

# Cartesian Closed Double Categories

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In memory of Marta Bunge



Murten 1984



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## Why double categories?

For spaces (N 78/82) and locales/toposes (N 81), we showed an inclusion  $Y \rightarrow B$  is exponentiable iff it is locally closed.

Marta wrote (March 97 email), “We are interested in things you have done about exponentiability and locally closed.”

1998 Montreal sabbatical (BN 00), we used exponentiability of locally closed subcategories to show  $\text{UFL}/B$  is a topos if  $B$  satisfies (IG). Also, learned of Street’s notes using  $\text{Lax}_N(B, \text{Prof}) \simeq \text{Cat}/B$  to show  $Y \rightarrow B$  is exponentiable in  $\text{Cat}$  iff it satisfies (FL).

Later generalized the equivalence for other exponentiability characterizations (N 10). Paré saw double categories there.

Used cotabulators in double categories to construct exponentials of locally closed inclusions for  $\text{Top}$ ,  $\text{Loc}$ ,  $\text{Topos}$ ,  $\text{Cat}$ ,  $\text{Pos}$  (N 12).

# Cartesian Closed Double Categories

Two approaches for cartesian closed 1-category  $C$

Objectwise: The functor  $( ) \times Y$  has a right adjoint  $( )^Y$ , for all  $Y$

As a bifunctor:  $C^{\text{op}} \times C \xrightarrow{[ , ]} C$  s.t.  $C(X \times Y, Z) \cong C(X, [Y, Z])$

We'll see that these approaches differ for double categories

## Outline

1. Double categories and examples
2. Lax double functors and adjoints
3. Cartesian closed double categories
4. More examples

# Double Categories

A double category  $\mathbb{D}$  is a pseudo internal category in  $\mathbf{CAT}$

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\circ} \mathbb{D}_1 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{\text{id} \bullet} \\ \xrightarrow{t} \end{array} \mathbb{D}_0$$

Objects  $X$  of  $\mathbb{D}_0$ , called objects of  $\mathbb{D}$

Morphisms  $X \xrightarrow{f} Y$  of  $\mathbb{D}_0$ , called horizontal morphisms of  $\mathbb{D}$

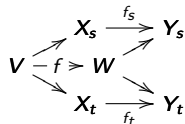
Objects  $X_s \xrightarrow{v} X_t$  of  $\mathbb{D}_1$ , called vertical morphism of  $\mathbb{D}$

Morphisms  $v \xrightarrow{\varphi} w$  of  $\mathbb{D}_1$ , called cells of  $\mathbb{D}$

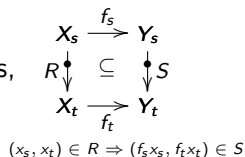
$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ v \downarrow & \varphi & \downarrow w \\ X_t & \xrightarrow{f_t} & Y_t \end{array}$$

## Examples

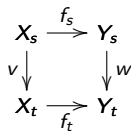
Span: sets, functions, spans,



Rel: sets, functions, relations,



Squ: sets, functions, functions,



Note:  $\text{Squ}_1 = \text{Sets}^\downarrow$ , the arrow category.

# Lax Functors

A lax functor  $F: \mathbb{D} \rightarrow \mathbb{E}$  consists of functors  $F_0: \mathbb{D}_0 \rightarrow \mathbb{E}_0$  and  $F_1: \mathbb{D}_1 \rightarrow \mathbb{E}_1$  compatible with  $s$  and  $t$ , and cells

$$\text{id}_{F_0 X} \rightarrow F_1(\text{id}_X) \quad \text{and} \quad F_1 w \odot F_1 v \rightarrow F_1(w \odot v)$$

satisfying naturality and coherence conditions.

Oplax and pseudo functors are defined with the cells in the opposite direction and invertible, respectively.

LxDbl denotes the 2-category double categories and lax functors

# Adjoints

## Lemma (GP 04)

*The following are equivalent for a lax functor  $\mathbb{D} \xrightarrow{F} \mathbb{E}$ , and functors  $\mathbb{E}_0 \xrightarrow{G_0} \mathbb{D}_0$  and  $\mathbb{E}_1 \xrightarrow{G_1} \mathbb{D}_1$  compatible with  $s$  and  $t$ .*

- ▶  $G$  is lax and  $F \dashv G$  in  $\text{LxDbl}$ .
- ▶  $F_0 \dashv G_0$ ,  $F_1 \dashv G_1$ , and  $G$  is lax.
- ▶  $F_0 \dashv G_0$ ,  $F_1 \dashv G_1$ , and  $F$  is oplax.

Why  $\text{LxDbl}$ ?

# (Lax) Cartesian Closed Categories

## Definition (A 18)

A double category  $\mathbb{D}$  is lax cartesian (AKA pre-cartesian) if  $\Delta: \mathbb{D} \rightarrow \mathbb{D} \times \mathbb{D}$  and  $!: \mathbb{D} \rightarrow \mathbb{1}$  have right adjoints  $\times$  and  $1$ .

## Definitions

1. A (lax) cartesian double category  $\mathbb{D}$  is (lax) cartesian closed if there is a lax functor  $[-, -]: \mathbb{D}^{\text{op}} \times \mathbb{D} \rightarrow \mathbb{D}$  inducing cartesian closed structures on  $\mathbb{D}_0$  and  $\mathbb{D}_1$ .
2. An object  $Y$  is (lax) exponentiable in  $\mathbb{D}$  if the lax functor  $(-) \times Y: \mathbb{D} \rightarrow \mathbb{D}$  exists and has a right adjoint in  $\text{LxDbl}$ .
3. A (lax) cartesian double category  $\mathbb{D}$  is objectwise (lax) cartesian closed if every object  $Y$  is (lax) exponentiable in  $\mathbb{D}$ .

Note: We'll see 3 is weaker than 1. Also, cartesian closure of  $\mathbb{D}_0$  and  $\mathbb{D}_1$  is not sufficient for 1 since compatibility with  $s, t$  may fail.



## Examples

$\mathbb{R}el$  is cartesian closed via  $\mathbb{R}el^{op} \times \mathbb{R}el \xrightarrow{[-,-]} \mathbb{R}el$  defined as follows

$$[Y, Z]_0 = Z^Y \quad \text{and} \quad [S, T]_1 = Z_s^{Y_s} \xrightarrow{T^S} Z_t^{Y_t}$$

where  $(\sigma_s, \sigma_t) \in T^S$  if  $(y_s, y_t) \in S \Rightarrow (\sigma_s y_s, \sigma_t y_t) \in T$

$$\begin{array}{ccc} Y_s & \xrightarrow{\sigma_s} & Z_s \\ S \downarrow & \subseteq & \downarrow T \\ Y_t & \xrightarrow{\sigma_t} & Z_t \end{array}$$

since

$$\begin{array}{ccc} X_s \times Y_s & \xrightarrow{f_s} & Z_s \\ R \times S \downarrow & \subseteq & \downarrow T \\ X_t \times Y_t & \xrightarrow{f_t} & Z_t \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} X_s & \xrightarrow{\hat{f}_s} & Z_s^{Y_s} \\ R \downarrow & \subseteq & \downarrow T^S \\ X_t & \xrightarrow{\hat{f}_t} & Z_t^{Y_t} \end{array}$$

where  $\hat{f}_s(x_s)(y_s) = f_s(x_s, y_s)$  and  $\hat{f}_t(x_t)(y_t) = f_t(x_t, y_t)$

## Examples, cont.

Span, like  $\mathbb{R}el$ , is cartesian closed via  $[Y, Z]_0 = Z^Y$ , and  $[V, W]_1$  given by

$$W^V \begin{array}{c} \nearrow Z_s^{Y_s} \\ \searrow Z_t^{Y_t} \end{array} \quad \text{where } W^V = \left\{ \begin{array}{ccc} & Y_s & \xrightarrow{\sigma_s} Z_s \\ & \nearrow \sigma & \nearrow W \\ v & & \\ & \searrow & \searrow \\ & Y_t & \xrightarrow{\sigma_t} Z_t \end{array} \right\}$$

$\mathbb{S}qu$  is objectwise cartesian closed;  $\mathbb{S}qu_0$  and  $\mathbb{S}qu_1$  are cartesian closed, but there is no bifunctor  $[-, -]: \mathbb{S}qu^{op} \times \mathbb{S}qu \rightarrow \mathbb{S}qu$ , since  $\mathbb{S}qu_1 = \mathit{Sets}^\downarrow$ , the arrow category, where

$$[v, w]_s = \left\{ \begin{array}{ccc} Y_s & \xrightarrow{\sigma_s} & Z_s \\ v \downarrow & & \downarrow w \\ Y_t & \xrightarrow{\sigma_t} & Z_t \end{array} \right\} \neq Z_s^{Y_s}$$

is not compatible with  $\mathbb{S}qu_1 \xrightarrow{s} \mathbb{S}qu_0$ .

## More Examples

$\mathbb{L}oc$ : locales, locale maps, lex maps,  $v f_s^* \geq f_t^* w$

$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ v \downarrow & \varphi & \downarrow w \\ X_t & \xrightarrow{f_t} & Y_t \end{array}$$

$\mathbb{T}op$ : spaces, continuous maps, lex maps on opens,  $v f_s^* \geq f_t^* w$

$\mathbb{C}at$ : categories, functors, profunctors,  $v \xrightarrow{\varphi} w(f_s -, f_t -)$

$\mathbb{P}os$ : posets, o.p. maps, order ideals,  $(f_s \times f_t)(v) \subseteq w$

Remark: These each have an object  $\mathbb{2}$  such that  $\mathbb{D}_1 \cong \mathbb{D}_0/\mathbb{2}$ , and were called glueing categories in (N 2012).

# Objectwise Cartesian Closed Double Categories

## Theorem (N 20)

*Suppose  $\mathbb{D}$  is a glueing category. Then  $Y$  is exponentiable in  $\mathbb{D}$  if and only if  $- \times Y: \mathbb{D} \rightarrow \mathbb{D}$  is oplax and  $Y$  is exponentiable in  $\mathbb{D}_0$ .*

## Proof.

Suppose  $Y$  is exponentiable in  $\mathbb{D}$ . Then  $Y$  is exponentiable in  $\mathbb{D}_0$ , since  $- \times Y$  has a right adjoint as an endofunctor of  $\mathbb{D}_0$ . The converse holds, as  $- \times Y$  on  $\mathbb{D}_1$  corresponds to pullback along  $2 \times Y \rightarrow 2$  on  $\mathbb{D}_0/2$ , which is exponentiable when  $Y$  is.  $\square$

## Corollary

1.  $Y$  is exponentiable in  $\mathbb{T}op$  iff  $\mathcal{O}(Y)$  is locally compact.
2.  $Y$  is exponentiable in  $\mathbb{L}oc$  iff  $Y$  is locally compact.
3.  $\mathbb{C}at$  and  $\mathbb{P}os$  are objectwise cartesian closed.

Note: But, we can do better for  $\mathbb{C}at$  and  $\mathbb{P}os$  ...

## Slices of Cat and Pos

$Y \xrightarrow{p} B$  is exponentiable Pos if it satisfies weak factorization lifting

$$\begin{array}{ccc} Y & & \\ \downarrow p & & \\ B & & \end{array} \quad \begin{array}{ccc} y & \xrightarrow{\alpha} & y' \\ \text{---} & \searrow & \nearrow \\ & y'' & \\ \text{---} & \swarrow & \searrow \\ py & \xrightarrow{p\alpha} & py' \\ \searrow \beta & & \swarrow \beta' \\ & b & \end{array} \quad (\text{WFL})$$

$Y \xrightarrow{p} B$  is exponentiable Cat if it satisfies the Giraud-Conduché factorization lifting condition (FL)=(WFL) + connectivity.

Example.  $\text{Cat}/2$  and  $\text{Pos}/2$  are cartesian closed.

### Corollary

*Cat and Pos are cartesian closed.*

### Proof.

$\text{Cat}_1 = \text{Cat}/2$  and  $\text{Pos}_1 = \text{Pos}/2$



## Slices of $\mathbb{C}at$ and $\mathbb{P}os$

Consider the double slice  $\mathbb{D} // B$ , where

$$(\mathbb{D} // B)_0 = \mathbb{D}_0 / B \quad \text{and} \quad (\mathbb{D} // B)_1 = \mathbb{D}_1 / \text{id}_B^\bullet$$

**Lemma (N 2020)**

$\mathbb{D} // B$  is a glueing category, whenever  $\mathbb{D}$  is.

**Corollary**

$\mathbb{C}at // 2$  is objectwise cartesian closed, but not cartesian closed.

Likewise, for  $\mathbb{P}os // 2$ .

**Proof.**

$(\mathbb{C}at // 2)_0$  and  $(\mathbb{P}os // 2)_0$  are cartesian closed, but not

$(\mathbb{C}at // 2)_1 \cong \mathbb{C}at / (2 \times 2)$  and  $(\mathbb{P}os // 2)_1 \cong \mathbb{P}os / (2 \times 2)$ . □

# Glueing Categories, Revisited

- ▶  $\mathbb{D}_0$  has finite limits
- ▶  $\text{id}^\bullet: \mathbb{D}_0 \rightarrow \mathbb{D}_1$  has a left adjoint  $\Gamma$  (i.e.,  $\mathbb{D}$  has cotabulators)
- ▶  $\Gamma_2: \mathbb{D}_1 \rightarrow \mathbb{D}_0/2$  is an equivalence, where  $2 = \Gamma(\text{id}_1^\bullet)$ , and

$$\begin{array}{ccc} X_s & \xrightarrow{i_s} & \Gamma v \\ \downarrow & & \downarrow \Gamma_2 v \\ 1 & \xrightarrow{i_s} & 2 \end{array} \quad \text{and} \quad \begin{array}{ccc} X_t & \xrightarrow{i_t} & \Gamma v \\ \downarrow & & \downarrow \Gamma_2 v \\ 1 & \xrightarrow{i_t} & 2 \end{array}$$

are pullbacks in  $\mathbb{D}_0$ , where  $v: X_s \rightarrow X_t$ .

- ▶  $\mathbb{D}$  is “horizontally invariant”

Examples.

Cat (and Pos):  $|\Gamma v| = |X_s| \sqcup |X_t|$  with morphisms (order) via  $v$

Top:  $\Gamma v = X_s \sqcup X_t$  with  $U$  open, if  $U_s, U_t$  open,  $U_t \subseteq v(U_s)$

Loc: cotabulators are given by Artin-Wraith glueing

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