Double categories and weak units

Simona Paoli[1]

[1]Department of Mathematics
University of Aberdeen (UK)

Virtual double categories workshop

- The classical notion of bicategory captures a 2-dimensional structure with weakly associative and weakly unital composition laws. This requires explicit description of the coherence axioms.
- In the modelling approach to weak 2-categories, a combinatorial machinery is set up so that compositions are weakly associative and weakly unital but the coherence axioms do not have to be stated explicitly.

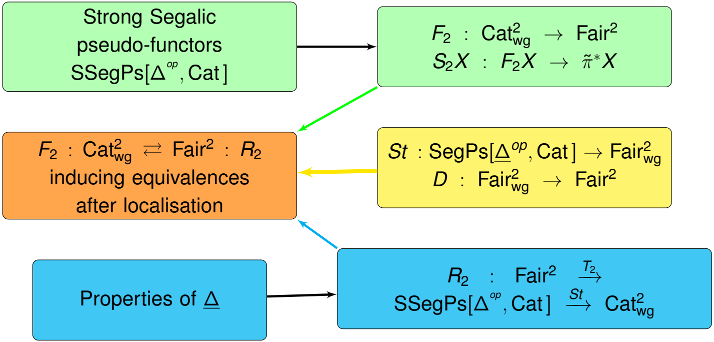
Models of weak 2-categories

- Several models of weak 2-categories exits and have been shown to be equivalent to bicategories. They also have higher dimensional generalizations.
- Weakly globular double categories $\mathsf{Cat}^2_{\mathsf{wg}}$, introduced by P. and Pronk, is a model based on a full subcategory of strict double categories.
- Fair 2-categories Fair^2, introduced by J.Kock, model weak 2-categories with strict associativity and weak unit laws.

Motivating question

- We aim to directly compare Fair^2 and Cat^2_{wg}, without using the equivalences of Fair^2 and Cat^2_{wg} with bicategories.
- This will highlight interesting features of weakly globular double categories and pave the way to higher dimensional generalizations (weak units conjecture).

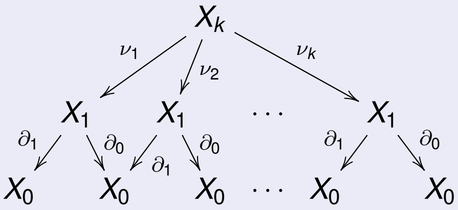
Overview



Plan:
- Background: the key players, $\mathrm{Cat}^2_{\mathrm{wg}}$, Fair^2, $\mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}]$
- From $\mathrm{Cat}^2_{\mathrm{wg}}$ to Fair^2
- From Fair^2 to $\mathrm{Cat}^2_{\mathrm{wg}}$
- Sketch of proof of main result

Let $X \in [\Delta^{op}, \mathcal{C}]$ be a simplicial object in a category $\mathcal{C}$ with pullbacks. Denote $X[k] = X_k$.

For each $k \geq 2$, let $\nu_i : X_k \to X_1$, $\nu_j = X(r_j)$, $r_j(0) = j - 1$, $r_j(1) = j$



There is a unique map, called Segal map

$$\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \ .$$

Segal maps and internal categories

- There is a nerve functor

$$N : \mathrm{Cat}\ \mathcal{C} \to [\Delta^{op}, \mathcal{C}]$$

$X \in \mathrm{Cat}\ \mathcal{C}$

$$NX \quad \cdots X_1 \times_{X_0} X_1 \times_{X_0} X_1 \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} X_1 \times_{X_0} X_1 \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow\end{smallmatrix} X_1 \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} X_0$$

Fact: $X \in [\Delta^{op}, \mathcal{C}]$ is the nerve of an internal category in $\mathcal{C}$ if and only if all the Segal maps $\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$ are isomorphisms.

Weakly globular double categories

$X \in [\Delta^{op}, \mathrm{Cat}]$ is in $\mathrm{Cat}^2_{\mathrm{wg}}$ if

i) The Segal maps are isomorphisms:

$$X_k \cong X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \qquad k \geq 2$$

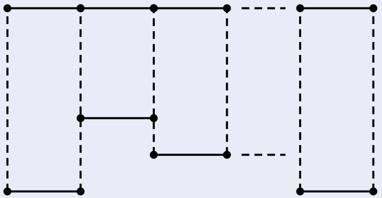
ii) Weak globularity condition: X_0 is an equivalence relation; thus $\gamma : X_0 \to X_0^d$ is an equivalence of categories, where X_0^d is the discrete category on the set of connected components of X_0.

iii) The induced Segal maps are equivalences of categories:

$$X_k \cong X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \xrightarrow{\simeq} X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \qquad k \geq 2$$

Weak globularity condition

- The set underlying X_0^d plays the role of set of objects.
- The induced Segal map condition is equivalent to



Truncation functor and hom category

- Let $p : \mathrm{Cat} \to \mathrm{Set}$ be the isomorphism classes of objects functor.
- There is a truncation functor

$$p^{(1)} : \mathrm{Cat}^2_{\mathrm{wg}} \to \mathrm{Cat}\,,$$

$(p^{(1)}X)_k = pX_k$ for all $k \geq 0$.

- Given $X \in \mathrm{Cat}^2_{\mathrm{wg}}$, $a, b \in X_0^d$ let $X(a, b)$ be the fibre at (a, b) of

$$X_1 \xrightarrow{(\partial_0, \partial_1)} X_0 \times X_0 \xrightarrow{(\gamma, \gamma)} X_0^d \times X_0^d .$$

2-Equivalences in $\mathsf{Cat}^2_{\mathsf{wg}}$

Definition

A morphism $F : X \to Y$ in $\mathsf{Cat}^2_{\mathsf{wg}}$ is a 2-equivalence if

(i) For all $a, b \in X_0^d$ $F(a, b) : X(a, b) \to Y(Fa, Fb)$ is an equivalence of categories.

(ii) $p^{(1)}F$ is an equivalence of categories.

Coloured categories

- A coloured category is a category $\mathcal{C}$ with a subcategory $\mathcal{W}$ containing all objects. The arrows of $\mathcal{W}$ are called coloured arrows.
- Morphisms of colored categories are colour-preserving functors.
- A coloured graph is a graph in which some of the edges have been singled out as colours.
- To form the free coloured category on a coloured graph take the free category on the whole graph an let $\mathcal{W}$ be the free category on the coloured part of the graph.

Coloured ordinals

- A (finite) coloured ordinal is a free coloured category on a (finite) linearly ordered coloured graph.

- Let $\mathbb{T}$ be the category of finite non-empty coloured ordinals

Morphisms are as usual ordinals for the dots but a link can be set but may not be broken.

- Functor $\pi : \mathbb{T} \to \Delta$ contracting all the links.

Semi-categories

- Let Δ_{mono} be obtained from Δ by removing the degeneracy maps.
- If $X \in [\Delta^{op}_{mono}, \mathsf{Set}]$ satisfies the Segal condition

$$X_k \cong X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \qquad k \geq 2$$

  then X is a semi-category.
- A coloured semi-category is a semi-category with a sub-semi-category comprising all objets. A morphism between coloured semi-categories is a colour preserving semi-functor.

Definition (J. Kock)

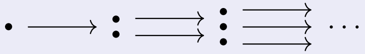
The fat delta $\underline{\Delta}$ is the category of all finite non-empty coloured semi-ordinals.

- One can naturally identify $\underline{\Delta} = \mathbb{T}_{mono}$.
- The functor $\pi : \mathbb{T} \to \Delta$ gives rise to a functor

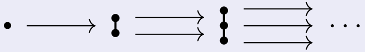
$$\pi : \underline{\Delta} = \mathbb{T}_{mono} \to \Delta_{mono} \hookrightarrow \Delta \ .$$

The fat delta, cont.

- There is a 'horizontal inclusion' $\Delta_{mono} \hookrightarrow \underline{\Delta}$ interpreting a semi-ordinal as a coloured semi-ordinal with nothing coloured



- There is a 'vertical inclusion' $\Delta_{mono} \hookrightarrow \underline{\Delta}$ interpreting a semi-ordinal as coloured semi-ordinal with everything coloured



- Let Cat be the coloured category with coloured arrows the equivalences of categories.

Definition (J. Kock)

A fair 2-category is a colour-preserving functor $X : \underline{\Delta}^{op} \to \mathrm{Cat}$ preserving discrete objects and pullbacks over discrete objects.

- Denote

$$\mathcal{O} = X_{\bullet}\,, \qquad \mathcal{A} = X_{\substack{\bullet \\ \bullet}}\,, \qquad \mathcal{U} = X_{\substack{\bullet \\ | \\ \bullet}}$$

and think of these as objects, arrows, weak identity arrows.

Remarks

- Let $\underline{m} \underset{\bullet}{+} \underline{n}$ be the pushout in $\underline{\Delta}^{op}$ of $\underline{m} \leftarrow \bullet \rightarrow \underline{n}$. Then we have a pullback

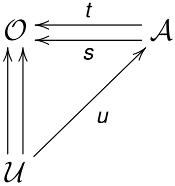
$$\begin{array}{ccc} X_{\underline{m} \underset{\bullet}{+} \underline{n}} & \longrightarrow & X_{\underline{n}} \\ \downarrow & & \downarrow \\ X_{\underline{m}} & \longrightarrow & \mathcal{O} \end{array}$$

- Hence, the restriction to either copy of $\Delta^{op}_{mono} \subset \underline{\Delta}^{op}$ is a Δ^{op}_{mono}-diagram satisfying the Segal condition: $\mathcal{A}$ and $\mathcal{U}$ are semi-categories.

- It can be shown that the two maps $\mathcal{U} \rightrightarrows \mathcal{O}$ coincide.

To give a fair 2-category X it is enough to give the following data:

a) A discrete category of objects $\mathcal{O} = X_{\bullet}$, a category of arrows $\mathcal{A} = X_{\substack{\bullet \\ \bullet}}$ and a category of weak units $\mathcal{U} = X_{\substack{\bullet \\ | \\ \bullet}}$ together with a commuting diagram



Fair 2-categories, cont.

b) Semi-category structures on $\mathcal{U} \rightrightarrows \mathcal{O}$ and $\mathcal{A} \rightrightarrows \mathcal{O}$ such that

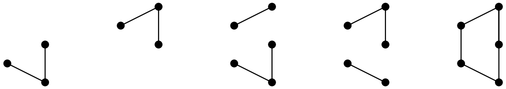
$$\begin{array}{ccc} \mathcal{U} & \rightrightarrows & \mathcal{O} \\ \downarrow & & \| \\ \mathcal{A} & \rightrightarrows & \mathcal{O} \end{array}$$

is a semi-functor.

c) The maps $\mathcal{U} \rightrightarrows \mathcal{O}$ as well as the composition maps

$$\mathcal{U} \times_{\mathcal{O}} \mathcal{A} \to \mathcal{A} \leftarrow \mathcal{A} \times_{\mathcal{O}} \mathcal{U}, \qquad \mathcal{U} \times_{\mathcal{O}} \mathcal{U} \to \mathcal{U}$$

which are images of



are equivalences of categories.

2-Equivalences in Fair^2

- There is a truncation functor

$$p^{(1)} : \mathsf{Fair}^2 \to \mathsf{Cat}$$

given by $(p^{(1)}X)_n = p(X_n)$ for all $n \in \Delta^{op}_{mono}$.

- Given $a, b \in X_0$ let $X(a, b)$ be the fiber at (a, b) of the map $X_1 \xrightarrow{(\partial_0, \partial_1)} X_0 \times X_0$.

Definition

A morphism $f : X \to Y$ in Fair^2 is a 2-equivalence if

(i) For all $a, b \in X_0$, $f_{(a,b)} : X(a, b) \to Y(fa, fb)$ is an equivalence of categories.

(ii) $p^{(1)} f$ is an equivalence of categories.

Segal-type models

$$\mathrm{Ta}^2 \hookrightarrow \mathrm{Ta}^2_{\mathrm{wg}} \hookleftarrow \mathrm{Cat}^2_{\mathrm{wg}}$$

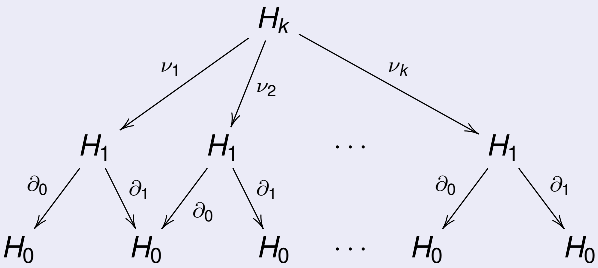
Definition

The category $\mathrm{Ta}^2_{\mathrm{wg}}$ of weakly globular Tamsamani 2-categories is the full subcategory of $[\Delta^{op}, \mathrm{Cat}\,]$ whose objects X are such that

i) X_0 is an equivalence relation.

ii) The induced Segal maps $\hat{\mu}_k : X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$ are equivalences of categories for all $k \geq 2$.

Let $H \in \mathrm{Ps}[\Delta^{op}, \mathrm{Cat}\,]$ be such that H_0 is discrete. The following diagram in Cat commutes



Hence there is a unique Segal map for all $k \geq 2$

$$H_k \to H_1 \times_{H_0} \overset{k}{\cdots} \times_{H_0} H_1 \ .$$

Segalic pseudo-functors

Definition

The category $\mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}]$ is the full subcategory of $\mathrm{Ps}[\Delta^{op}, \mathrm{Cat}]$ whose objects H are such that

i) H_0 is discrete.

ii) All Segal maps are isomorphisms for all $k \geq 2$

$$H_k \cong H_1 \times_{H_0} \overset{k}{\cdots} \times_{H_0} H_1 \ .$$

Theorem

There is a functor

$$Tr_2 : \mathrm{Ta}^2_{\mathrm{wg}} \to \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}]$$

$$(Tr_2 X)_k = \begin{cases} X_0^d, & k = 0 \\ X_1, & k = 1 \\ X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1, & k > 1 \,. \end{cases}$$

Further, the strictification functor $St : \mathrm{Ps}[\Delta^{op}, \mathrm{Cat}] \to [\Delta^{op}, \mathrm{Cat}]$ *restricts to a functor*

$$St : \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}] \to \mathrm{Cat}^2_{\mathrm{wg}} \,.$$

Strong Segalic pseudo-functors

- The inclusion functor $i : \Delta^{op}_{mono} \to \Delta^{op}$ induces a functor $i^* : \mathrm{Ps}[\Delta^{op}, \mathrm{Cat}] \to \mathrm{Ps}[\Delta^{op}_{mono}, \mathrm{Cat}]$.

Definition

A Segalic pseudo-functor $X \in \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}]$ is called strong if $i^* X \in [\Delta^{op}_{mono}, \mathrm{Cat}]$. A morphism of strong Segalic pseudo-functors is a pseudo-natural transformation F in $\mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}]$ such that $i^* F$ is a natural transformation in $[\Delta^{op}_{mono}, \mathrm{Cat}]$.

- We denote by $\mathrm{SSegPs}[\Delta^{op}, \mathrm{Cat}]$ the category of strong Segalic pseudo-functors, so that

$$i^* : \mathrm{SSegPs}[\Delta^{op}, \mathrm{Cat}] \to [\Delta^{op}_{mono}, \mathrm{Cat}] \ .$$

$\mathsf{Cat}^2_{\mathsf{wg}}$ and strong Segalic pseudo-functors

Proposition

The restriction to $\mathsf{Cat}^2_{\mathsf{wg}} \subset \mathsf{Ta}^2_{\mathsf{wg}}$ *of the functor* $Tr_2 : \mathsf{Ta}^2_{\mathsf{wg}} \to \mathsf{SegPs}[\Delta^{op}, \mathsf{Cat}]$ *is a functor*

$$Tr_2 : \mathsf{Cat}^2_{\mathsf{wg}} \to \mathsf{SSegPs}[\Delta^{op}, \mathsf{Cat}].$$

- To show that $i^* Tr_2 X \in [\Delta^{op}_{mono}, \mathsf{Cat}]$ we show that

$$\partial'_i = Tr_2 \partial_i : (Tr_2 X)_n \to (Tr_2 X)_{n-1}$$

satisfy the semi-simplicial identities $\partial'_i \partial'_j = \partial'_{j-1} \partial'_i, \quad i < j$.

Idea of proof

- The induced Segal maps ($k \geq 2$)

$$\hat{\mu}_k : X_k = X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 = (Tr_2 X)_k$$

  is injective on objects, thus $\nu_k \hat{\mu}_k = \mathrm{Id}$, where ν_k is the pseudo-inverse.

- Thus for instance for $k > 2$

$$(Tr_2 X)_{k+1} \xrightarrow{\partial'_j} (Tr_2 X)_k \xrightarrow{\partial'_i} (Tr_2 X)_{k-1}$$

$$\partial'_i \partial'_j = \hat{\mu}_{k-1} \partial_i \nu_k \hat{\mu}_k \partial_j \nu_{k+1} = \hat{\mu}_{k-1} \partial_i \partial_j \nu_{k+1} =$$

$$= \hat{\mu}_{k-1} \partial_{j-1} \partial_i \nu_{k+1} = \hat{\mu}_{k-1} \partial_{j-1} \nu_k \hat{\mu}_k \partial_i \nu_{k+1} = \partial'_{j-1} \partial'_i \ .$$

Theorem

There is a functor

$$F_2 : \mathsf{Cat}^2_{\mathsf{wg}} \to \mathsf{Fair}^2$$

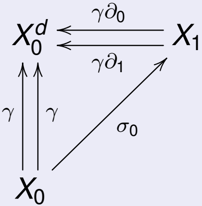
such that $(F_2X)_0 = X_0^d$, $p^{(1)}X = p^{(1)}F_2X$ *and, for each* $a, b \in X_0^d$, $X(a,b) \cong F_2X(a,b)$.

F_2 *preserves* 2*-equivalences.*

- Given $X \in \mathrm{Cat}^2_{\mathrm{wg}}$ define

$$(F_2 X)_{\bullet} = X_0^d, \qquad (F_1 X)_{\substack{\bullet\\ \bullet}} = X_1, \qquad (F_2 X)_{\substack{\bullet\\ |\\ \bullet}} = X_0$$

with the commuting diagram



where $\partial_0, \partial_1 : X_1 \to X_0$ (resp. $\sigma_0 : X_0 \to X_1$) are the face (resp. degeneracy) operators in X.

Idea of proof, cont.

- Since $i^* Tr_2 X \in [\Delta^{op}_{mono}, \mathrm{Cat}]$, $i^* Tr_2 X$ is a semi-category object internal to Cat ,

$$X_1 \times_{X_0^d} X_0 \longrightarrow X_1 \underset{\gamma\partial_1}{\overset{\gamma\partial_0}{\rightrightarrows}} X_0^d \, .$$

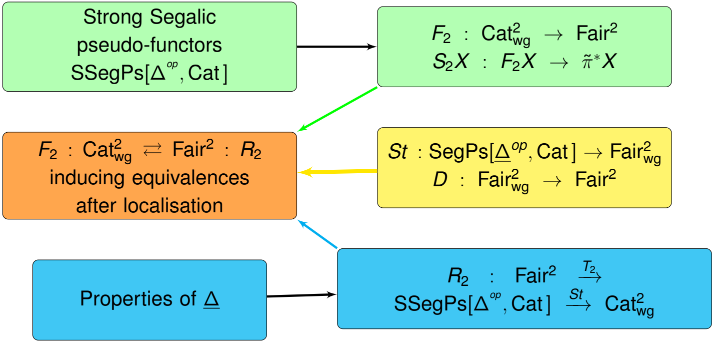
which also restricts to a semi-category structure internal to Cat

$$X_0 \times_{X_0^d} X_0 \longrightarrow X_0 \underset{\gamma}{\overset{\gamma}{\rightrightarrows}} X_0^d \, .$$

- γ as well as the following composition maps are equivalences of categories

$$X_0 \times_{X_0^d} X_0 \to X_0, \qquad X_0 \times_{X_0^d} X_1 \to X_1, \qquad X_1 \times_{X_0^d} X_0 \to X_1$$

Overview



Plan:
- Background: the key players, Cat^2_{wg}, Fair^2, $\mathsf{SegPs}[\underline{\Delta}^{op}, \mathsf{Cat}]$
- From Cat^2_{wg} to Fair^2
- From Fair^2 to Cat^2_{wg}
- Sketch of proof of main result

Proposition

There is a functor

$$T_2 : \mathsf{Fair}^2 \to \mathsf{SSegPs}[\Delta^{op}, \mathsf{Cat}]$$

such that, for each $X \in \mathsf{Fair}^2$, $(T_2X)_0 = X_0$, $(T_2X)_1 = X_1$ *and* $(T_2X)_r = X_1 \times_{X_0} \overset{r}{\cdots} \times_{X_0} X_1$ *for* $r \geq 2$.

- For each $\underline{k} \in \underline{\Delta}$ and $X \in \text{Fair}^2$ there is an equivalence of categories
$$\alpha_{\underline{k}} : X_{\pi(\underline{k})} \leftrightarrows X_{\underline{k}} : \beta_{\underline{k}}$$
such that $\beta_{\underline{k}}\alpha_{\underline{k}} = \text{Id}$.

- Let $\underline{f} : \underline{n} \to \underline{m}$ and $\underline{f}' : \underline{n}' \to \underline{m}'$ be maps in $\underline{\Delta}^{op}$ with $\pi\underline{f} = \pi\underline{f}'$. Then, if $X \in \text{Fair}^2$, $\beta_{\underline{m}} X(\underline{f})\alpha_{\underline{n}} = \beta_{\underline{m}'} X(\underline{f}')\alpha_{\underline{n}'}$.

The functor T_2, cont.

- Given $X \in \text{Fair}^2$ and $n \in \Delta^{op}$, let $(T_2 X)_n = X_{\underline{n}}$.

- Given $f : n \to m$ in Δ^{op}, choose $\underline{f} : \underline{n} \to \underline{m}$ in $\underline{\Delta}^{op}$ with $\pi\underline{f} = f$ and let $T_2 f$ be given by the composite

$$X_n \xrightarrow{\alpha_{\underline{n}}} X_{\underline{n}} \xrightarrow{\underline{f}} X_{\underline{m}} \xrightarrow{\beta_{\underline{m}}} X_m \ .$$

- From the previous slide, this is well defined.

- Given $n \xrightarrow{f} m \xrightarrow{g} s$ in Δ^{op}, to define $T_2(gf)$ we need maps in $\underline{\Delta}^{op}$

$$\underline{n} \xrightarrow{\underline{f}} \underline{m} \xrightarrow{\underline{g}} \underline{s}, \pi(\underline{f}) = f, \pi(\underline{g}) = g,$$

so that $T_2(gf)$ is the composite $X_n \xrightarrow{\alpha_{\underline{n}}} X_{\underline{n}} \xrightarrow{\underline{gf}} X_{\underline{s}} \xrightarrow{\beta_{\underline{m}}} X_m$.

- The existence of the liftings $\underline{f}, \underline{g}$ of f and g is not obvious.

- Main issue: one can easily find maps

$$\underline{n} \xrightarrow{\underline{f}'} \underline{m} \qquad \underline{m}' \xrightarrow{\underline{g}'} \underline{s} \qquad \pi(\underline{f}') = f,\ \pi(\underline{g}') = g$$

but why can we ensure that we can find maps such that $\underline{m} = \underline{m}'$?

Lifting sequences of maps in delta

Proposition

Given maps in Δ

$$n_1 \xrightarrow{f_1} n_2 \xrightarrow{f_2} n_3 \rightarrow \cdots \xrightarrow{f_k} n_{k+1}$$

there are maps in $\underline{\Delta}$

$$\underline{n}_1 \xrightarrow{\underline{f}_1} \underline{n}_2 \xrightarrow{\underline{f}_2} \underline{n}_3 \rightarrow \cdots \xrightarrow{\underline{f}_k} \underline{n}_{k+1}$$

with $\pi \underline{f}_j = f_j$.

- The proof is by induction on k and depends on properties of $\underline{\Delta}$ in relation to Δ.

Definition

Let $R_2 : \mathsf{Fair}^2 \to \mathsf{Cat}^2_{\mathsf{wg}}$ be the composite

$$\mathsf{Fair}^2 \xrightarrow{T_2} \mathsf{SSegPs}[\Delta^{op}, \mathsf{Cat}\,] \xrightarrow{St} \mathsf{Cat}^2_{\mathsf{wg}}\ ,$$

where St is the restriction to $\mathsf{SSegPs}[\Delta^{op}, \mathsf{Cat}\,]$ of the functor $St\ : \mathsf{SegPs}[\Delta^{op}, \mathsf{Cat}\,] \to \mathsf{Cat}^2_{\mathsf{wg}}$.

Theorem (P.)

The functors

$$F_2 : \mathsf{Cat}^2_{\mathsf{wg}} \rightleftarrows \mathsf{Fair}^2 : R_2$$

induce an equivalence of categories after localization with respect to the 2-equivalences

$$\mathsf{Cat}^2_{\mathsf{wg}}/\!\sim\ \simeq\ \mathsf{Fair}^2/\!\sim .$$

Method of proof

- Given $X \in \mathrm{Cat}^2_{\mathrm{wg}}$, we produce a 2-equivalence in $\mathrm{Cat}^2_{\mathrm{wg}}$ between X and R_2F_2X.
- Given $Y \in \mathrm{Fair}^2$, we produce a zig-zag of 2-equivalences in Fair^2 between Y and F_2R_2Y.
- The construction of these maps requires a new player, the category $\mathrm{Fair}^2_{\mathrm{wg}}$ of weakly globular fair 2-categories.

Proof of main result: comparing X and R_2F_2X

- Recall $R_2 : \mathsf{Fair}^2 \xrightarrow{T_2} \mathsf{SSegPs}[\Delta^{op}, \mathsf{Cat}] \xrightarrow{St} \mathsf{Cat}^2_{\mathsf{wg}}$ and $F_2 : \mathsf{Cat}^2_{\mathsf{wg}} \to \mathsf{Fair}^2$.
- Given $X \in \mathsf{Cat}^2_{\mathsf{wg}}$ there is a levelwise equivalence pseudo-natural transformation in $T_2F_2X \to X$ in $\mathsf{Ps}[\Delta^{op}, \mathsf{Cat}]$.
- By adjunction, this corresponds to a levelwise equivalence natural transformation in $[\Delta^{op}, \mathsf{Cat}]$

$$R_2F_2X = St\, T_2F_2X \to X\ .$$

- In particular, this is a 2-equivalence in Fair^2 between X and R_2F_2X. Hence $X \cong R_2F_2X$ in $\mathsf{Cat}^2_{\mathsf{wg}}/\sim$.

Proof of main result: comparing Y and $R_2 F_2 Y$

- Given $Y \in \text{Fair}^2$, there is a levelwise equivalence pseudo-natural transformation $F_2 R_2 Y \to Y$ in $\text{Ps}[\underline{\Delta}^{op}, \text{Cat}]$.
- By adjunction, this gives a natural transformation in $[\underline{\Delta}^{op}, \text{Cat}]$ $St\, F_2 R_2 Y \to Y$.
- Since $F_2 St\, T_2 Y \in \text{Fair}^2$ then $F_2 R_2 Y \in \text{SegPs}[\underline{\Delta}^{op}, \text{Cat}]$ so $St\, F_2 R_2 Y \in \text{Fair}^2_{\text{wg}}$.
- So we have a zig-zag of 2-equivalences in $\text{Fair}^2_{\text{wg}}$

$$F_2 R_2 Y \leftarrow St\, F_2 R_2 Y \to Y$$

Comparing Y and $R_2 F_2 Y$, cont.

- There is a functor $D : \mathrm{Fair}^2_{\mathrm{wg}} \to \mathrm{Fair}^2$ which preseves 2-equivalences and is identity on Fair^2.

- From the zig-zag of 2-equivalences in $\mathrm{Fair}^2_{\mathrm{wg}}$

$$F_2 R_2 Y \leftarrow St\, F_2 R_2 Y \to Y$$

  we obtain the zig-zag of 2-equivalences in Fair^2

$$F_2 R_2 Y = DF_2 R_2 Y \leftarrow DSt\, F_2 R_2 Y \to DY = Y\ .$$

- It follows that $Y \cong R_2 F_2 Y$ in $\mathrm{Fair}^2/\sim$.

Summary

- Several models of weak 2-categories, in particular the Segal-type models and fair 2-categories.
- Direct comparison between weakly globular double categories and fair 2-categories.
- New light on weakly globular double categories, as encoding weak units.
- Lifting of strings of maps from Δ to $\underline{\Delta}$; category $\mathrm{Fair}^2_{\mathrm{wg}}$.
- Potential for higher dimensional generalisations.