Para construction for double categories

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Third Virtual Workshop on Double Categories

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of Sciences and Arts

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- MOTIVATION
- (PRE) MONOIDAL double categories
- PARA CONSTRUCTION
  - ACTION of double categories
  - How PARA generalizes Kleisli
- TYPES OF ACTIONS: lax monoidal, bistrong, commutative
- MAIN RESULT

MOTIVATION I

Motivation

Hugo Paquet, Philip Saville : "premonoidal bicategories"

monoidal

ps. functor $- \otimes -$

ps nat tr. $\alpha_{-,-,-}$

1 pentagonator
3 triangulators
} modifications

+3 ax's

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premonoidal

ps. functors $A \times -, - \times B$ $\quad A \times B = A \otimes B$

ps. nat. tr. $\alpha_{A,B,-}$ $\quad$ 4 pentagonators

$\alpha_{-,B,C}$ $\quad$ 6 triangulators

$\alpha_{A,-,C}$ $\quad$ (modifications + axioms)

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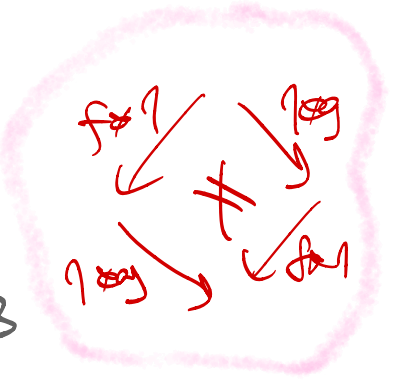
6 triangulators

(modifications + axioms)

$\lambda_{A,B,C}{}^{A} -, \lambda_{A}{}^{X} -, \rho_{A}{}^{X} -$ central

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[BF] Double cat. equivalence

$$2\text{-}\mathbb{P}s^{st}_{lax}(\mathbb{A} \times \mathbb{B}, \mathbb{C}) \simeq \mathbb{P}s_{lax}(\mathbb{A} \times \mathbb{B}, \mathbb{C})$$

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binary → n-ary

$\mathbb{A}\times\mathbb{B} \to \mathbb{A}_1 \times \dots \times \mathbb{A}_n$

Motivation

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$$g_n\text{-}\mathbb{P}s^{st}_{lax}(\mathbb{D}\times\ldots\times\mathbb{D};\mathbb{D}) \simeq \mathbb{P}s_{lax}(\mathbb{D}\times\ldots\times\mathbb{D},\mathbb{D})$$

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$\Leftrightarrow$

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MOTIVATION II

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H. Paquet, P. Saville: "Effectful semantics in 2-dimensional categories: strong, commutative, and concurrent pseudomonads", LICS '24

Theorems 7.3 – 7.5

For a pseudomonad T on a monoidal bicategory $\mathcal{B}$ TFAE:

1. monoidal structures on T
2. commutative —11—
3. monoidal structures on $\mathcal{B}_T$ extending that on $\mathcal{B}$.

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$K : \mathcal{B} \to \mathcal{B}_T$

iso $K(-) \otimes K(-) \Rightarrow K(- \otimes -)$

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Def. 1.2. "Para construction" pseudodouble category

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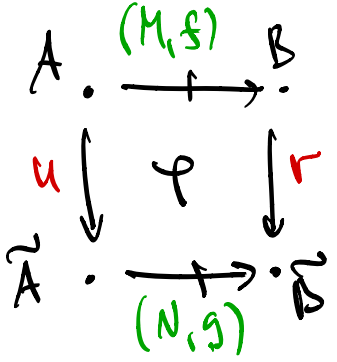
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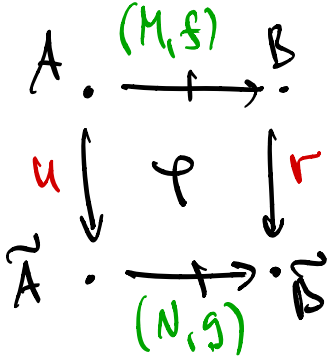
$f : M \triangleright A \to B$ morphism in $\mathcal{C}$

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- dual
- $\mathcal{M} = *$

} Para generalizes Kleisli construction

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C.5.21.

If the action $\triangleright : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ is lax monoidal ($\mathcal{M}$ braided),
then $\mathrm{Para}_{\mathcal{M}}(\mathcal{C})$ is a monoidal ps.double category.

CONJECTURE FOR THE RESEARCH

GOAL

A1 There is a "purely central" promonoidal double cat. structure on $\mathbb{D}$ $\iff$ there is a monoidal d. cat. structure on $\mathbb{D}$.

A2 For a "bistrong" double monad T on $\mathbb{D}$, the Kleisli d.cat $\mathbb{K}\ell_T(\mathbb{D})$ is promonoidal.

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IDEA: generalize
- (bicategory $\rightsquigarrow$ d.cat.
- ps.monad T $\rightsquigarrow$ action $\triangleright : |\mathcal{M}| \times \mathbb{D} \to \mathbb{D}$
- Kleisli $\mathcal{B}_T$ $\rightsquigarrow$ coPara: $\mathrm{coPara}_{|\mathcal{M}|}(\mathbb{D})$

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GOAL

CONJECTURE.

Given an action $\triangleright : \mathbb{M} \times \mathbb{D} \to \mathbb{D}$ ($\mathbb{M}$ braided, $\mathbb{D}$ monoidal),
TFAE:

1. the action $\triangleright$ is lax monoidal
2. — " — bistrong and commutative
3. $\mathrm{colax}_{\mathbb{M}}(\mathbb{D})$ is monoidal (extending $\otimes$ of $\mathbb{D}$)
4. $\mathrm{colax}_{\mathbb{M}}(\mathbb{D})$ is "purely central" premonoidal.

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(PRE) MONOIDAL DOUBLE CATEGORIES

Monoidal p. cats.

Structure: 1D d. cat. + pseudo double functors $\otimes : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$, $I : \mathbb{1} \to \mathbb{D}$

+ invertible vertical transformations

$$\mathcal{L} : \otimes(\mathrm{Id} \times \otimes) \Rightarrow \otimes(\otimes \times \mathrm{Id})$$

$$\lambda : \otimes(I \times \mathrm{Id}) \Rightarrow \mathrm{Id}$$

$$\rho : \otimes(\mathrm{Id} \times I) \Rightarrow \mathrm{Id}$$

+ pentagonal and 3 triangular axioms.

PREMONOIDAL P. CATS.

(DF) $\mathbb{D}$ d. cat. + pseudodouble functors $A\times -$, $- \times B : \mathbb{D} \to \mathbb{D}$, $A,B \in \mathbb{D}$

s.t. $A \times B = B \times A$

+ invertible vertical transformations

$I : \mathbb{1} \to \mathbb{D}$

$\alpha_{A,B,-} : A\times(B\times -) \Rightarrow (A\times B)\times -$

$\alpha_{-,B,C} : -\times(B\times C) \Rightarrow (-\times B)\times C$

$\alpha_{A,-,C} : A\times(-\times C) \Rightarrow (A\times -)\times C$

$\lambda : I\times - \Rightarrow Id$

$\rho : - \times I \Rightarrow Id$

s.t. $\alpha_{A,B,C}, \lambda_A, \rho_A$ are "inversely central"

+ 4 pentagons + 6 triangles

HORIZONTAL

VERSIONS OF

THE DEFINITIONS

HORIZONTALLY MONOIDAL P. CATS.

$\mathbb{D}$ d. cat. + pseudo double functors $\otimes : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$

$I : \mathbb{1} \to \mathbb{D}$

+ horizontal pseudonatural equivalences

$\mathcal{L} : \otimes \times \otimes \Rightarrow \otimes(\otimes \times Id)$

$\lambda : \otimes(I \times Id) \Rightarrow Id$

$\rho : \otimes(Id \times I) \Rightarrow Id$

+ invertible horizontal modifications

$\Pi, \tilde{\chi}, \tilde{\delta}, \tilde{\mu}$

+ axioms ~(TA1)–(TA3) ("Π is a left and right normalized 4-cocycle").

HORIZONTALLY PREMONOIDAL P. CATS.

(DF) ID d.cat. + pseudodouble functors $A\times -$, $-\times B : \mathbb{D} \to \mathbb{D}$, $A,B \in \mathbb{D}$

s.t. $A\times B = B \times A$

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$\alpha_{A,-,C} : A\times(-\times C) \Rightarrow (A\times -)\times C$

$\lambda : I\times - \Rightarrow Id$

$\rho : - \times I \Rightarrow Id$

s.t. $\alpha_{A,B,C}$, λ_A, ρ_A are central

+ 4 pentagonators + 6 triangulators (10 horizontal modifications)

+ 3 axioms

LIFTING

VERTICAL STRUCTURES

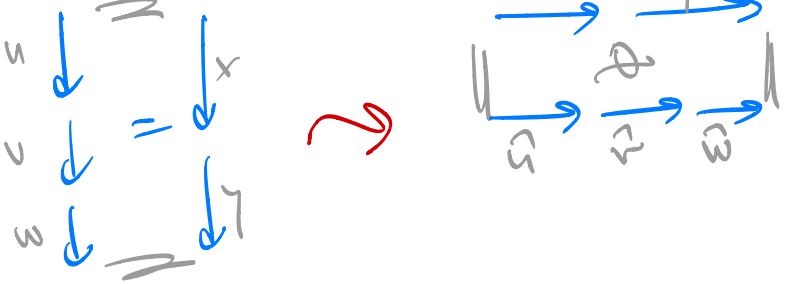
TO HORIZONTAL ONES

Assuming the corresponding 1V-cell category has companions :

(+ ifftable)

invertible vertical transformations ⇝ horizontal pseudofunctorial equivalences

identity vertical modifications ⇝ horizontal modifications



PARA GNSTRUCTION

ACTIONS OF DOUBLE CATEGORIES

Def. Let $\mathbb{M}$ be monoidally monoidal.

An action of $\mathbb{M}$ on $\mathbb{D}$ is given by

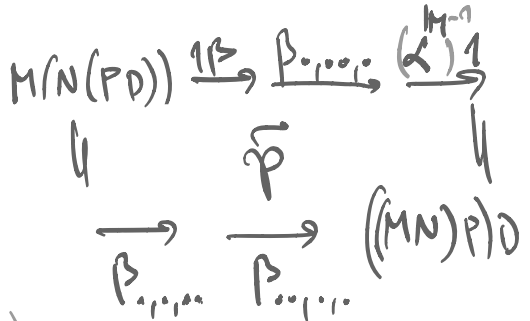
1) a pseudo double functor $\triangleright : \mathbb{M} \times \mathbb{D} \to \mathbb{D}$,

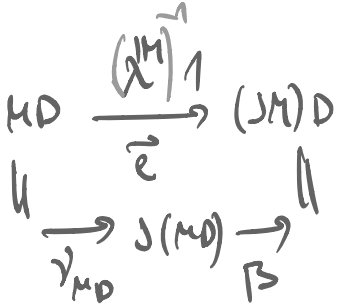
2) horiz. ps. nat. transformations (with components)

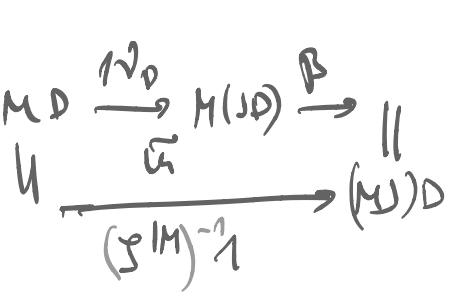
$$\beta_{M,N,D} : M \triangleright (N \triangleright D) \to (M \otimes N) \triangleright D$$

$$\nu_M : M \to \mathbb{J} \triangleright M$$

3) and horiz. modifications







4) satisfying 3 axs.

PARA

CONSTRUCTION

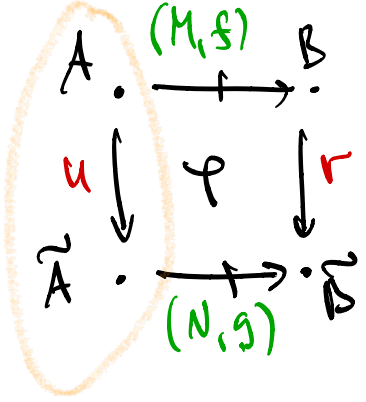
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Let $\mathbb{M}$ be horizontally monoidal with an action $\bullet : \mathbb{M} \times \mathbb{D} \to \mathbb{D}$ on $\mathbb{D}$.

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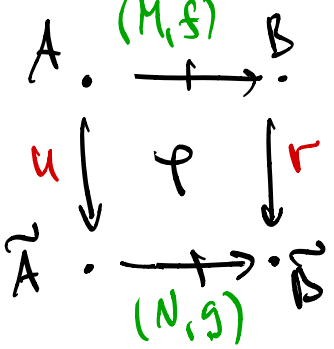


- the tight category is the one of $\mathbb{D}$

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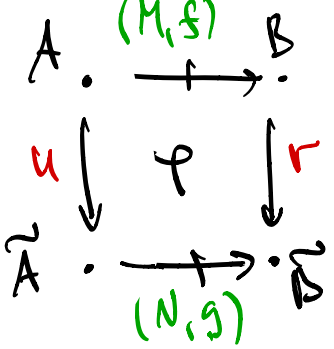
- the tight category is the one of $\mathbb{D}$
- loose cells are loose cells from $\mathbb{D}$ of the form:

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- the tight category is the one of $\mathbb{D}$
- loose cells are loose cells from $\mathbb{D}$ of the form:

$$f : M \triangleright A \to B$$

- 2-cells are given by 2-cells from $\mathbb{D}$:

$$\begin{array}{ccc} M \triangleright A & \xrightarrow{f} & B \\ {\scriptstyle m \triangleright u}\downarrow & \varphi & \downarrow{\scriptstyle r} \\ N \triangleright \tilde{A} & \xrightarrow[g]{} & \tilde{B} \end{array}$$

Composition

horizontal:

$g \circ f := $

$$(N \circ M) \triangleright A \xrightarrow{\vec{\alpha}} N \triangleright (M \triangleright A) \xrightarrow{1 \triangleright f} N \triangleright B \xrightarrow{g} C$$

$\Psi \circ \Phi := $

$$\begin{array}{ccccccc}
(NM)A & \xrightarrow{\vec{\alpha}} & N(MA) & \xrightarrow{1f} & NB & \xrightarrow{g} & C \\
{\scriptstyle (nm)u}\downarrow & \vec{\alpha}^{n,m,u} & \downarrow{\scriptstyle n(mu)} & \Phi & \downarrow{\scriptstyle nv} & \Psi & \downarrow{\scriptstyle w} \\
(\tilde{N}\tilde{M})\tilde{A} & \xrightarrow[\vec{\alpha}]{} & \tilde{N}(\tilde{M}\tilde{A}) & \xrightarrow[1h]{} & \tilde{N}\tilde{B} & \xrightarrow[z]{} & \tilde{C}
\end{array}$$

Vertical: stacking



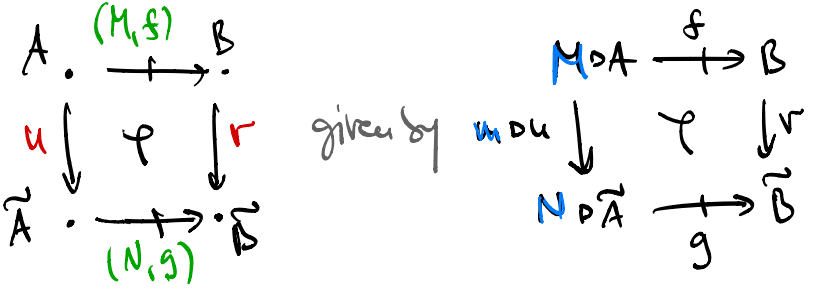
$$\begin{array}{ccc}
M \triangleright A & \xrightarrow{f} & B \\
{\scriptstyle m \triangleright u}\downarrow & \Phi & \downarrow{\scriptstyle v} \\
\tilde{M} \triangleright \tilde{A} & \xrightarrow[h]{} & \tilde{B}
\end{array}$$

HOW PAPA

GENERALIZES

KLEISLI

How Para generalizes Kleisli



$\mathbb{D} : \mathbb{M} \times \mathbb{D} \to \mathbb{D}$ $\longleftrightarrow$ $\mathbb{M} \to \mathrm{Fun}(\mathbb{D}, \mathbb{D})$

action d. functor — monoidal d. functor

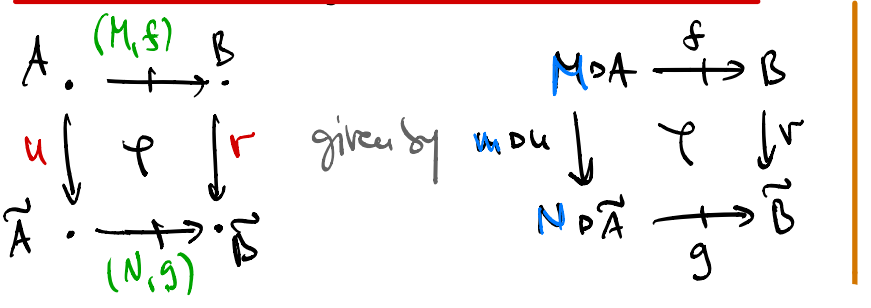
for $\mathbb{M} = \mathbb{X}$: $\rightsquigarrow$ endo d. functor $T : \mathbb{D} \to \mathbb{D}$

$\beta : M \odot (N \odot D) \to (M \cdot N) \odot D$ $\rightsquigarrow$ n.t. $T(T(D)) \to T(D)$

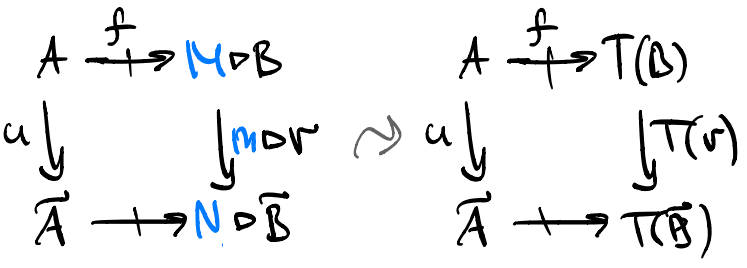
$\gamma : D \to J D$ $\rightsquigarrow$ $D \to T(D)$

$\vec{\rho}, \vec{\ell}, \vec{\alpha}$ $\rightsquigarrow$ associativity + 2 unit laws of a double monad

How Para generalizes Kleisli



dually: CoPara



Kleisli

$\odot : \mathbb{M} \times \mathbb{D} \to \mathbb{D}$ $\longleftrightarrow$ $\mathbb{M} \to \mathrm{Fun}(\mathbb{D}, \mathbb{D})$

action d. functor — monoidal d. functor

for $\mathbb{M} = \mathbb{1}$: $\rightsquigarrow$ endo d. functor $T : \mathbb{D} \to \mathbb{D}$

$\beta : M \odot (N \odot D) \to (M \otimes N) \odot D$ $\rightsquigarrow$ n.t. $T(T(D)) \to T(D)$

$\gamma : D \to \mathbb{J} D$ $\rightsquigarrow$ $D \to T(D)$

$\vec{\rho}, \vec{\ell}, \vec{\alpha}$ $\rightsquigarrow$ associativity + 2 unit laws of a double monad

TYPES OF ACTIONS:

- lax monoidal
- bistrong
- commutative

Lax monoidal actions

Let both $\mathbb{M}$ and $\mathbb{D}$ be lax monoidal, and $\mathbb{M}$ braided.

Lax monoidal actions

Let both $\mathbb{M}$ and $\mathbb{D}$ be lax monoidal, and $\mathbb{M}$ braided.

l.ps.nat.tr. $\phi : - \otimes - \Rightarrow (- \otimes -)\tau : \mathbb{M} \times \mathbb{M} \to \mathbb{M} \times \mathbb{M}$ + 2 lax. modif.



Lax monoidal actions

Let both $\mathbb{M}$ and $\mathbb{D}$ be horiz. monoidal, and $\mathbb{M}$ braided.

An action $\triangleright : \mathbb{M} \times \mathbb{D} \to \mathbb{D}$ is **lax monoidal** if:

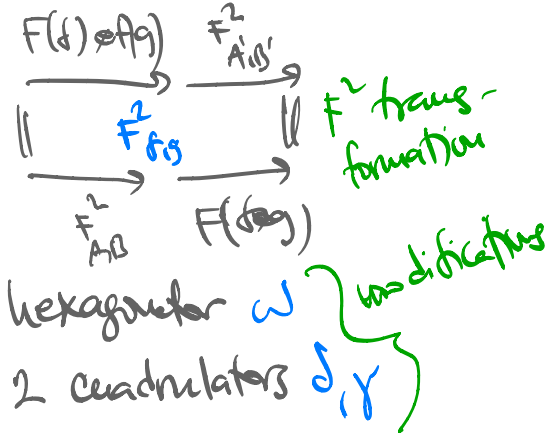
- the ps. double functor $\triangleright$ is lax monoidal
- the transf. β, ν are monoidal horiz. ps-trans. s.t. the horiz. modif. β^2, β^0, ν^2, ν^0 are invertible
- the horiz. modifications $\vec{\rho}, \vec{\ell}, \vec{\alpha}$ are monoidal.

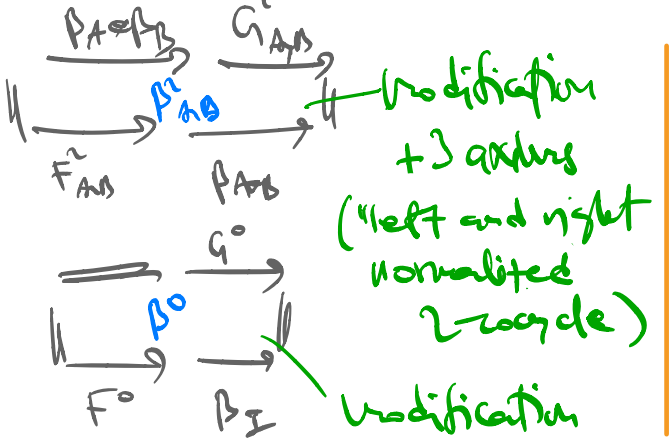
Lax monoidal actions

Let $\mathbb{M}$ and $\mathbb{D}$ be bicat. monoidal, and $\mathbb{M}$ braided.

An action $\triangleright : \mathbb{M} \times \mathbb{D} \to \mathbb{D}$ is **lax monoidal** if:

- the ps. dable functor $\triangleright$ is lax monoidal
- the transf. β, γ are monoidal horiz. ps-nat. transf. s.t. the horiz. modif. β^2, β^0, γ^2, γ^0 are invertible
- the horiz. modifications $\vec{\rho}, \vec{\ell}, \vec{\alpha}$ are monoidal.





$\vec{\rho}, \vec{\ell}, \vec{\alpha}$ obey 3 axioms

$\vec{\rho}$ is a left and right monoidated 4-cocycle

Bistrong and commutative actions

Both $\mathbb{M}, \mathbb{D}$ monoidal + action $\triangleright : \mathbb{M} \times \mathbb{D} \to \mathbb{D}$.

Left strength on $\triangleright$: lax ps.nat.tr. $t^{\mathbb{M}}_{A,B} : A \otimes (M \triangleright B) \to M \triangleright (A \otimes B)$ + mod. x, y, z, w + ax's

left strength $s^{\mathbb{M}}_{A,B} : (M \triangleright A) \otimes B \to M \triangleright (A \otimes B)$

bistrong action: h. mod. $\mathfrak{Q}$ (relatively s, t)

commutative action: h. mod. $\mathfrak{C}$ (relatively s, t, β, ϕ) (for $\mathbb{M}$ braided)

ϕ: braiding

FIRST EQUIVALENCE RESULT:



$\mathbb{D}$ is lax monoidal

$\mathbb{D}$ is bistrong and commutative

Additional assumptions: symmetric braiding $\mathbb{L}$

symmetric unit object of $\mathbb{M}$

MAIN EQUIVALENCE RESULT

MAIN THEOREM

For $\mathbb{M}, \mathbb{D}$ l.monoidal, $\mathbb{M}$ braided (with symmetric braiding and unit object), with lax action $\triangleright : \mathbb{M} \times \mathbb{D} \to \mathbb{D}$, TFAE:

1. the action $\triangleright$ is lax monoidal
2. — " — strong and commutative
3. $\mathrm{coPar}_{\mathbb{M}}(\mathbb{D})$ is monoidal (extending $\otimes$ of $\mathbb{D}$)
4. $\mathrm{coPar}_{\mathbb{M}}(\mathbb{D})$ is "purely central" premonoidal.

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Strategy of the PROOF.

(2 ⇒ 4)

(A2) For a "bistrong" double monad T on $\mathbb{D}$, the Kleisli d.cat $\mathbb{K}\ell_T(\mathbb{D})$ is promonoidal.

(TE) Bistrong lax action ⇝ $\mathrm{coPar}_{\mathbb{M}}(\mathbb{D})$ is promonoidal

Commutative action ⇒ "pure centrality". ✓

MAIN THEOREM

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Strategy of the PROOF.

$(4 \Rightarrow 3)$ (+1) There is a "purely central" premonoidal double cat. structure on $\mathbb{D} \iff$ there is a monoidal d. cat. structure on $\mathbb{D}$.

Horizontal version is true, too: $\mathbb{g}\text{-}\mathbb{Ps}^{st}_{lap}(\mathbb{D}\times\dots\times\mathbb{D};\mathbb{D}) \simeq \mathbb{Ps}_{lap}(\mathbb{D}\times\dots\times\mathbb{D},\mathbb{D})$

For the extension, the iso is given via $(-\otimes-)^\circ$.

MAIN THEOREM

For $\mathbb{M}, \mathbb{D}$ l.monoidal, $\mathbb{M}$ braided (with symmetric braiding and unit object), with lax action $\triangleright : \mathbb{M} \times \mathbb{D} \to \mathbb{D}$, TFAE:

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$P : \mathbb{D} \to \mathrm{coPar}_{\mathbb{M}}(\mathbb{D})$

$f \mapsto \nu f$

$A \xrightarrow{f} B \xrightarrow{\nu_B} J \triangleright B$

Strategy of the Proof.

(3 ⇒ 2) Extension (i.e.) $P(-) \otimes^P P(-) \Rightarrow P(- \otimes -)$ induces left and right strength.

The monoidal str. of $\mathrm{coPar}_{\mathbb{M}}(\mathbb{D})$ induces a left and right action of $\mathbb{D}$ on $\mathrm{coPar}_{\mathbb{M}}(\mathbb{D})$.

The associativity of the action induces an equivalence $\kappa : (- \triangleright P(-)) \circ - \Rightarrow - \triangleright (P(-) \circ -)$.

One defines $\gamma := \kappa_{1,1',1}$ and $c := \gamma \dfrac{\lambda x \otimes^P}{\alpha \circ \lambda x \otimes^P}$. ⊠

THANK YOU

FOR

FOLLOWING!