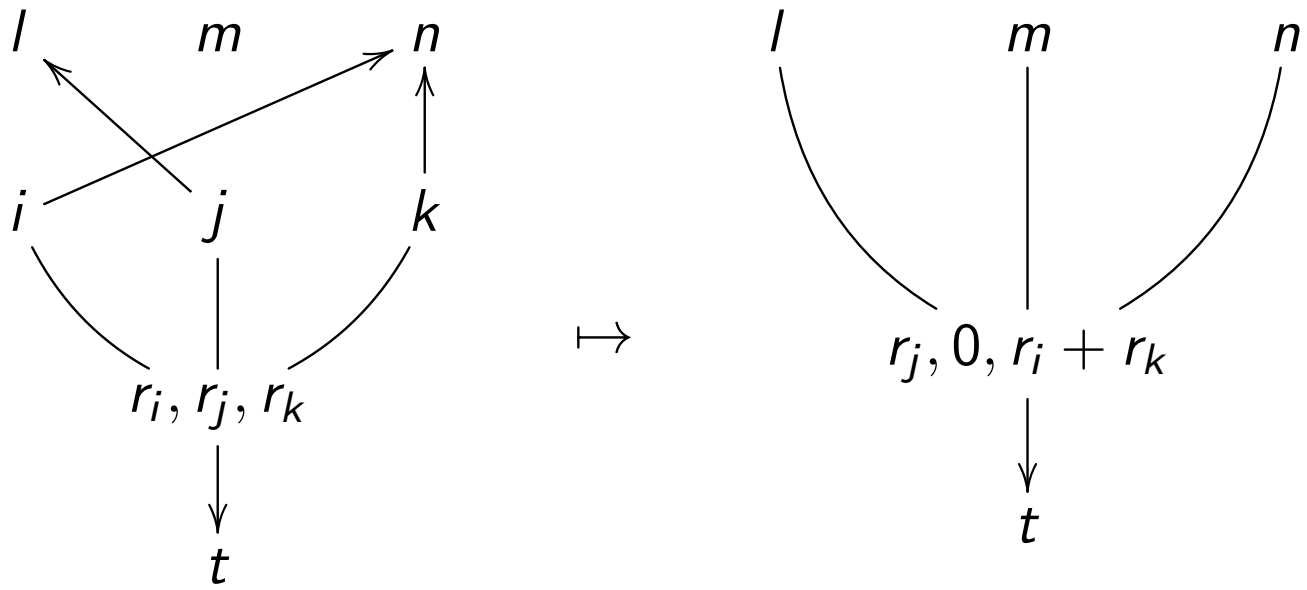
Accordance with the classical definition

The distributive law, along with the algebra for a monad condition, are exactly the compatibility conditions between composition and sums which are "classically" used to define cartesian multicategories.

Example: a view on ri(n)gs

Recall that an elementary operation $\alpha : I \to 1$ in $\mathbb{R}_{\triangleright}$ is a family $r : I \to R$ of elements of the monoid R.

If R *is a rig*, we obtain the span evaluation $\sum_f \alpha$ by adding the elements in the fiber of $f : I \to J$:



The distributive law in $\mathbb{R}_{\triangleright}$

The distributive law in $\mathbb{R}_{\triangleright}$ expresses the distributivity conditions of multiplication-composition over the addition $\sum$.

Theorem

Cartesian structures on the cocartesian operad $\mathbb{R}_{\triangleright}$ correspond to rig structures on the monoid R.

Characterization of rigs

Rigs are cartesian and cocartesian operads.

Theorem

Cartesian and cocartesian multicategories correspond to categories enriched in commutative monoids.

Cartesian multicategories as virtual product categories

Theorem

For a cartesian multicategory $\mathbb{M}$, the following are equivalent:

1. $\mathbb{M}$ *is representable.*
2. $\mathbb{M}$ *has universal products.*
3. $\mathbb{M}$ *has algebraic products.*

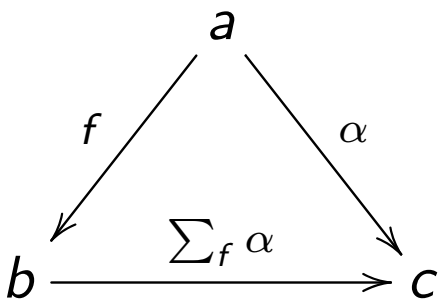
Corollary

For a category enriched in commutative monoids, the following are equivalent:

1. *it has sums;*
2. *it has products;*
3. *it has algebraic products (biproducts).*

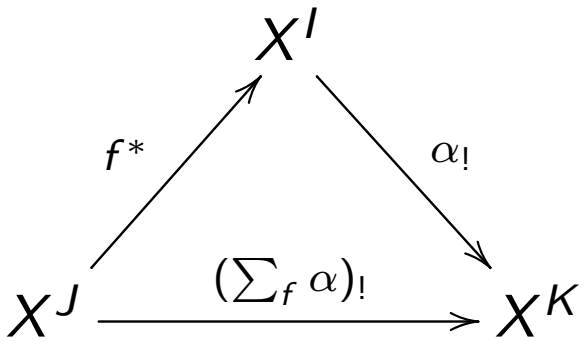
Algebras for a cartesian operad

An algebra in $\mathbb{A}$ for the cartesian theory $\mathbb{M}$,
is a plain algebra which is also a morphism cartesian multicategories.
That is, $\mathbb{A}$ has itself a cartesian structure which is preserved by $\mathbb{A} \to \mathbb{M}$.
This amounts to the existence of the following triangles for any span in $\mathbb{A}$:



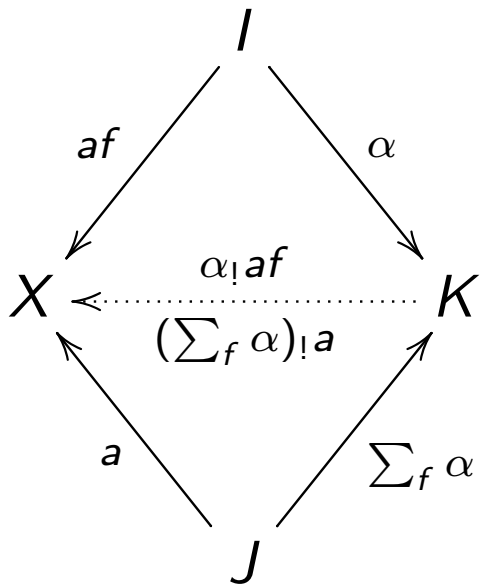
Algebras for a cartesian operad: alternative view

Equivalently, algebra (on a set X) for a cartesian operad $\mathbb{M}$
can be defined as a morphism of cartesian operads $\mathbb{M} \to \mathbb{E}\mathrm{nd}\,(\mathrm{X})$
Indeed, since $\sum_f \alpha$ in $\mathbb{E}\mathrm{nd}\,(\mathrm{X})$ is defined by $\sum_f \alpha = \alpha \circ f^*$ we have:



Algebras for a cartesian operad

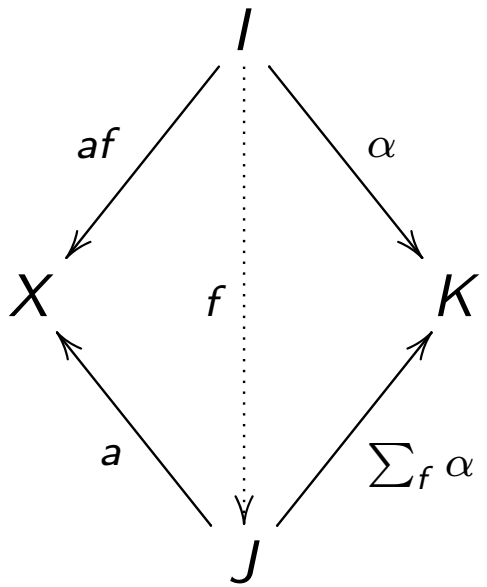
Putting it in the form of span evaluation, the triangle says that two spans below are evaluated to the same family



Free cartesian algebras

The free cartesian algebra on a set X for the cartesian operad $\mathbb{M}$ can be obtained as classes of spans, similarly to the free ordinary algebras.

But now we consider equivalences classes of spans, considering two spans equivalent if there is a mediating f as below:



A view on modules

For a monoid R, the algebras for $\mathbb{R}_{\triangleright}$ on a set X
are commutative monoids on which R acts by homomorphisms.

Modules

If R is a rig, the algebras for the cartesian operad $\mathbb{R}_{\triangleright}$ are the R-modules.

Presevation of sums

Since $\sum$ in $\mathbb{R}_{\triangleright}$ is the sum in the rig R
and $\sum$ in $\mathbb{E}\mathrm{nd}\,(\mathrm{X})$ is given by duplication of variables,
we get the module condition $(r+s)x = rx + sx$.

Free cartesian muticategories

cart $\mathbb{M}$ is the *free cartesian muticategory* on $\mathbb{M}$.
Its algebras for correspond to the plain algebras for $\mathbb{M}$.

Example

If $\mathbb{B}\mathrm{ij}$ is the initial operad, **cart** $\mathbb{B}\mathrm{ij} = \mathbb{S}\mathrm{ect}$.
Its algebras are sets.

Example

If $\mathbb{R}_{\triangleright}$ is the cocartesian operad on the monoid R,
cart $\mathbb{R}_{\triangleright}$ is the cocartesian operad on the free rig on R.

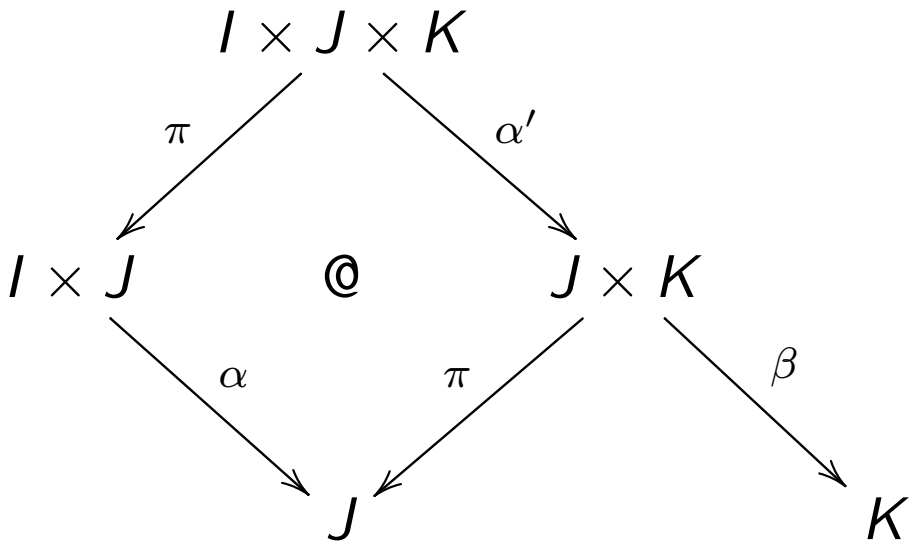
Example

In particular, since $\mathbb{P}\mathrm{b} = 1_{\triangleright}$, we have that **cart** $\mathbb{P}\mathrm{b} = \mathbb{N}_{\triangleright}$.

Relations with Lawvere theories

Given a cartesian operad $\mathbb{M}$, we have the category **Mat** $\mathbb{M}$:

- objects are finite sets;
- arrows $\underline{\alpha} : I \to J$ are operations $\alpha : I \times J \to J$ in $\mathbb{M}$;
- composition is defined by $\underline{\beta} \circ \underline{\alpha} = \underline{\sum_J \beta \circ \alpha'}$



Relations with Lawvere theories

The Lawvere theory for R-modules

For a rig R, $\mathbf{Mat}\,\mathbb{R}_{\triangleright}$ is the category of *matrices* on R.

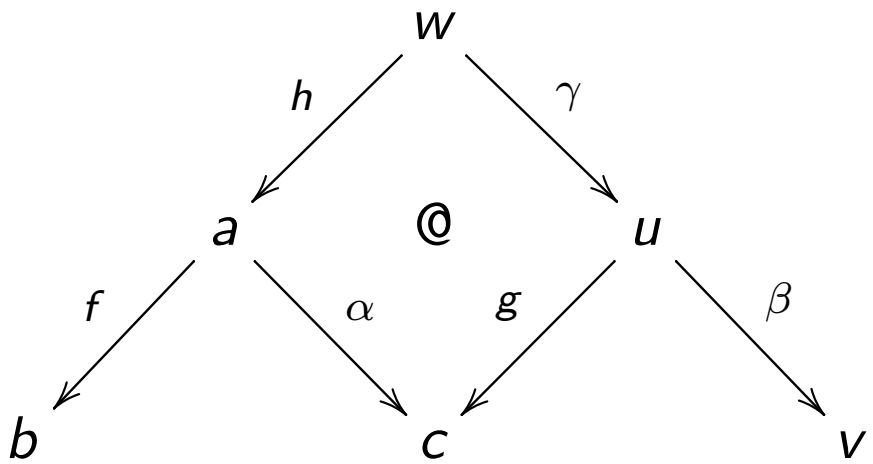
The free semiadditive category on $\mathcal{C}$

For a category $\mathcal{C}$ enriched in commutative monoids,
$\mathbf{Mat}\,\mathbb{C}_{\triangleright}$ is the category of *matrices* on $\mathcal{C}$.

Relations with Lawvere theories

The Lawvere theory associated to a free cartesian theory

It is easy to see that the Lawvere theory associated to **cart** $\mathbb{M}$ is simply the category **Span** $\mathbb{M}$ of (not enhanced) spans in $\mathbb{M}$:



Relations with Lawvere theories

The Lawvere theory for sets

$$\mathbf{Mat}\,(\mathbf{cart}\,\mathbb{B}\mathrm{ij}\,) = \mathbf{Mat}\,\mathbb{S}\mathrm{ect} = \mathbf{Span}\,\mathbb{B}\mathrm{ij} = \mathbf{Set}_{\mathrm{f}}^{\mathrm{op}}$$

The Lawvere theory for commutative monoids

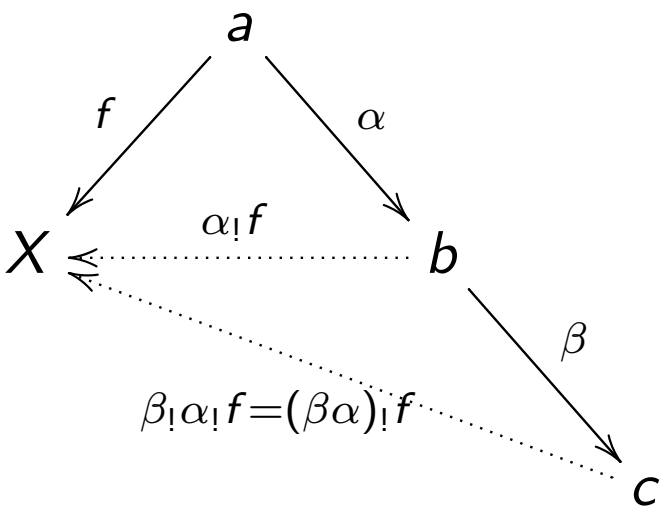
$$\mathbf{Mat}\,(\mathbf{cart}\,\mathbb{P}\mathrm{b}\,) = \mathbf{Mat}\,\mathbb{N}_{\triangleright} = \mathbf{Span}\,\mathbb{P}\mathrm{b} = \mathbf{Span}\,\mathbf{Set}_{\mathrm{f}}$$

The Lawvere theory for monoids

$$\mathbf{Mat}\,(\mathbf{cart}\,\mathbb{T}\mathrm{ot}\,) = \mathbf{Span}\,\mathbb{T}\mathrm{ot}$$

Comparison of two "dual" span monads

It is interesting to compare the monad **Span**
on sets (or better, on $\mathbb{B}\mathrm{ij}$ algebras)
and the monad **cart** on symmetric multicategories:



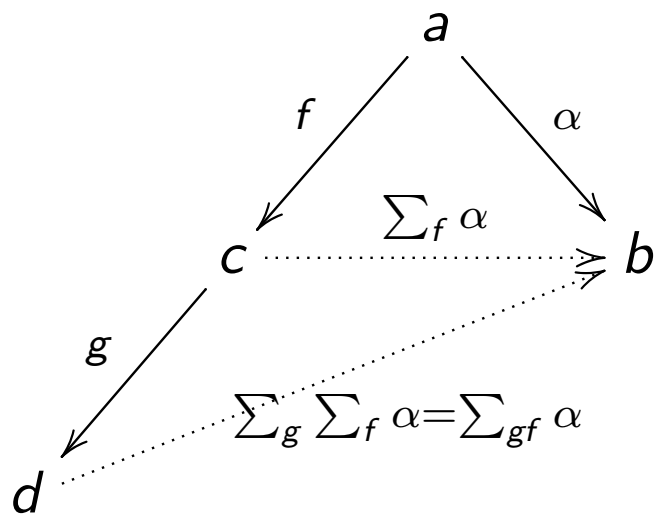


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3 Algebras

4 Cartesian algebraic theories

5 Conclusions

Conclusions

The unbiased approach to multicategories, based on double categories, provides a natural framework for categorical algebra
and, consequently, enlighten several of its aspects, such as:

- the relations between algebraic theories and their algebras;
- the relations between operadic and cartesian theories;
- the fruitful view of cartesian theories themselves as multicategories rather than as mere categories as in Lawvere theories;
- the concrete view of the monads associated free theories and algebras, namely as spans in a double category.