DOUBLY WEAK DOUBLE CATEGORIES

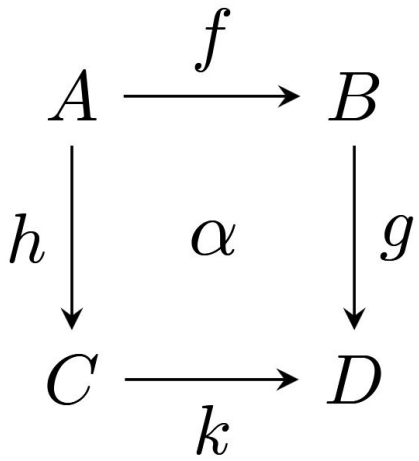
with Aaron Fairbanks

joint work with Michael Shulman

arXiv:2506.23651

Review

A **double category** is a structure like a 2-category but with two sorts of arrow (**horizontal** and **vertical**) and with square-shaped **cells** between them.



Review

Double categories are typically **weak** in one direction.

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- Horizontal arrow composition is *weakly* associative.

(Like in a *weak* 2-category, a.k.a. bicategory.)

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- Horizontal arrow composition is *weakly* associative.
  (Like in a *weak* 2-category, a.k.a. bicategory.)
- Vertical arrow composition is *strictly* associative.
  (Like in a *strict* 2-category.)

Review

Double categories are typically **weak** in one direction.

- Horizontal arrow composition is *weakly* associative.
  (Like in a *weak* 2-category, a.k.a. bicategory.)

- Vertical arrow composition is *strictly* associative.
  (Like in a *strict* 2-category.)

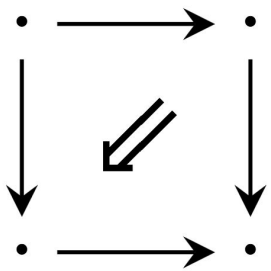
These are also called **pseudo double categories**.

The problem of doubly weak double categories

In practice, we find several examples of “double categories that are weak in both directions”.

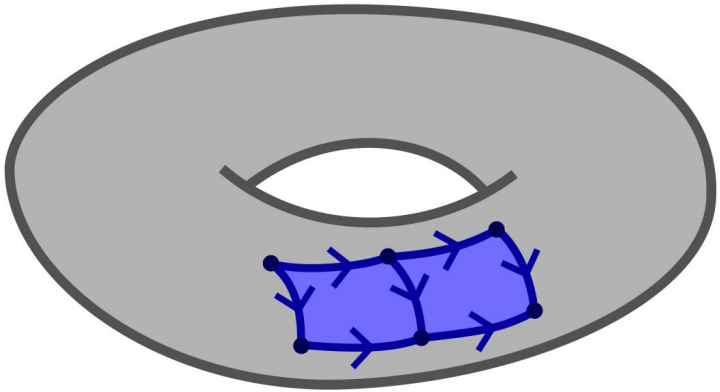
Examples

The squares in a strict 2-category form a strict double category.



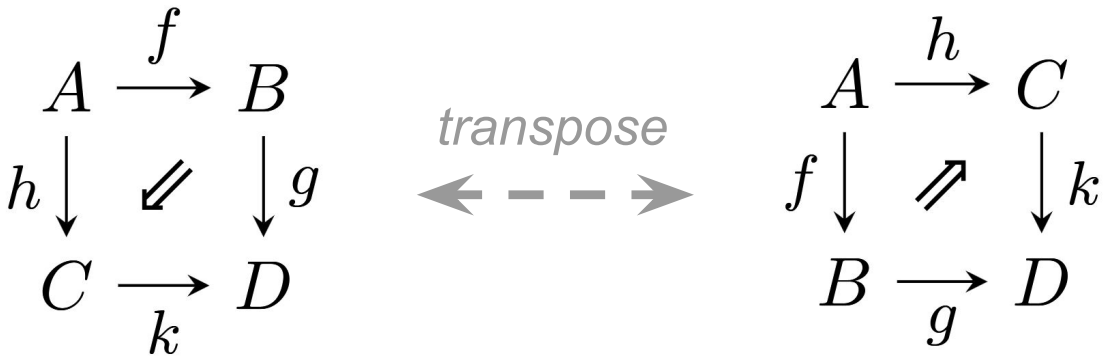
The squares in a bicategory should form a doubly weak double category.

A topological space should have a “fundamental doubly weak double groupoid”.



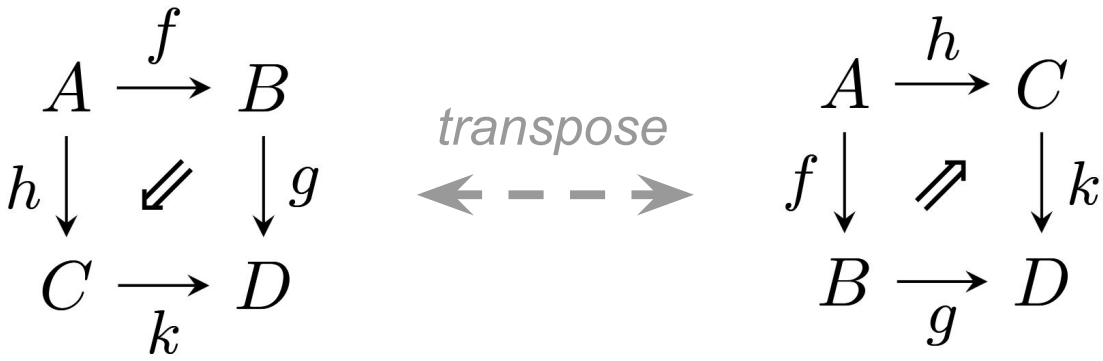
(Defining composition of square cells in a topological space is arguably more primitive than that of globular cells.)

The *transpose* of a pseudo double category does not exist as a pseudo double category.



It should exist as a doubly weak double category.

The *transpose* of a pseudo double category does not exist as a pseudo double category.



It should exist as a doubly weak double category.

The transpose of any doubly weak double category should still be a doubly weak double category.

If C and D are strict 2-categories,
we have a strict “Hom” double category:

- objects are functors $C \longrightarrow D$
- arrows are lax and colax transformations
- cells are modifications

If D is a bicategory, this should be
a doubly weak double category.

Similarly…

If T is a 2-monad on a strict 2-category, we have a strict double category T-Alg:

- objects are T-algebras
- arrows are lax and colax morphisms
- cells are 2-cells with compatibility law

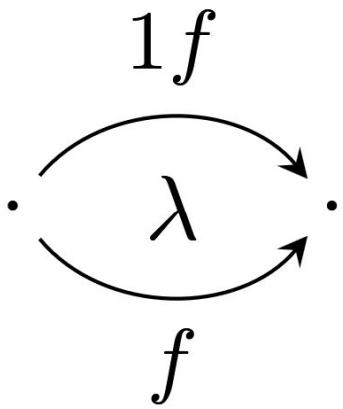
If T is a pseudomonad on a bicategory, this should be a doubly weak double category.

Is there a definition of doubly weak double category?

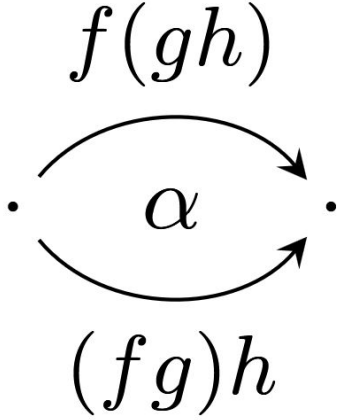
Problem

Trying to generalize the familiar definition leads into deadlock around coherence conditions…

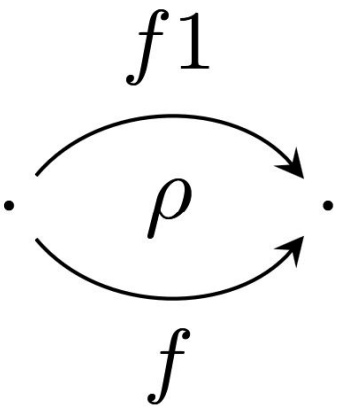
Coherence cells in a bicategory (globular)



left unitor

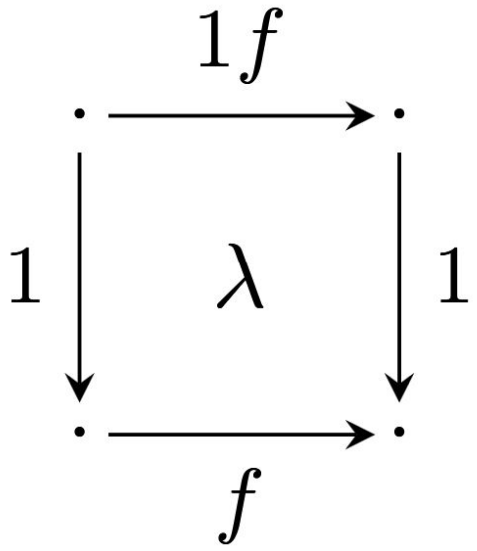


associator

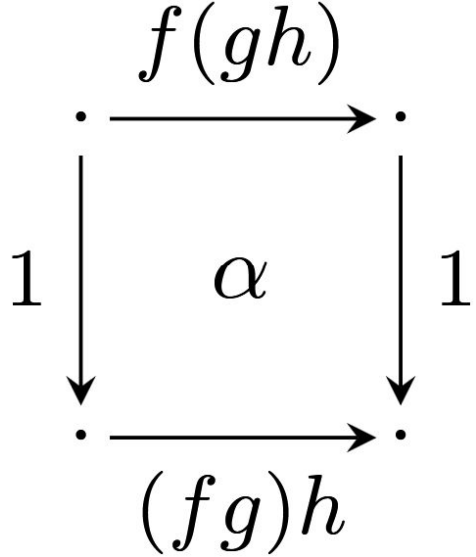


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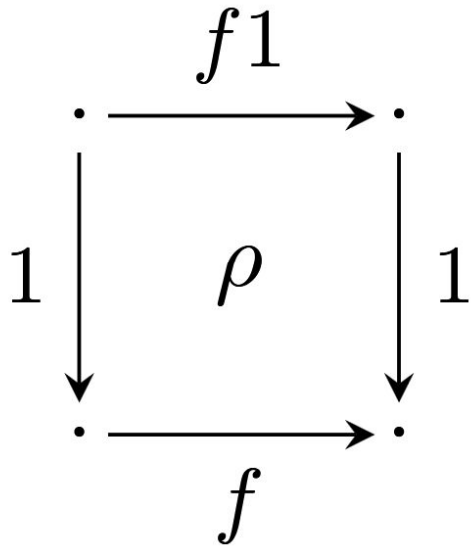
Coherence cells in a double category (square)



left unitor

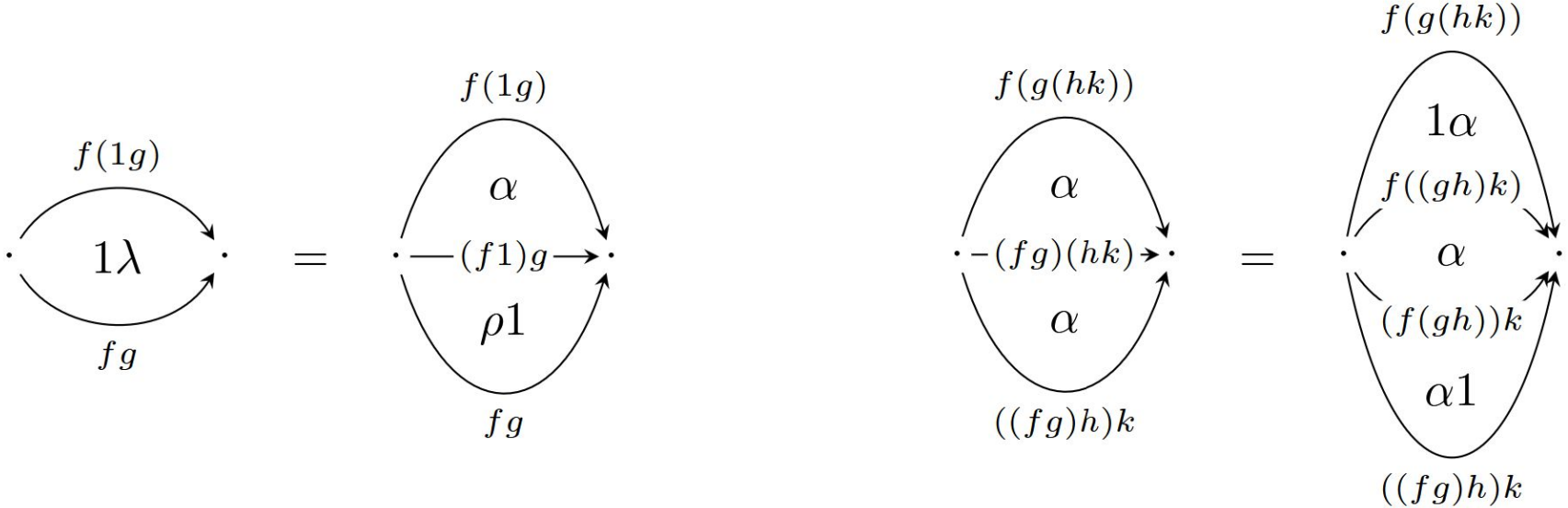


associator



right unitor

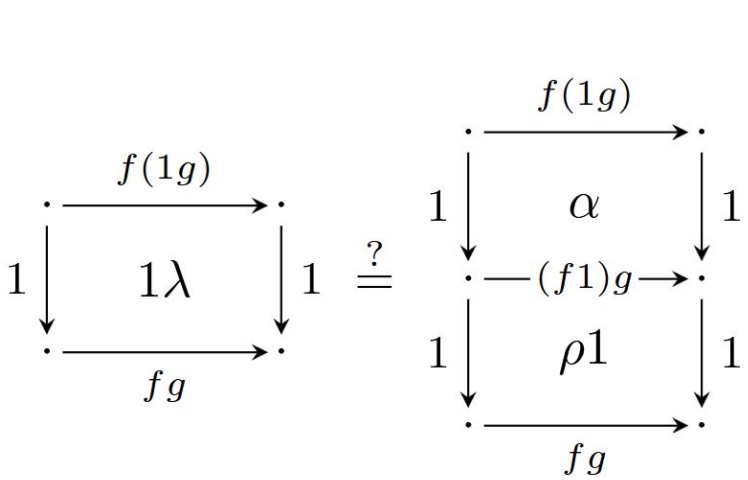
Coherence laws in a bicategory



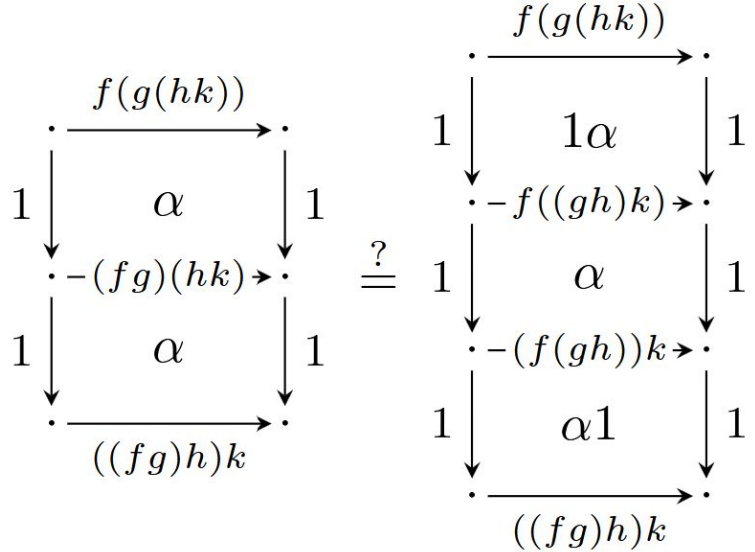
triangle equation

pentagon equation

Coherence laws in a doubly weak double category?

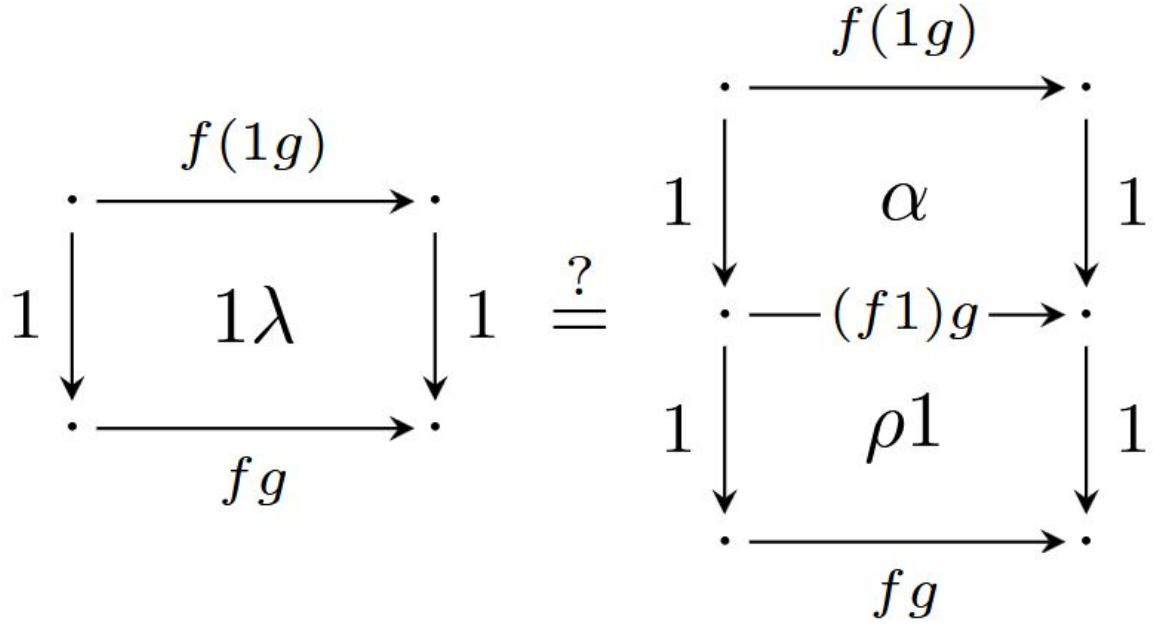


triangle equation?



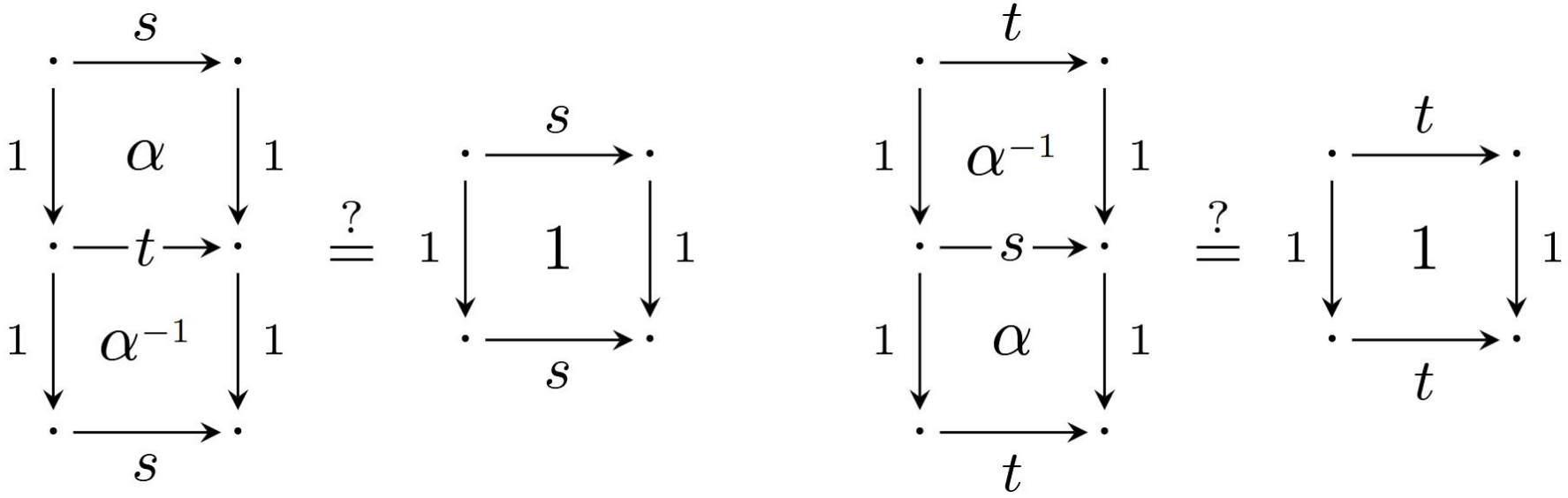
pentagon equation?

The equations don’t make sense anymore.

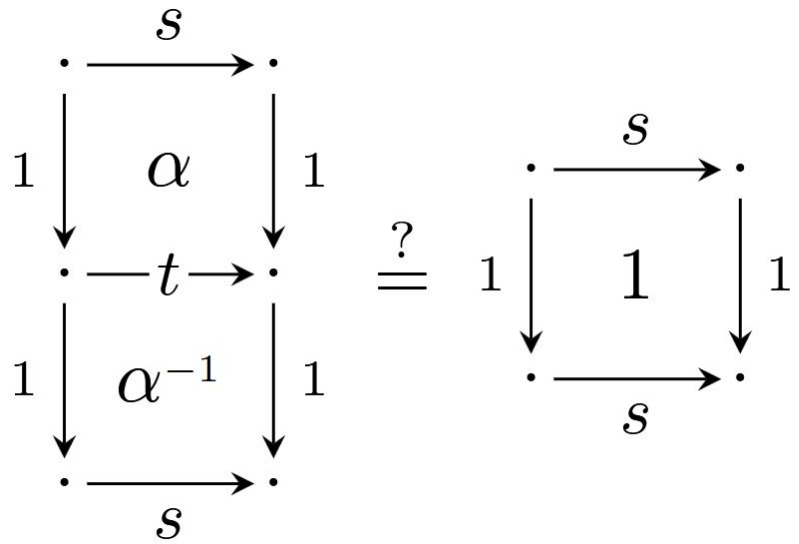


(cells have different vertical boundaries)

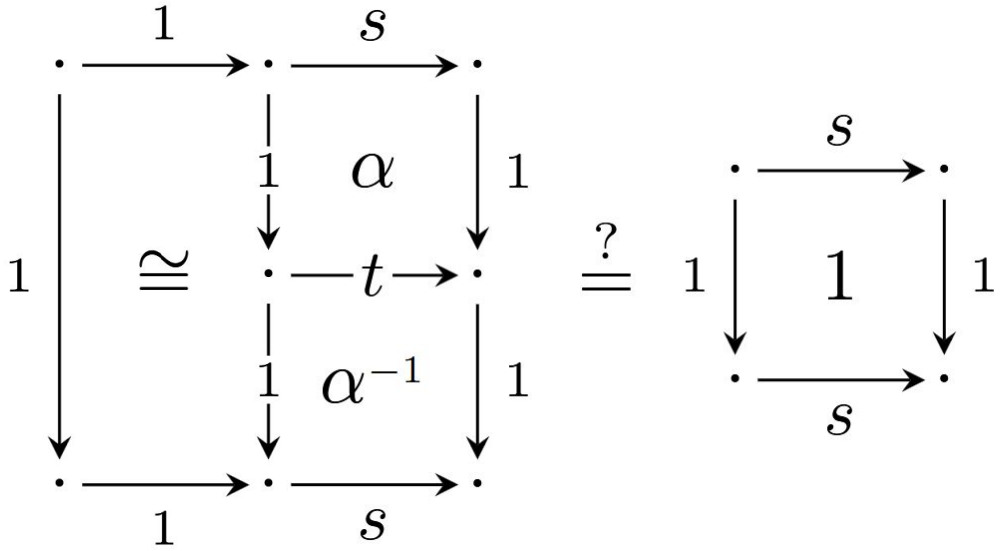
Even the definition of “inverse” doesn’t make sense anymore.



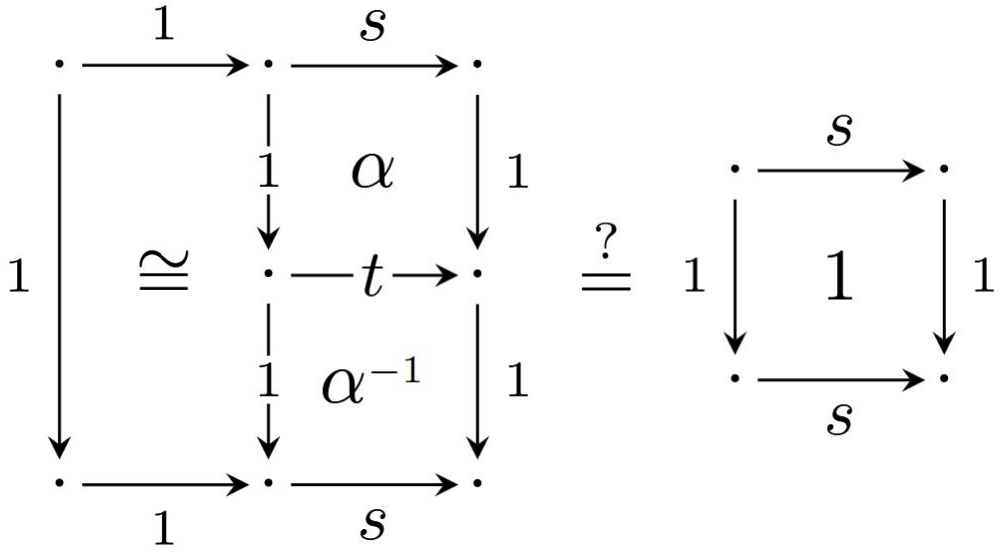
Applying coherence cells to fix vertical boundaries would mess up the horizontal boundaries…



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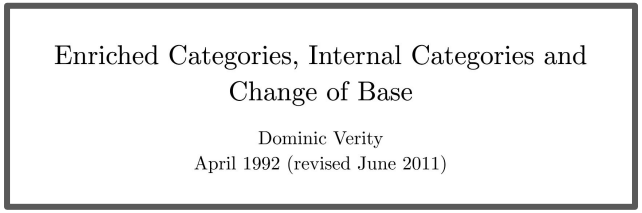
Applying coherence cells to fix vertical boundaries would mess up the horizontal boundaries…



…and so on, *ad infinitum*.

At least two solutions have been proposed.

Verity proposed *double bicategories*
in his 1992 thesis


Enriched Categories, Internal Categories and Change of Base

Dominic Verity
April 1992 (revised June 2011)

Garner proposed *cubical bicategories*
in a 2010 message to the Categories List


Re: Composing modifications

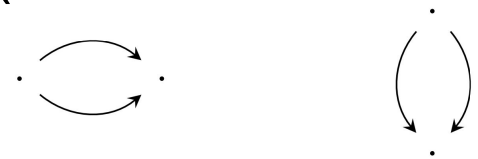
Richard Garner **5 Mar 2010**

Verity's idea

Include bicategories
as part of the definition
(in addition to square cells).

Verity double bicategory

- Two bicategories (“horizontal” / “vertical”) sharing objects



- Square cells with “composition” and “identity” operations



- Operations where bicategory cells “act” on square cells



- Long list of axioms

Verity double bicategory

Pros	Cons

Verity double bicategory

Pros

- Describes examples occurring in practice

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- Has underlying vertical / horizontal bicategories

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- Complicated definition (seems *ad hoc*?)
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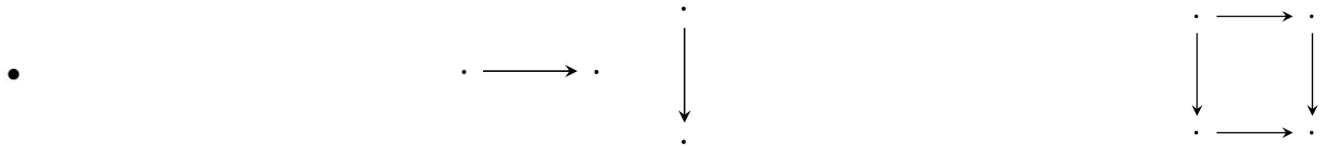
(for example)

Garner's idea

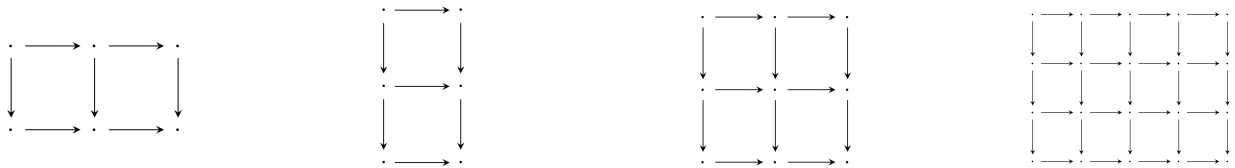
Allow every reasonable way of composing squares, and impose every reasonable axiom.

Garner cubical bicategory

- Objects, horizontal & vertical arrows, and squares



- Usual composition and identity operations for arrows
- Composition operation for every possible grid of squares



and every possible way of composing up its boundaries

- General associativity laws for grid compositions

Garner cubical bicategory

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Garner cubical bicategory

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Garner cubical bicategory

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- Definition is “natural” (not *ad hoc* seeming)

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(for example)

Is there a definition of doubly weak double category that solves these problems?

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Yes.

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doubly weak double category
that solves these problems?

Yes.

From a suitable perspective, it is
precisely the double-categorical analogue
of a weak 2-category (bicategory).

Plan

Give an alternative, equivalent definition of weak 2-category, which generalizes well to double categories.

Rethinking bicategories

Idea

A weak 2-category is
like a strict 2-category but
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Let’s make this idea precise.

An **implicit 2-category** consists of…

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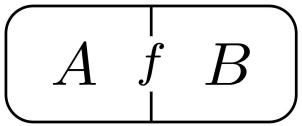
- A set of **objects**

An **implicit 2-category** consists of…

- A set of **objects**
- For all pairs of objects *A* and *B*,
  a set of **arrows** from *A* to *B*

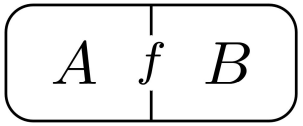
$$A \xrightarrow{f} B$$

or

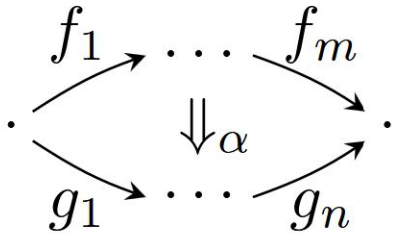


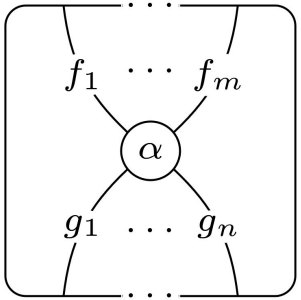
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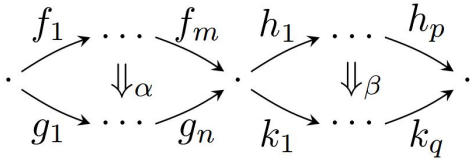
- For all parallel *paths* of arrows *f* and *g*,
  a set of **2-cells** from *f* to *g*

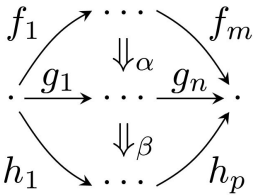


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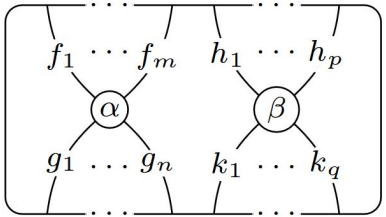
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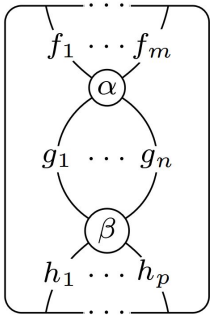
- **Composition** operations on 2-cells (horizontal and vertical)





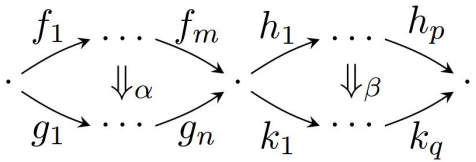
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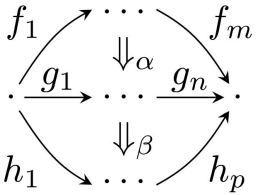




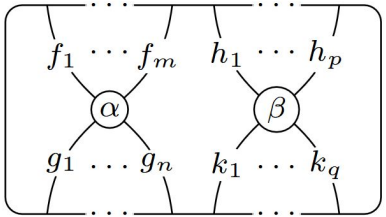
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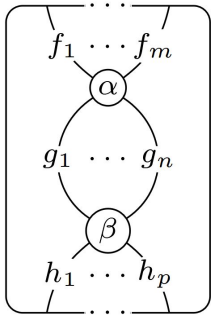
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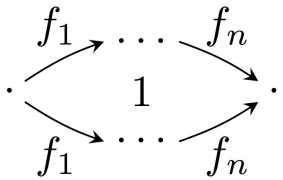


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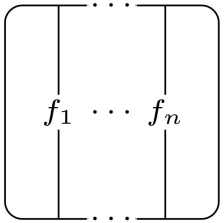




- **Identity** 2-cells

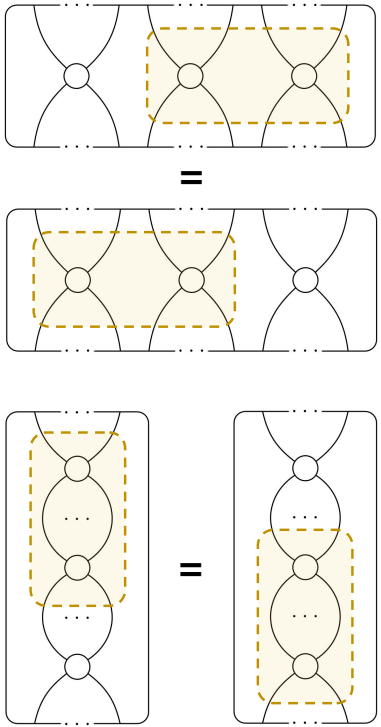


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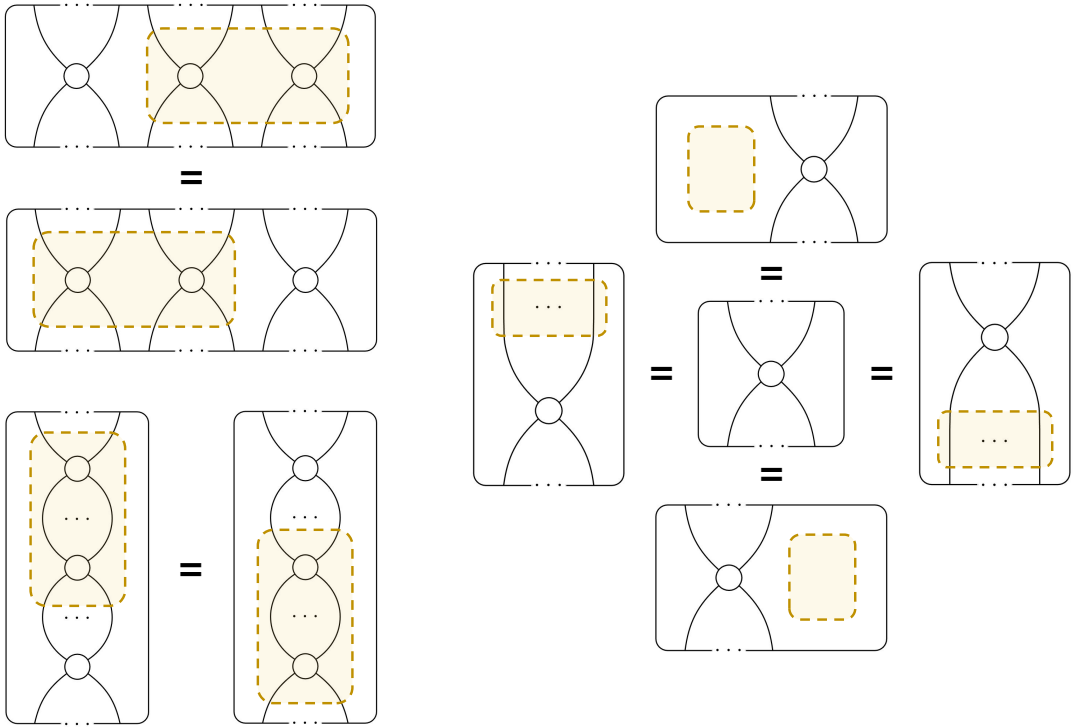
An **implicit 2-category** consists of…

- **Associativity**



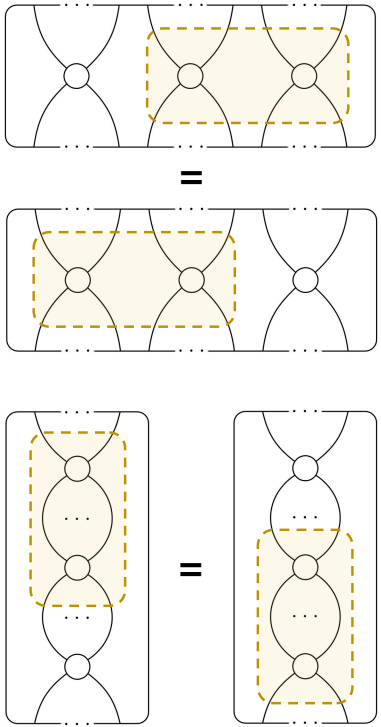
An **implicit 2-category** consists of…

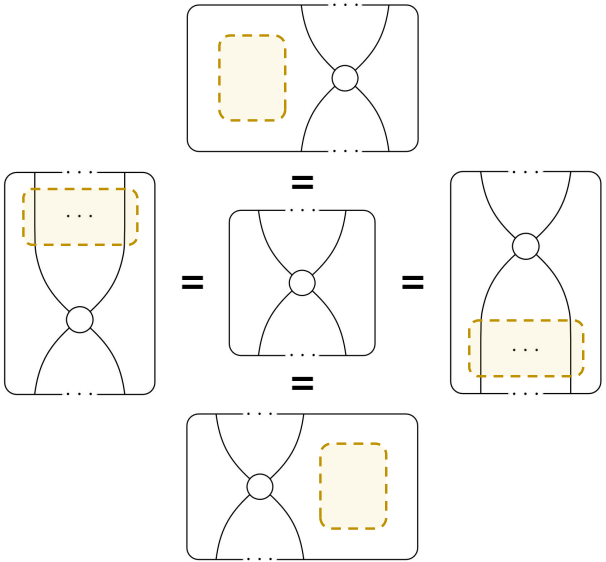
- **Associativity**, **unit**

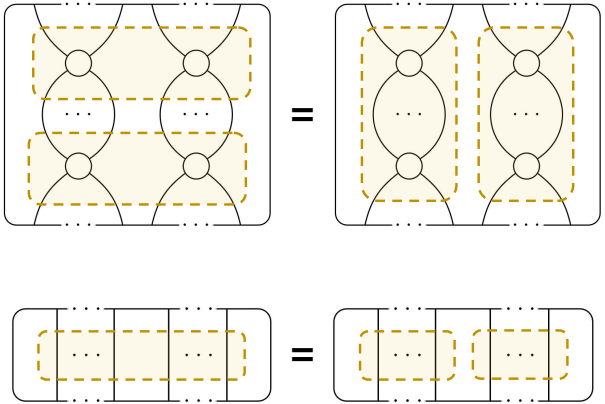


An **implicit 2-category** consists of…

- **Associativity**, **unit**, and **interchange** laws







One-sentence definition:

An **implicit 2-category**
is a strict 2-category
whose underlying category
of arrows is free.

A **functor** of implicit 2-categories

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- maps objects to objects

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Implicit 2-categories are described by an essentially algebraic theory.

(Like categories are.)

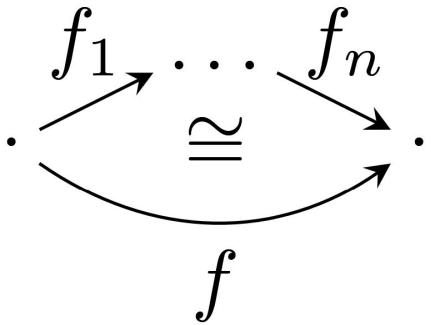
Functors of implicit 2-categories are homomorphisms of algebraic structure.

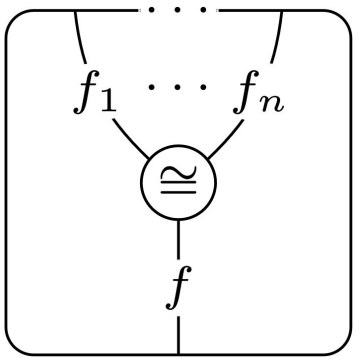
One-sentence definitions:

An **implicit 2-category** is a strict 2-category whose underlying category of arrows is free.

An **implicit 2-category functor** a strict 2-functor that sends generating arrows to generating arrows.

An implicit 2-category is **representable** if each path of arrows is isomorphic to a single arrow.

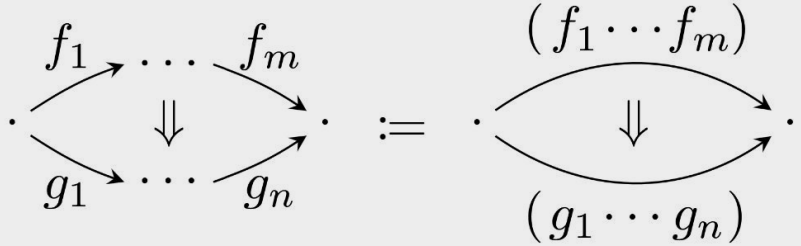




Every bicategory has an **underlying** implicit 2-category, which is representable.

The objects and arrows are the same.

The 2-cells in the implicit 2-category are given by 2-cells between composite arrows in the bicategory.



[in implicit 2-cat] [in bicategory]

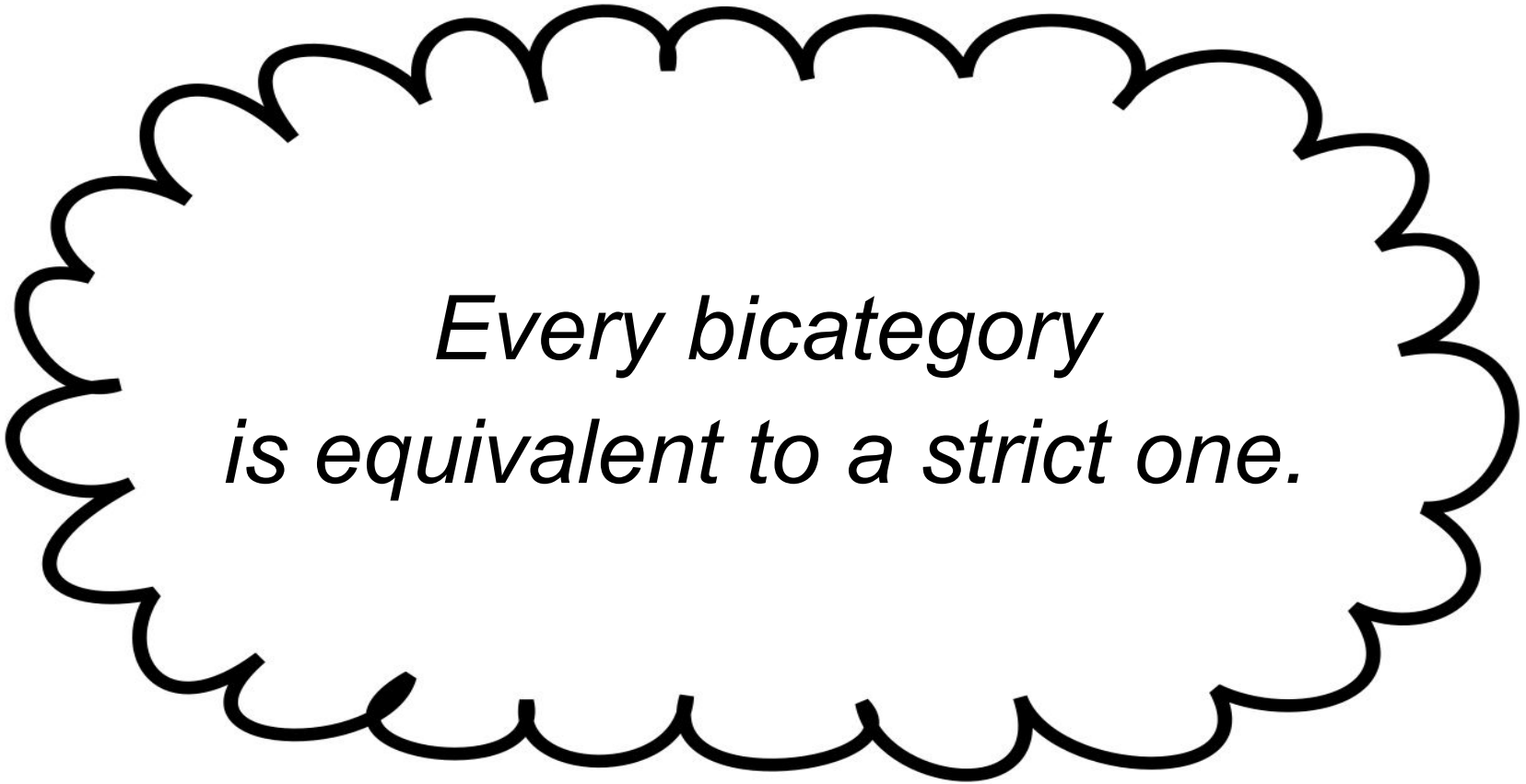
Every bicategory has an **underlying** implicit 2-category, which is representable.

Useful fact:

A pseudofunctor between bicategories is exactly the same as a functor between underlying implicit 2-categories.

Proposition

Bicategories are equivalent to representable implicit 2-categories.


Every bicategory
is equivalent to a strict one.

Important distinction:

The category of bicategories is *not* equivalent to the category of strict 2-categories.

$$\mathbf{Bicat} \not\simeq \mathbf{2\text{-}Cat}$$

The category of bicategories *is* equivalent to the category of representable implicit 2-categories.

Reasons to use this definition of bicategory

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- No coherence axioms

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- No coherence axioms
- Easy to explain to people
- Already implicit in string diagrams
- No equality between paths of arrows
- Makes bicategory theory easier

The original definition of bicategory does have its own advantages.

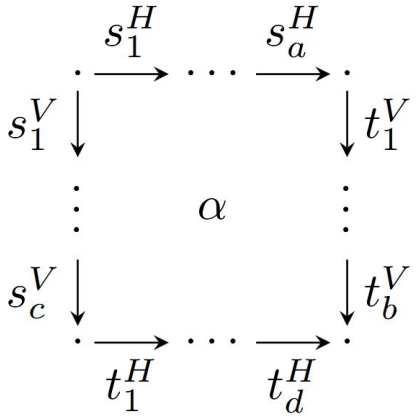
- Finite axiomatization
- Written in the language of categories, functors, and natural transformations
  - Special case of a “pseudo enriched category”

    (in the monoidal bicategory Cat)

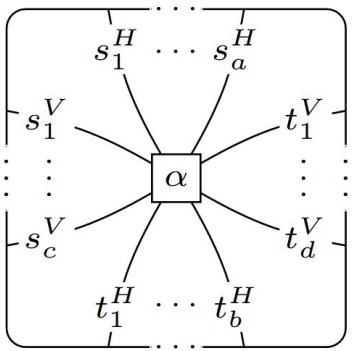
Another advantage
of the “implicit” definition
is that it generalizes to
double categories.

One-sentence definition:

An **implicit double category** is a strict double category whose underlying categories of arrows (vertical and horizontal) are free.

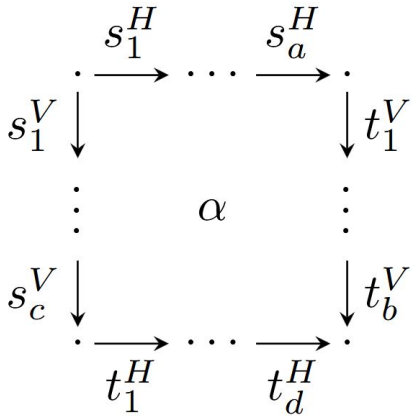


or

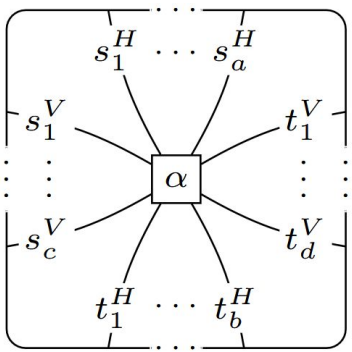


One-sentence definition:

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or



(This is also an essentially algebraic structure in its own right.)

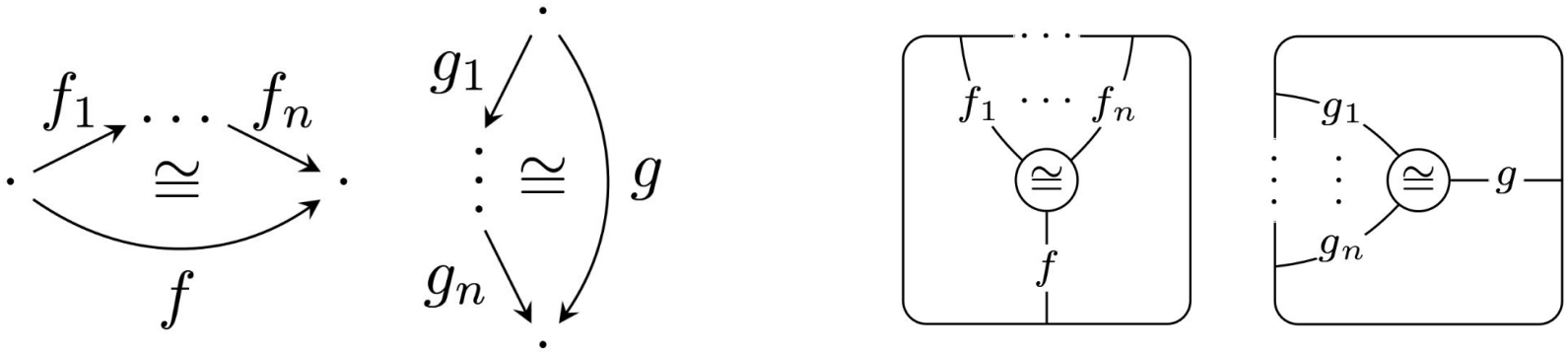
One-sentence definitions:

An **implicit double category** is a strict double category whose underlying categories of arrows (vertical and horizontal) are free.

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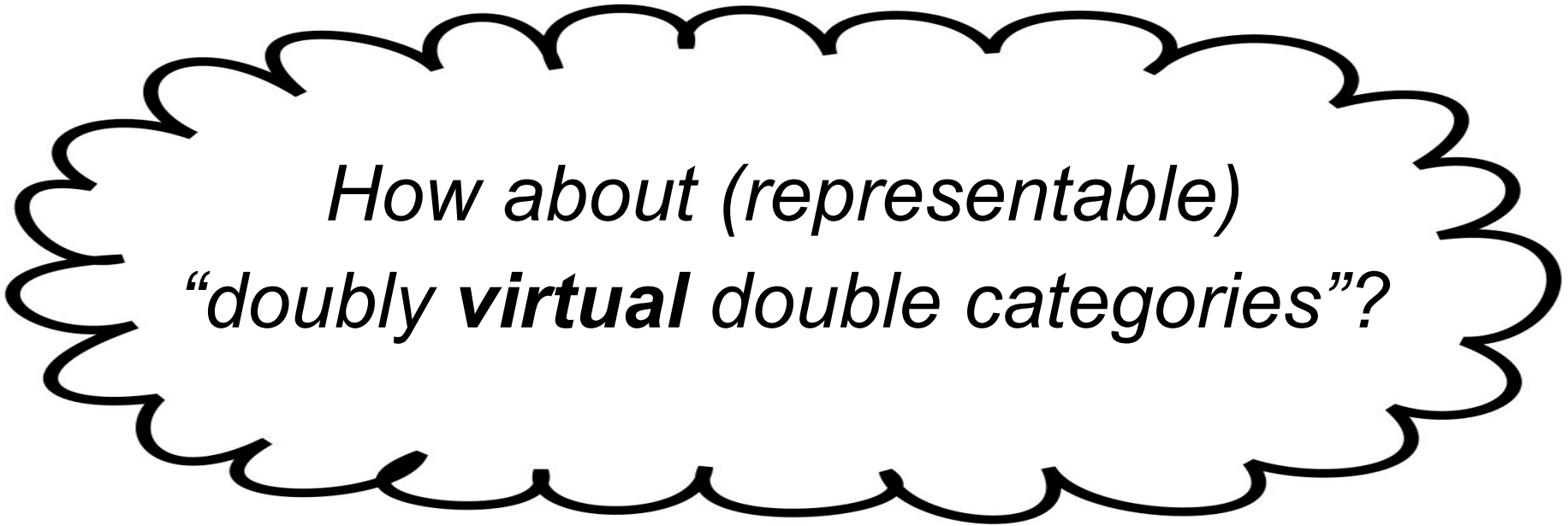
An implicit double category is **representable** if each path of arrows (horizontal or vertical) is isomorphic to a single arrow.

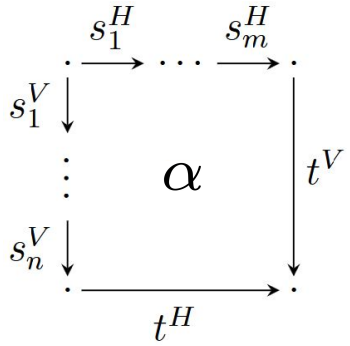


Definition

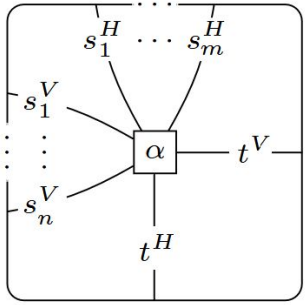
A **doubly weak double category**
is a representable
implicit double category.

Some of the audience:

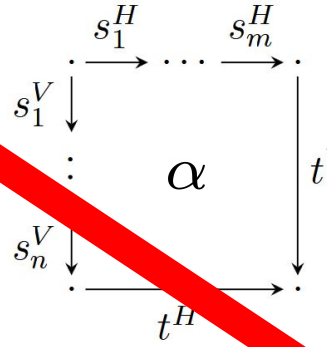




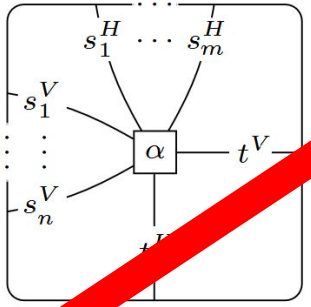
or



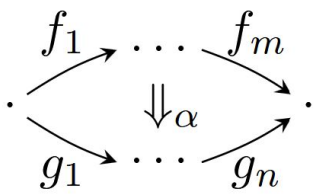
How about (representable)
*"doubly **virtual** double categories"?*

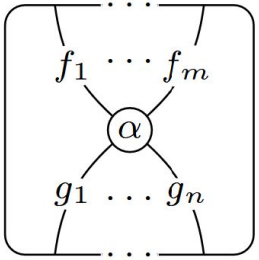


or



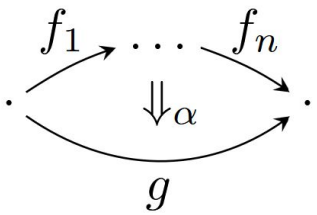
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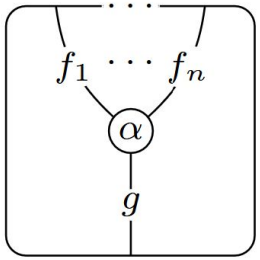




Implicit 2-category

- Morphisms classify pseudo functors
- Representability is described in terms of internal isomorphisms



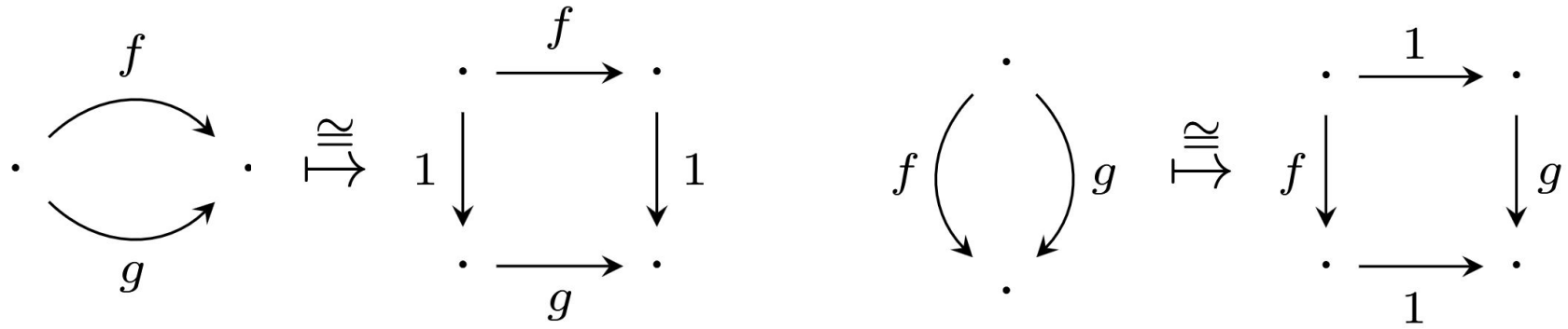


Virtual 2-category

- Morphisms classify lax functors
- Representability is only described in terms of universal properties

Other definitions

A Verity double bicategory is **tidy** if there are canonical bijections



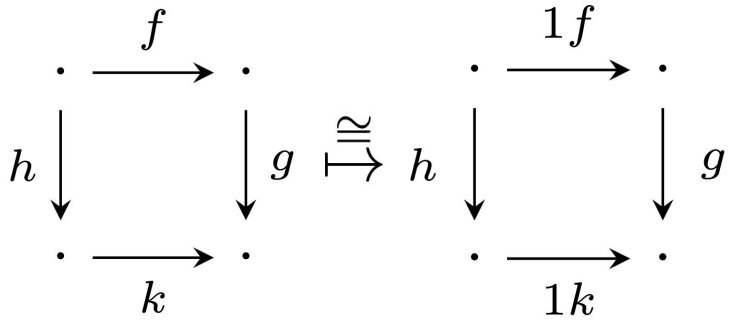
(This condition is also mentioned in Verity's thesis.)

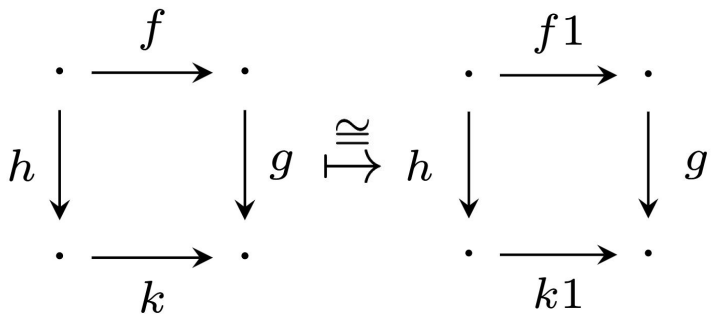
Proposition

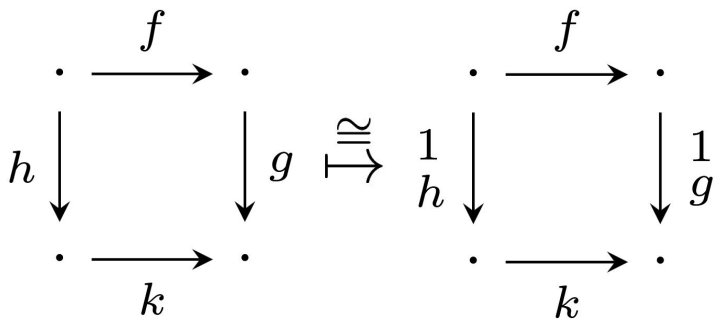
Doubly weak double categories
are equivalent to
tidy Verity double bicategories.

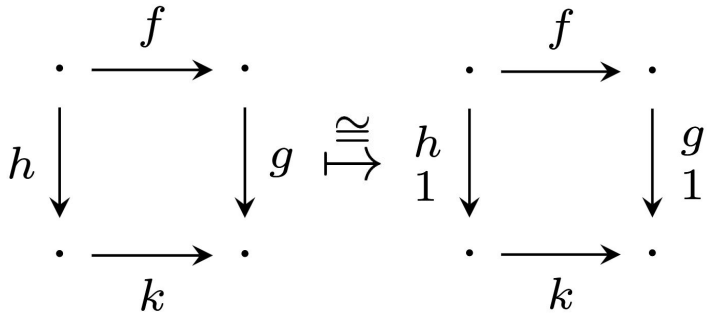
(requires elaboration on morphisms)

A Garner cubical bicategory is **tidy** if there are canonical bijections









Proposition

Doubly weak double categories
are equivalent to
tidy Garner cubical bicategories

(requires elaboration on morphisms)

Tidy double bicategories constitute a *finite* definition of doubly weak double category.

Tidy cubical bicategories constitute a definition that uses *only square cells*.

Can we combine both these features?

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Tidy cubical bicategories constitute a definition that uses *only square cells*.

Can we combine both these features?

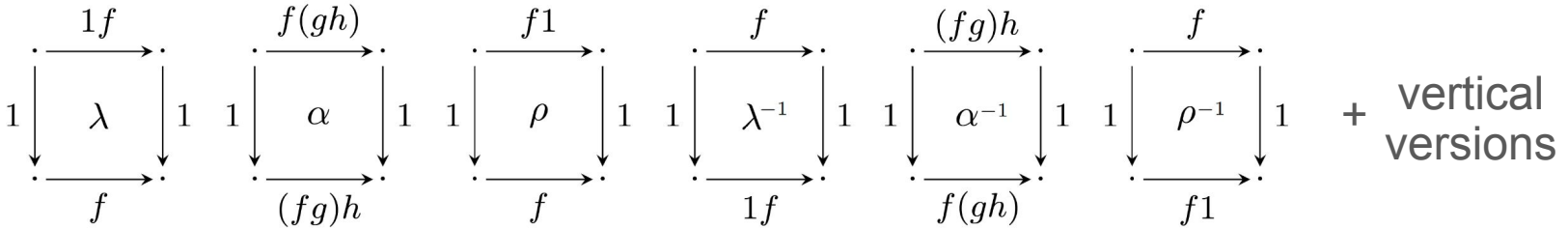
Yes.

A new squares-only definition (also equivalent)

- Objects, horizontal & vertical arrows, and squares with usual composition and identity operations

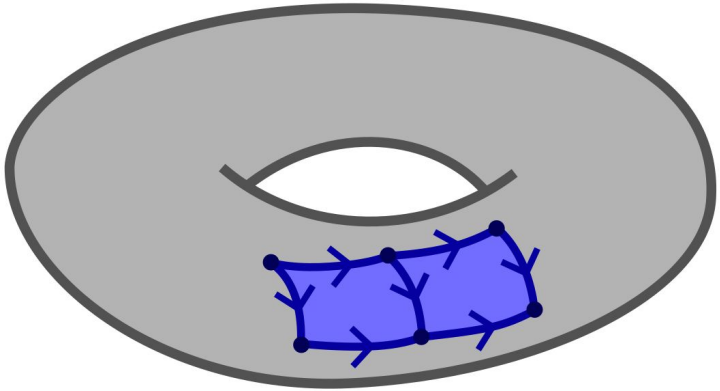


- Horizontal & vertical **coherence squares**

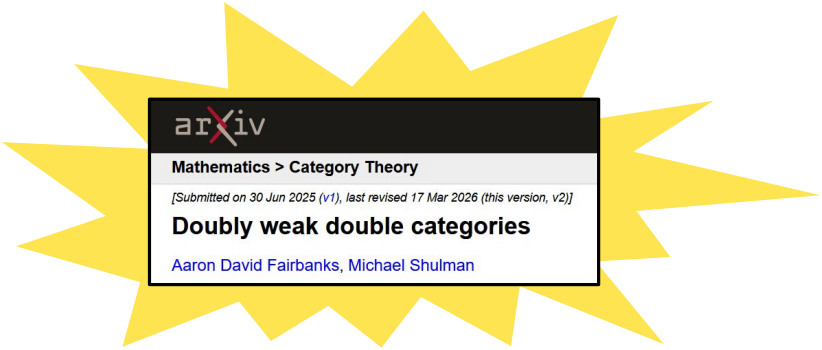


- Long list of conditions (like tidy double bicategories)

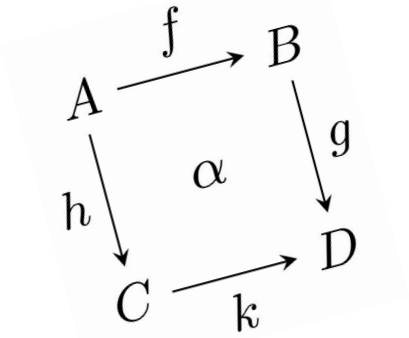
For example, this squares-only definition is well-suited to defining the **doubly weak double groupoid** of a topological space.

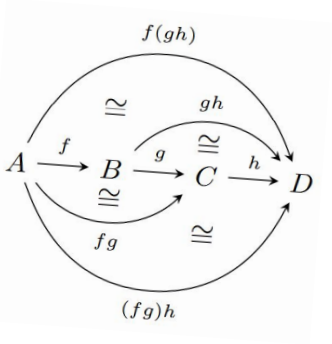


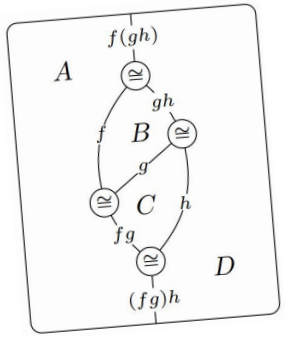
Paper advertisement

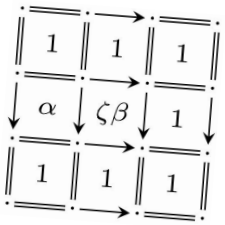


- All content from this talk
- **(2-)monadicity** results
- Another definition with squares + **monogons**
- Double-categorical **strictification** theorem
- Appendix on **higher morphisms** (more to do)

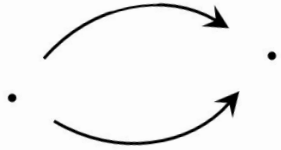

A
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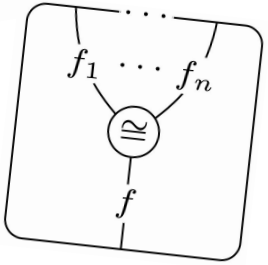

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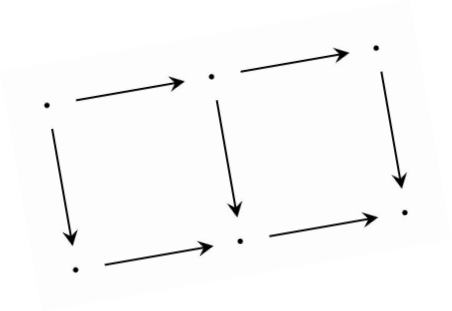

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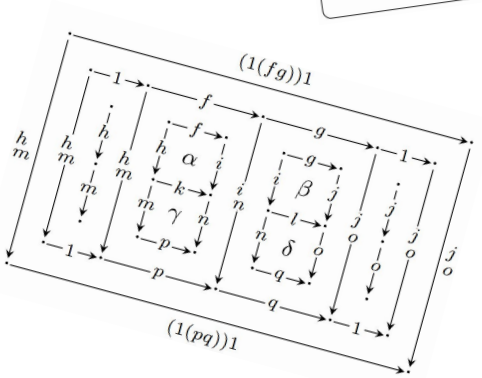

1
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Thank you




f1
fn
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(1(fg))1
(1(pq))1

