Cartesian Closed Double Categories

Susan Niefield

Union College
Schenectady, NY

In memory of Marta Bunge

Murten 1984

Tours 1994

Why double categories?

For spaces (N 78/82) and locales/toposes (N 81), we showed an inclusion $Y \rightarrowtail B$ is exponentiable iff it is locally closed.

Marta wrote (March 97 email), "We are interested in things you have done about exponentiabilty and locally closed."

1998 Montreal sabbatical (BN 00), we used exponentiability of locally closed subcategories to show UFL/B is a topos if B satisfies (IG). Also, learned of Street's notes using $\mathrm{Lax}_{\mathrm{N}}(B, \mathrm{Prof}) \simeq \mathrm{Cat}/B$ to show $Y \rightarrowtail B$ is exponentiable in Cat iff it satisfies (FL).

Later generalized the equivalence for other exponentiabilty characterizations (N 10). Paré saw double categories there.

Used cotabulators in double categories to construct exponentials of locally closed inclusions for Top, Loc, Topos, Cat, Pos (N 12).

Cartesian Closed Double Categories

Two approaches for cartesian closed 1-category $\mathbb{C}$

Objectwise: The functor $(\ ) \times Y$ has a right adjoint $(\ )^Y$, for all Y

As a bifunctor: $\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \xrightarrow{[\ ,\ ]} \mathbb{C}$ s.t. $\mathbb{C}(X \times Y, Z) \cong \mathbb{C}(X, [Y, Z])$

We'll see that these approaches differ for double categories

Outline

1. Double categories and examples
2. Lax double functors and adjoints
3. Cartesian closed double categories
4. More examplesss

Double Categories

A double category $\mathbb{D}$ is a pseudo internal category in CAT

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\odot} \mathbb{D}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{D}_0 \qquad \mathbb{D}_0 \xrightarrow{\mathrm{id}^\bullet} \mathbb{D}_1$$

Objects X of $\mathbb{D}_0$, called objects of $\mathbb{D}$

Morphisms $X \xrightarrow{f} Y$ of $\mathbb{D}_0$, called horizontal morphisms of $\mathbb{D}$

Objects $X_s \overset{v}{\bullet\!\!\rightarrow} X_t$ of $\mathbb{D}_1$, called vertical morphism of $\mathbb{D}$

Morphisms $v \xrightarrow{\varphi} w$ of $\mathbb{D}_1$, called cells of $\mathbb{D}$

$$\begin{tikzcd} X_s \arrow[r, "f_s"] \arrow[d, "v"'] \arrow[dr, phantom, "\varphi"] & Y_s \arrow[d, "w"] \\ X_t \arrow[r, "f_t"'] & Y_t \end{tikzcd}$$

Examples

$\mathbb{S}\mathrm{pan}$: sets, functions, spans,
$$\begin{tikzcd} & X_s \arrow[rr, "f_s"] & & Y_s \\ V \arrow[ur] \arrow[dr] \arrow[rr, "f"] & & W \arrow[ur] \arrow[dr] & \\ & X_t \arrow[rr, "f_t"'] & & Y_t \end{tikzcd}$$

$\mathbb{R}\mathrm{el}$: sets, functions, relations,
$$\begin{tikzcd} X_s \arrow[r, "f_s"] \arrow[d, "R"'] \arrow[dr, phantom, "\subseteq"] & Y_s \arrow[d, "S"] \\ X_t \arrow[r, "f_t"'] & Y_t \end{tikzcd}$$
$(x_s, x_t) \in R \Rightarrow (f_s x_s, f_t x_t) \in S$

$\mathbb{S}\mathrm{qu}$: sets, functions, functions,
$$\begin{tikzcd} X_s \arrow[r, "f_s"] \arrow[d, "v"'] & Y_s \arrow[d, "w"] \\ X_t \arrow[r, "f_t"'] & Y_t \end{tikzcd}$$

Note: $\mathbb{S}\mathrm{qu}_1 = \mathrm{Sets}^{\downarrow}$, the arrow category.

Lax Functors

A lax functor $F\colon \mathbb{D} \longrightarrow \mathbb{E}$ consists of functors $F_0\colon \mathbb{D}_0 \longrightarrow \mathbb{E}_0$ and $F_1\colon \mathbb{D}_1 \longrightarrow \mathbb{E}_1$ compatible with s and t, and cells

$$\mathrm{id}^{\bullet}_{F_0 X} \longrightarrow F_1(\mathrm{id}^{\bullet}_X) \quad \text{and} \quad F_1 w \odot F_1 v \longrightarrow F_1(w \odot v)$$

satisfying naturality and coherence conditions.

Oplax and pseudo functors are defined with the cells in the opposite direction and invertible, respectively.

LxDbl denotes the 2-category double categories and lax functors

Adjoints

Lemma (GP 04)

The following are equivalent for a lax functor $\mathbb{D} \xrightarrow{F} \mathbb{E}$, and functors $\mathbb{E}_0 \xrightarrow{G_0} \mathbb{D}_0$ and $\mathbb{E}_1 \xrightarrow{G_1} \mathbb{D}_1$ compatible with s and t.

- *G is lax and $F \dashv G$ in LxDbl.*
- *$F_0 \dashv G_0$, $F_1 \dashv G_1$, and G is lax.*
- *$F_0 \dashv G_0$, $F_1 \dashv G_1$, and F is oplax.*

Why LxDbl?

(Lax) Cartesian Closed Categories

Definition (A 18)

A double category $\mathbb{D}$ is lax cartesian (AKA pre-cartesian) if $\Delta \colon \mathbb{D} \longrightarrow \mathbb{D} \times \mathbb{D}$ and $! \colon \mathbb{D} \longrightarrow \mathbb{1}$ have right adoints $\times$ and 1.

Definitions

1. A (lax) cartesian double category $\mathbb{D}$ is (lax) cartesian closed if there is a lax functor $[-,-] : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \longrightarrow \mathbb{D}$ inducing cartesian closed structures on $\mathbb{D}_0$ and $\mathbb{D}_1$.
2. An object Y is (lax) exponentiable in $\mathbb{D}$ if the lax functor $(-) \times Y \colon \mathbb{D} \longrightarrow \mathbb{D}$ exists and has a right adjoint in LxDbl.
3. A (lax) cartesian double category $\mathbb{D}$ is objectwise (lax) cartesian closed if every object Y is (lax) exponentiable in $\mathbb{D}$.

Note: We'll see 3 is weaker than 1. Also, cartesian closure of $\mathbb{D}_0$ and $\mathbb{D}_1$ is not sufficient for 1 since compatibility with s, t may fail.

Examples

$\mathbb{R}\mathrm{el}$ is cartesian closed via $\mathbb{R}\mathrm{el}^{\mathrm{op}} \times \mathbb{R}\mathrm{el} \xrightarrow{[-,-]} \mathbb{R}\mathrm{el}$ defined as follows

$$[Y,Z]_0 = Z^Y \quad \text{and} \quad [S,T]_1 = {Z_s}^{Y_s} \xrightarrow{T^S} {Z_t}^{Y_t}$$

where $(\sigma_s, \sigma_t) \in T^S$ if $(y_s, y_t) \in S \Rightarrow (\sigma_s y_s, \sigma_t y_t) \in T$

$$\begin{tikzcd}
Y_s \arrow[r, "\sigma_s"] \arrow[d, "S"', "\bullet" marking] \arrow[dr, phantom, "\subseteq"] & Z_s \arrow[d, "T", "\bullet" marking] \\
Y_t \arrow[r, "\sigma_t"'] & Z_t
\end{tikzcd}$$

since

$$\begin{tikzcd}
X_s \times Y_s \arrow[r, "f_s"] \arrow[d, "R \times S"', "\bullet" marking] \arrow[dr, phantom, "\subseteq"] & Z_s \arrow[d, "T", "\bullet" marking] \\
X_t \times Y_t \arrow[r, "f_t"'] & Z_t
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
X_s \arrow[r, "\hat{f}_s"] \arrow[d, "R"', "\bullet" marking] \arrow[dr, phantom, "\subseteq"] & {Z_s}^{Y_s} \arrow[d, "T^S", "\bullet" marking] \\
X_t \arrow[r, "\hat{f}_t"'] & {Z_t}^{Y_t}
\end{tikzcd}$$

where $\hat{f}_s(x_s)(y_s) = f_s(x_s, y_s)$ and $\hat{f}_t(x_t)(y_t) = f_t(x_t, y_t)$

Examples, cont.

$\mathbb{S}\mathrm{pan}$, like $\mathbb{R}\mathrm{el}$, is cartesian closed via $[Y, Z]_0 = Z^Y$, and $[V, W]_1$ given by

$$
\begin{tikzcd}[row sep=small]
 & {Z_s}^{Y_s} \\
W^V \arrow[ur] \arrow[dr] & \\
 & {Z_t}^{Y_t}
\end{tikzcd}
\qquad \text{where} \quad W^V = \left\{
\begin{tikzcd}[row sep=small]
 & Y_s \arrow[r, "\sigma_s"] & Z_s \\
V \arrow[ur] \arrow[r, dashed, "\sigma"] \arrow[dr] & W \arrow[ur] \arrow[dr] & \\
 & Y_t \arrow[r, "\sigma_t"'] & Z_t
\end{tikzcd}
\right\}
$$

$\mathbb{S}\mathrm{qu}$ is objectwise cartesian closed; $\mathbb{S}\mathrm{qu}_0$ and $\mathbb{S}\mathrm{qu}_1$ are cartesian closed, but there is no bifunctor $[-,-]\colon \mathbb{S}\mathrm{qu}^{\mathrm{op}} \times \mathbb{S}\mathrm{qu} \longrightarrow \mathbb{S}\mathrm{qu}$, since $\mathbb{S}\mathrm{qu}_1 = \mathit{Sets}^{\downarrow}$, the arrow category, where

$$
[v, w]_s = \left\{
\begin{tikzcd}
Y_s \arrow[r, "\sigma_s"] \arrow[d, "v"'] & Z_s \arrow[d, "w"] \\
Y_t \arrow[r, "\sigma_t"'] & Z_t
\end{tikzcd}
\right\} \neq {Z_s}^{Y_s}
$$

is not compatible with $\mathbb{S}\mathrm{qu}_1 \xrightarrow{s} \mathbb{S}\mathrm{qu}_0$.

More Examples

$$\begin{tikzcd} X_s \arrow[r, "f_s"] \arrow[d, "v"'] \arrow[dr, phantom, "\varphi"] & Y_s \arrow[d, "w"] \\ X_t \arrow[r, "f_t"'] & Y_t \end{tikzcd}$$

$\mathbb{L}\mathrm{oc}$: locales, locale maps, lex maps, $vf_s^* \geq f_t^* w$

$\mathbb{T}\mathrm{op}$: spaces, continuous maps, lex maps on opens, $vf_s^* \geq f_t^* w$

$\mathbb{C}\mathrm{at}$: categories, functors, profunctors, $v \xrightarrow{\varphi} w(f_s-, f_t-)$

$\mathbb{P}\mathrm{os}$: posets, o.p. maps, order ideals, $(f_s \times f_t)(v) \subseteq w$

Remark: These each have an object $\mathbb{2}$ such that $\mathbb{D}_1 \cong \mathbb{D}_0/\mathbb{2}$, and were called glueing categories in (N 2012).

Objectwise Cartesian Closed Double Categories

Theorem (N 20)

Suppose $\mathbb{D}$ is a glueing category. Then Y is exponentiable in $\mathbb{D}$ if and only if $- \times Y : \mathbb{D} \longrightarrow \mathbb{D}$ is oplax and Y is exponentiable in $\mathbb{D}_0$.

Proof.

Suppose Y is exponentiable in $\mathbb{D}$. Then Y is exponentiable in $\mathbb{D}_0$, since $- \times Y$ has a right adjoint as an endofunctor of $\mathbb{D}_0$. The converse holds, as $- \times Y$ on $\mathbb{D}_1$ corresponds to pullback along $\mathbb{2} \times Y \longrightarrow \mathbb{2}$ on $\mathbb{D}_0/\mathbb{2}$, which is exponentiable when Y is. $\square$

Corollary

1. *Y is exponentiable in $\mathbb{T}\mathrm{op}$ iff $\mathcal{O}(Y)$ is locally compact.*
2. *Y is exponentiable in $\mathbb{L}\mathrm{oc}$ iff Y is locally compact.*
3. *$\mathbb{C}\mathrm{at}$ and $\mathbb{P}\mathrm{os}$ are objectwise cartesian closed.*

Note: But, we can do better for $\mathbb{C}\mathrm{at}$ and $\mathbb{P}\mathrm{os}$ ...

Slices of $\mathbb{C}\mathrm{at}$ and $\mathbb{P}\mathrm{os}$

$Y \xrightarrow{p} B$ is exponentiable $\mathbb{P}\mathrm{os}$ if it satisfies weak factorization lifting

$$\begin{tikzcd} Y \arrow[d, "p"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} y \arrow[rr, "\alpha"] \arrow[dr, dotted] & & y' \\ & y'' \arrow[ur, dotted] & \\ py \arrow[rr, "p\alpha"] \arrow[dr, "\beta"'] & & py' \\ & b \arrow[ur, "\beta'"'] & \end{tikzcd} \qquad (WFL)$$

$Y \xrightarrow{p} B$ is exponentiable $\mathbb{C}\mathrm{at}$ if it satisfies the Giraud-Conduché factorization lifting condition (FL)=(WFL) + connectivity.

Example. $\mathbb{C}\mathrm{at}/\mathbb{2}$ and $\mathbb{P}\mathrm{os}/\mathbb{2}$ are cartesian closed.

Corollary

$\mathbb{C}\mathrm{at}$ and $\mathbb{P}\mathrm{os}$ are cartesian closed.

Proof.

$\mathbb{C}\mathrm{at}_1 = \mathbb{C}\mathrm{at}/\mathbb{2}$ and $\mathbb{P}\mathrm{os}_1 = \mathbb{P}\mathrm{os}/\mathbb{2}$ $\square$

Slices of $\mathbb{C}\mathrm{at}$ and $\mathbb{P}\mathrm{os}$

Consider the double slice $\mathbb{D}//B$, where

$$(\mathbb{D}//B)_0 = \mathbb{D}_0/B \quad \text{and} \quad (\mathbb{D}//B)_1 = \mathbb{D}_1/\operatorname{id}_B^\bullet$$

Lemma (N 2020)

$\mathbb{D}//B$ is a glueing category, whenever $\mathbb{D}$ is.

Corollary

$\mathbb{C}\mathrm{at}//2$ is objectwise cartesian closed, but not cartesian closed. Likewise, for $\mathbb{P}\mathrm{os}//2$.

Proof.

$(\mathbb{C}\mathrm{at}//2)_0$ and $(\mathbb{P}\mathrm{os}//2)_0$ are cartesian closed, but not $(\mathbb{C}\mathrm{at}//2)_1 \cong \mathrm{Cat}/(2 \times 2)$ and $(\mathbb{P}\mathrm{os}//2)_1 \cong \mathrm{Pos}/(2 \times 2)$. $\square$

Glueing Categories, Revisited

- $\mathbb{D}_0$ has finite limits
- $\mathrm{id}^\bullet \colon \mathbb{D}_0 \longrightarrow \mathbb{D}_1$ has a left adjoint Γ (i.e., $\mathbb{D}$ has cotabulators)
- $\Gamma_{\mathbb{2}} \colon \mathbb{D}_1 \longrightarrow \mathbb{D}_0/\mathbb{2}$ is an equivalence, where $\mathbb{2} = \Gamma(\mathrm{id}^\bullet_1)$, and

$$\begin{tikzcd} X_s \arrow[r, "i_s"] \arrow[d] & \Gamma v \arrow[d, "\Gamma_{\mathbb{2}} v"] \\ 1 \arrow[r, "i_s"'] & \mathbb{2} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_t \arrow[r, "i_t"] \arrow[d] & \Gamma v \arrow[d, "\Gamma_{\mathbb{2}} v"] \\ 1 \arrow[r, "i_t"'] & \mathbb{2} \end{tikzcd}$$

  are pullbacks in $\mathbb{D}_0$, where $v \colon X_s \xrightarrow{\bullet} X_t$.
- $\mathbb{D}$ is "horizontally invariant"

Examples.

$\mathbb{C}\mathrm{at}$ (and $\mathbb{P}\mathrm{os}$): $|\Gamma v| = |X_s| \sqcup |X_t|$ with morphisms (order) via v

$\mathbb{T}\mathrm{op}$: $\Gamma v = X_s \sqcup X_t$ with U open, if U_s, U_t open, $U_t \subseteq v(U_s)$

$\mathbb{L}\mathrm{oc}$: cotabulators are given by Artin-Wraith glueing

- E. Aleiferi, Cartesian Double Categories with an Emphasis on Characterizing Spans, Dalhousie PhD Thesis, 2018, https://arxiv.org/abs/1809.06940.
- M. Bunge and S. B. Niefield, Exponentiability and single universes, JPAA 148 (2000), 217–250.
- M. Grandis and R. Paré, Adjoints for double categories, Cahiers 45 (2004), 193–240.
- S. B. Niefield, Cartesian inclusions: locales and toposes, Comm. in Alg. 9 (1981), 1639–1671.
- S. B. Niefield, Cartesianness: topological spaces, uniform spaces, and affine schemes, JPAA 23 (1982), 147–167.
- S. B. Niefield, Lax presheaves and exponentiability, TAC 24, (2010), 288–301.
- S. B. Niefield, The glueing construction and double categories, JPAA 216 (2012), 1827–1836.
- S. B. Niefield, Exponentiability in double categories and the glueing construction, TAC 35 (2020), 1208–1226.