FROM ZERO-DIMENSIONAL TO CONTINUOUS DUALITIES

A DOUBLE-CATEGORICAL ACCOUNT

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SLIDES: HACKMD.IO/P/L8TPPJXLRBEZYFH1GG-EOW

OVERVIEW

1. **Duality theory**: Priestley duality
2. **Topology**: quotienting Cantor space
3. **Category theory**: splitting idempotents
4. **Double categories**: functions and relations
5. **Domain theory**: round filters as adjoint relations

1. DUALITY THEORY

STONE AND PRIESTLEY DUALITY

Stone spaces $\simeq$ Boolean algebras$^{\mathrm{op}}$

$$X \mapsto \mathsf{Stone}(X, 2) = \text{clopens} \qquad B \mapsto \mathsf{BA}(B, 2) = \text{ultrafilters}$$

Zero-dimensional: the topology is generated by clopens.

Priestley spaces $\simeq$ distributive lattices$^{\mathrm{op}}$

$$X \mapsto \mathsf{Pri}(X, 2) = \text{clopen upsets} \qquad A \mapsto \mathsf{DL}(A, 2) = \text{prime filters}$$

Now $2 = \{0 < 1\}$ is ordered. Both directions: homming into 2.

ASIDE: IND AND PRO COMPLETIONS

Finite structures:

$$\mathsf{FinSet} \simeq \mathsf{FinBA}^{\mathrm{op}} \qquad \mathsf{FinPos} \simeq \mathsf{FinDL}^{\mathrm{op}}$$

$$f : X \to Y \;\mapsto\; f^{-1} : 2^Y \to 2^X$$

Pro on spaces, Ind on algebras:

$$\mathsf{Stone} \simeq \mathsf{Pro}(\mathsf{FinSet}) \simeq \mathsf{Ind}(\mathsf{FinBA})^{\mathrm{op}} \simeq \mathsf{BA}^{\mathrm{op}}$$

$$\mathsf{Pri} \simeq \mathsf{Pro}(\mathsf{FinPos}) \simeq \mathsf{Ind}(\mathsf{FinDL})^{\mathrm{op}} \simeq \mathsf{DL}^{\mathrm{op}}$$

DUALITY IS ORDER-ENRICHED

Maps are ordered pointwise.

$\mathsf{Pri} \simeq \mathsf{DL}^{\mathrm{op}}$ reverses 1-cells but not 2-cells.

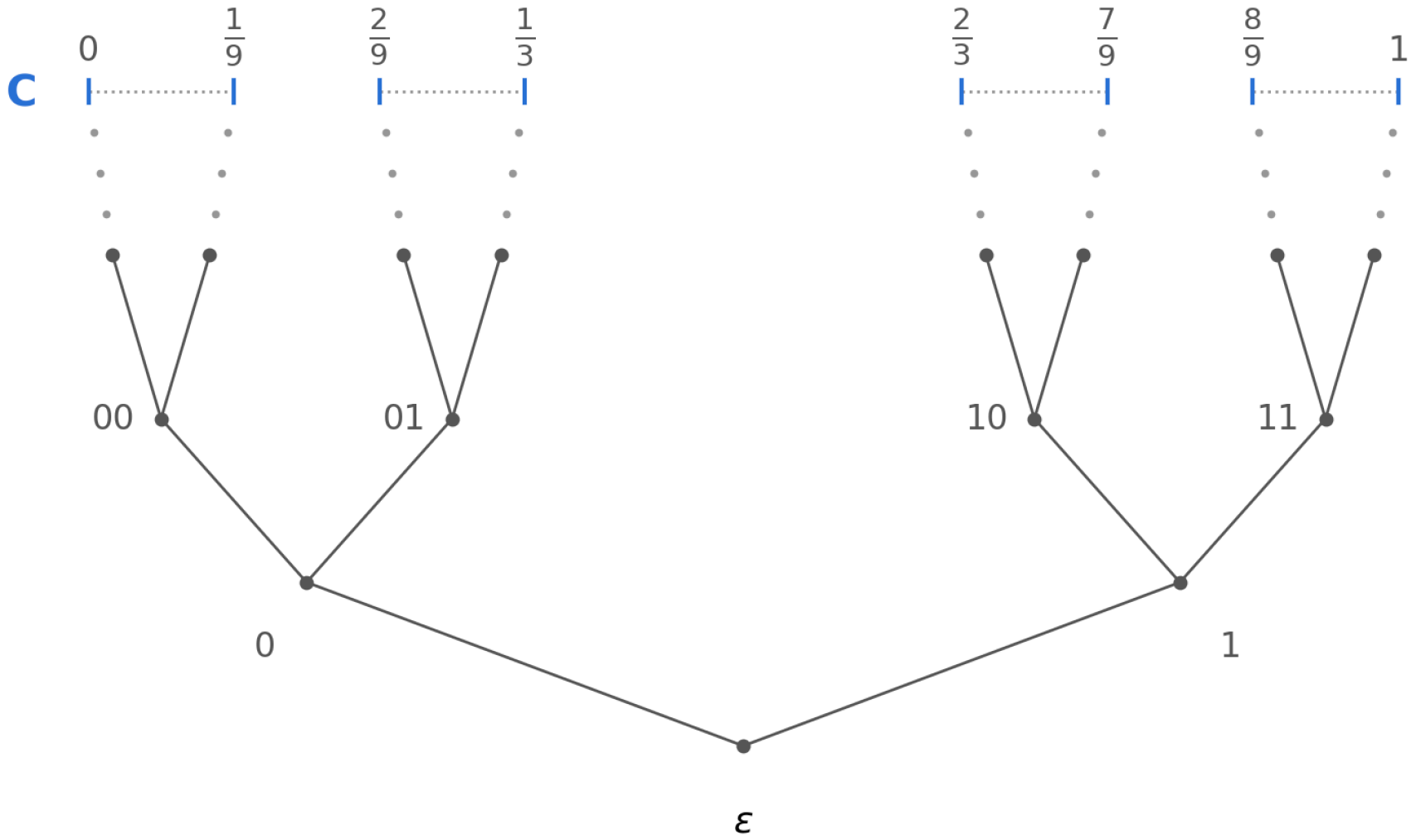
$$f \leq g \quad \Longleftrightarrow \quad 2^f \leq 2^g$$

where $2^f = f^{-1}$.

2. TOPOLOGY

CANTOR SPACE

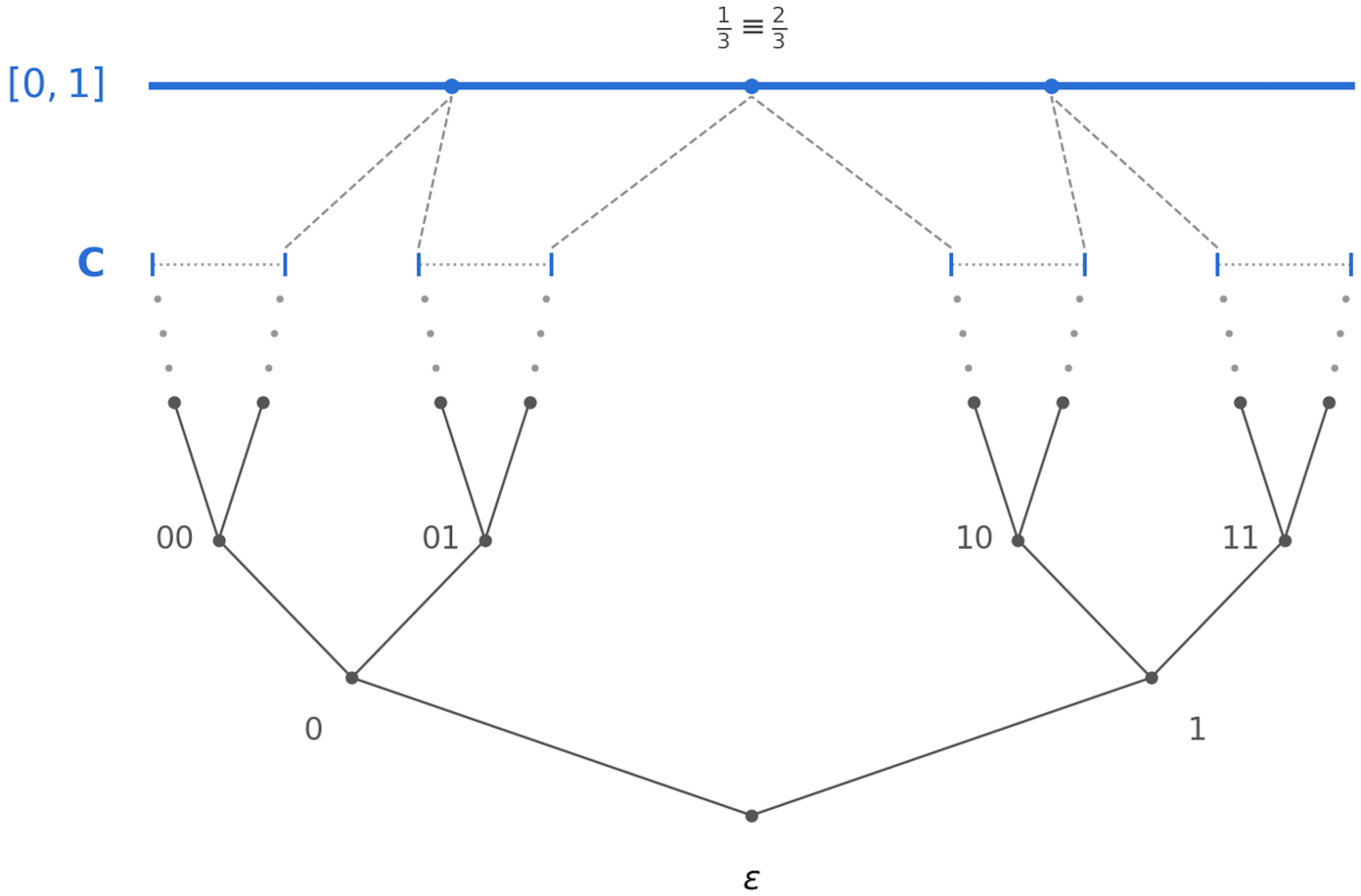
$\mathbf{C} = \{0,1\}^\omega$, the middle-thirds set in $[0,1]$.



UNIT INTERVAL AS QUOTIENT OF CANTOR SPACE

$\equiv$ glues the two endpoints of each gap: $0111\ldots \equiv 1000\ldots$

$$\mathbf{C}/\equiv \;\cong\; [0,1]$$



THE DUAL OF THE UNIT INTERVAL?

The dual $2^{\mathbf{C}}$ of Cantor space $\mathbf{C}$ is the free Boolean algebra on countably many generators.

The unit interval can be represented as $(\mathbf{C}, \equiv)$.

What is the dual of $\equiv$?

Btw, $\equiv$ is idempotent and above identity (a monad).

THE DUAL OF A RELATION?

$R : X \nrightarrow Y$ dualises to $2^R : 2^X \nrightarrow 2^Y$:

$$a \; 2^R \; b \quad \Longleftrightarrow \quad R[a] \subseteq b$$

"all R-successors of a are in b" (modal logic, weakest preconditions)

The dual of the identity is $\subseteq$.

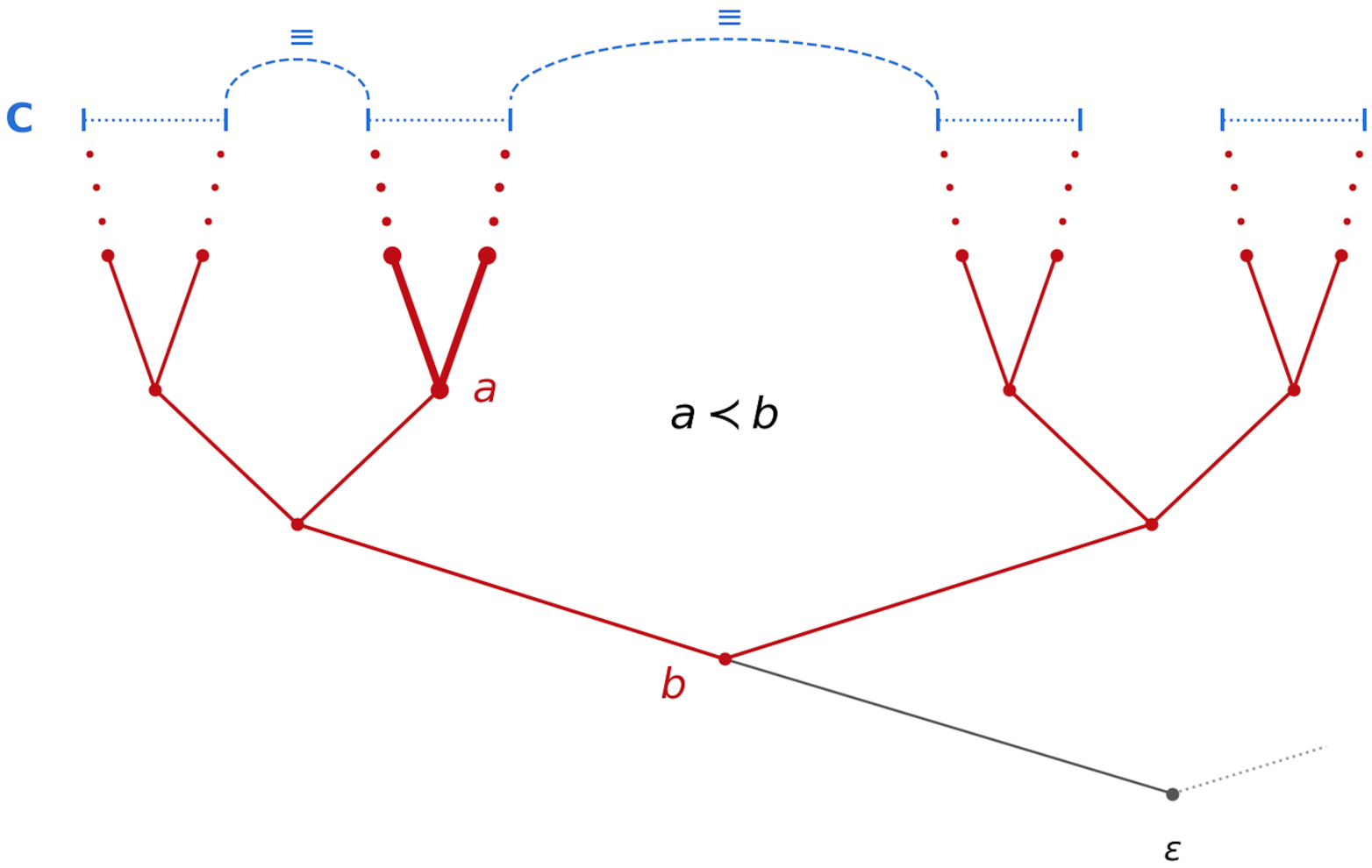
Duality reverses the order on relations:

spaces	algebras
above identity	below identity
transitive	interpolative
idempotent	idempotent
monad	comonad

THE DUAL OF $\equiv$ IS WAY-BELOW ($\prec$)

$2^{\mathbf{C}}$: the clopens of $\mathbf{C}$, or the basic opens of the product topology on $\{0,1\}^{\omega}$.

$$a \prec b \iff \equiv[a] \subseteq b \iff \text{(non-trivial) endpoints of } b \text{ are not in } a$$



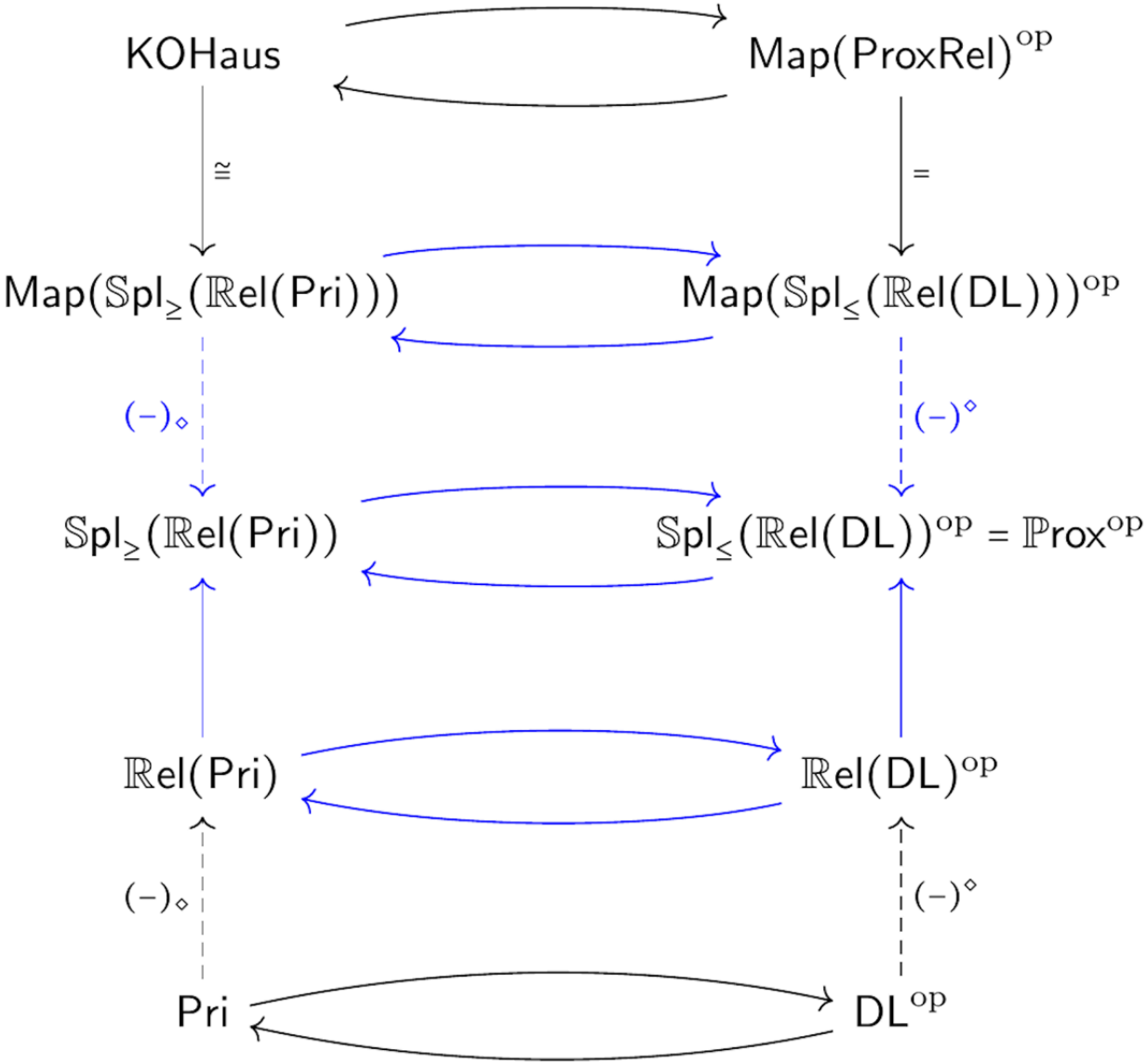
STEPPING STONE

Space-side references:

- Every compact Hausdorff space is a quotient of a Stone space (Manes 1969).
- Every continuous lattice is a quotient of an algebraic lattice (Scott 1972).
- Every compact ordered space is a quotient of a Priestley space (Flagg 1997).
- Compact ordered spaces are the exact completion of Priestley spaces. (Aravantinos-Sotiropoulos 2022).

Question: What is a good framework to extend zero dimensional dualities to continuous dualities?

3. CATEGORY THEORY


KOHaus
Map(ProxRel)^op
≅
=
Map(Spl≥(Rel(Pri)))
Map(Spl≤(Rel(DL)))^op
(−)⋄
(−)⋄
Spl≥(Rel(Pri))
Spl≤(Rel(DL))^op = Prox^op
Rel(Pri)
Rel(DL)^op
(−)⋄
(−)⋄
Pri
DL^op

4. DOUBLE CATEGORIES

WHY DOUBLE CATEGORIES?

- Duality needs the functions: $\mathsf{Pri} \simeq \mathsf{DL}^{\mathrm{op}}$.
- Quotients and proximities are relations: $\equiv$, $\prec$.
- A double category holds both functions (arrows) and relations (proarrows).
- We only need strict ultraflat double categories (Grandis 2019):
- Arrow and proarrow projections are order-enriched categories.

THE DOUBLE CATEGORY OF RELATIONS

$\mathbb{R}\mathsf{el}(\mathcal{C})$: arrows are functions, proarrows $\nrightarrow$ are $\mathcal{C}$-relations, squares are

$$\begin{array}{ccc} C & \xrightarrow{f} & C' \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R'} \\ D & \underset{g}{\to} & D' \end{array} \qquad\qquad c\ R\ d \ \Rightarrow\ f(c)\ R'\ g(d)$$

Strict and flat: being a square is a property.

THE DUAL OF A RELATION, AS A SQUARE

Writing 1 for the free distributive lattice over 1 generator, $\mathbf{Pri} \simeq \mathbf{DL}^{\mathrm{op}}$ implies

$$\begin{array}{ccc} X & \xrightarrow{a} & 2 \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ Y & \xrightarrow[b]{} & 2 \end{array} \qquad\qquad \begin{array}{ccc} 2^X & \xleftarrow{a} & 1 \\ {\scriptstyle 2^R}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ 2^Y & \xleftarrow[b]{} & 1 \end{array}$$

$$x\ R\ y \Rightarrow a(x) \leq b(y) \quad \Longleftrightarrow \quad a\ 2^R\ b$$

$$x\ R\ y \Rightarrow a(x) \leq b(y) \quad \Longleftrightarrow \quad R[a] \subseteq b$$

DUALITY

$$\mathbb{R}\mathsf{el}(\mathsf{Pri}) \simeq \mathbb{R}\mathsf{el}(\mathsf{DL})^{\mathrm{op}}$$

$^{\mathrm{op}}$ reverses arrows and squares but not proarrows.

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle S} \\ B & \xrightarrow[g]{} & B' \end{array} \qquad \begin{array}{ccc} 2^A & \xleftarrow{2^f} & 2^{A'} \\ {\scriptstyle 2^R}\downarrow & & \downarrow{\scriptstyle 2^S} \\ 2^B & \xleftarrow[2^g]{} & 2^{B'} \end{array}$$

COMPANIONS, CONJOINTS, MAPS

$$f_\diamond = \{(c,d) \mid f(c) \leq d\} \qquad f^\diamond = \{(d,c) \mid d \leq f(c)\} \qquad f_\diamond \dashv f^\diamond$$

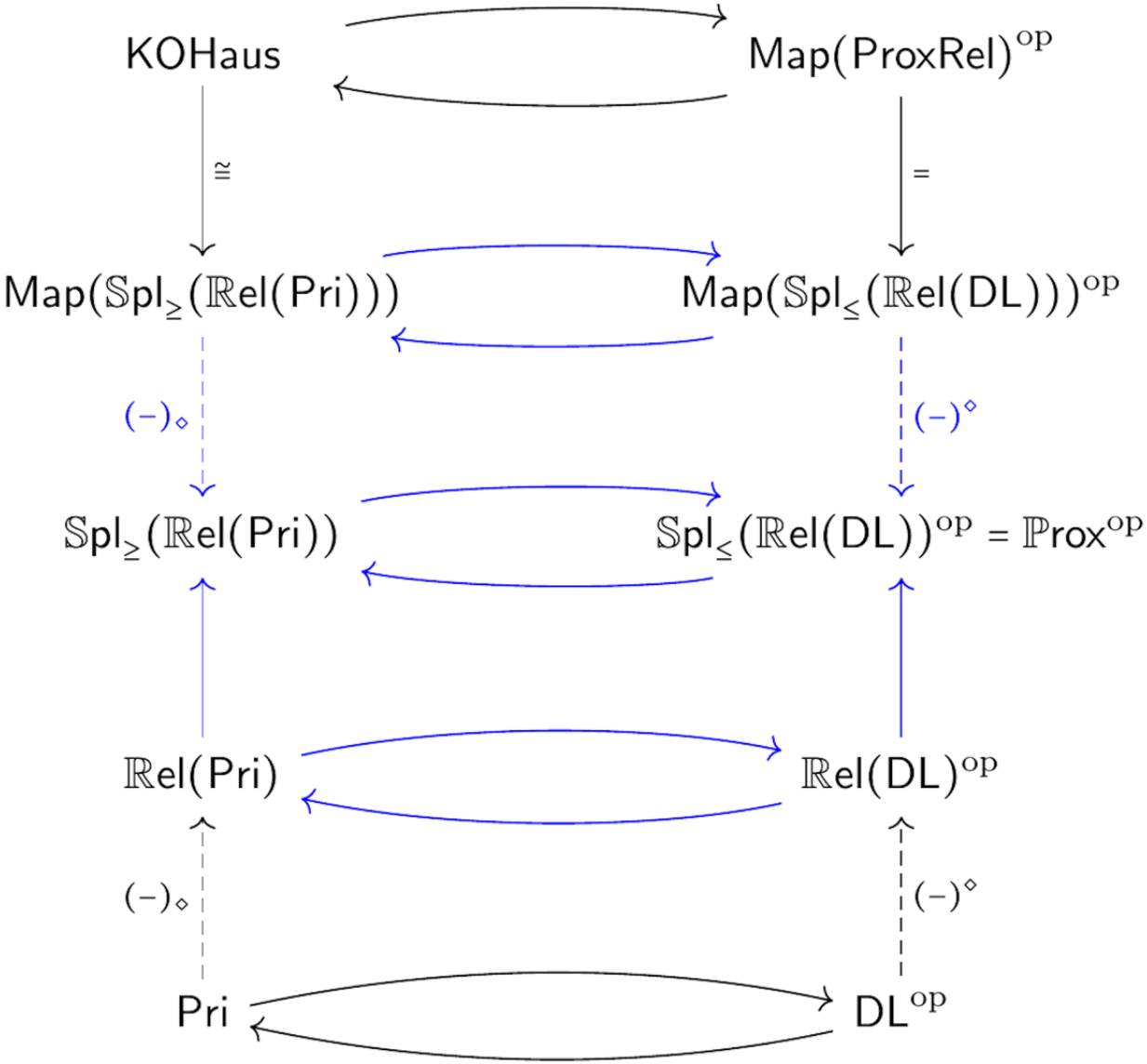
Functions are left adjoint relations.

$\mathsf{Map}(\mathbb{C})$ keeps the left adjoint proarrows of $\mathbb{C}$.

$$(-)_\diamond : \mathsf{Pri} \longrightarrow \mathsf{Rel}(\mathsf{Pri})$$

$$(-)^\diamond : \mathsf{DL}^{\mathrm{op}} \longrightarrow \mathsf{Rel}(\mathsf{DL})$$

5. DOMAIN THEORY

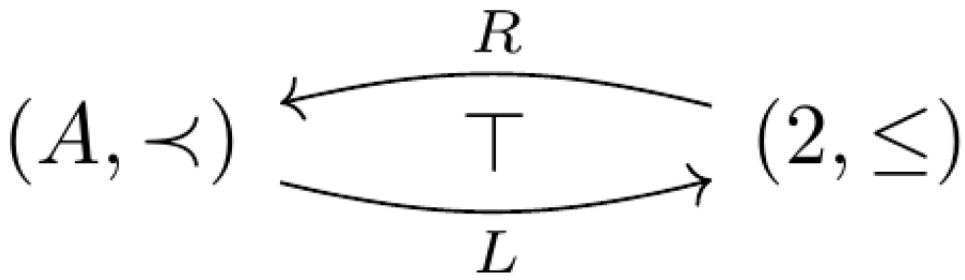

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POINTS ARE ADJOINT RELATIONS

Priestley: points are DL-morphism $A \to 2$.

Now: points are left-adjoint Prox-relations

$$L \dashv R \;:\; (A, \prec) \nrightarrow (2, \leq)$$



$(L0,\ 1R)$ is a cut:

- $L0$ is a round ideal, $1R$ a round filter
- disjoint: $L0 \cap 1R = \emptyset$
- tight: $a \prec a' \;\Rightarrow\; a \in L0$ or $a' \in 1R$

REFERENCES

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- Manes 1969
- Priestley 1970
- Scott 1972
- Smyth 1992
- Jung, Sünderhauf 1996
- Flagg 1997
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- Grandis, Paré 2004
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