Formalising size in formal category theory

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Virtual Double Categories Workshop
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Plan

1. Motivation: unifying the notions of subobject classifier and Yoneda embedding
2. Augmented virtual double categories
3. (Co-)cartesian cells
4. Formal Yoneda embeddings
5. Some formal results involving a formal notion of size

Power sets

The power set PA of a set A *classifies relations* $J \subseteq A \times B$:

$$\begin{array}{ccc} A & \overset{\forall J}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\;} & B \\ & {\scriptstyle y}\searrow \quad \swarrow {\scriptstyle \exists! J^{\lambda}} & \\ & PA & \end{array}$$

$$y(a) = \{a\}$$

$$J^{\lambda}(b) = \{a \in A \mid aJb\}$$

Uniqueness of J^{λ}? How to fill the triangle with a cell?

Power sets

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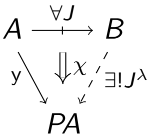


$$\mathsf{y}(a) = \{a\}$$

$$J^\lambda(b) = \{a \in A \mid aJb\}$$

Regarding $PA = (PA, \subseteq)$ as a partial order the cell χ is the equality of relations

$$aJb \iff \{a\} \subseteq J^\lambda(b) \iff (\mathsf{y}\, a) \subseteq (J^\lambda b)$$

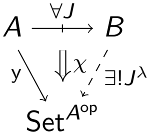
χ is *cartesian*: it exhibits J as the restriction of $\subseteq$ along y and J^λ.

Notes

- In topos theory the above (roughly) generalises as the notion of *power object*.
- Taking $A = 1$ gives the *subobject classifier* $\mathsf{t}\colon 1 \to P1 =: \Omega$.

Power sets versus Yoneda embeddings

The category $\mathsf{Set}^{A^{\mathrm{op}}}$ of presheaves on a locally small category A *classifies profunctors* $J\colon A^{\mathrm{op}} \times B \to \mathsf{Set}$:



$$\mathrm{y}(a) = A(-, a)$$

$$J^{\lambda}(b) = J(-, b)$$

χ exhibits J as the "restriction of $\mathsf{Set}^{A^{\mathrm{op}}}$ along y and J^{λ}":

$$J(a, b) \cong \mathsf{Set}^{A^{\mathrm{op}}}\Big(A(-, a), J(-, b)\Big) = \mathsf{Set}^{A^{\mathrm{op}}}(\mathrm{y}\, a, J^{\lambda} b)$$

Unifying the notions of power object and Yoneda embedding

Aim: A formal notion that unifies the notions of power object and Yoneda embedding.

Requirements on the setting

- "double dimensional" ($J\colon A \nrightarrow B$ and $f\colon A \to C$)
- for Yoneda embeddings: $\mathsf{Set}^{A^{\mathrm{op}}}$ is likely to be locally large
  - $\Rightarrow$ Set-profunctors might not compose
  - $\Rightarrow$ $\mathsf{Set}^{A^{\mathrm{op}}}$ might not have a unit (hom) Set-profunctor

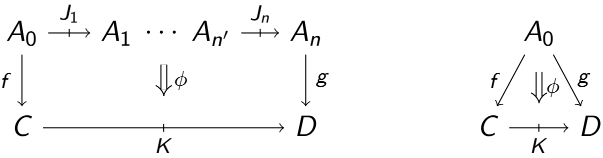
Natural setting is that of *augmented virtual double categories*.

Virtual double categories

Definition (Burroni)

A *virtual double category* $\mathcal{K}$ has

- objects A, B, ...
- vertical morphisms $f: A \to C$, ...
- horizontal morphisms $J: A \nrightarrow B$, ...
- *unary* cells



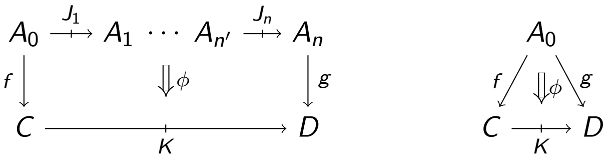
- vertical composition of morphisms $g \circ f$ and cells $\psi \circ (\phi_1, \ldots, \phi_n)$
- vertical identity morphisms id_A and cells id_J

Augmented virtual double categories

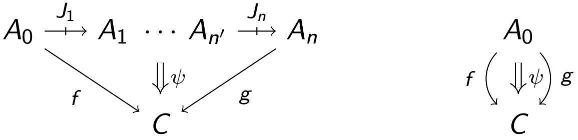
Definition

An *augmented virtual double category* $\mathcal{K}$ has

- objects, vertical morphisms and horizontal morphisms
- *unary* cells



- *nullary* cells



- vertical composition and vertical identities

Examples

- Prof of Set-profunctors between locally large categories

- sProf of *small* profunctors $J\colon A \nrightarrow B$ between locally small categories
(each $J(-, b)$ is a small presheaf, i.e. a small colimit of representable presheaves)

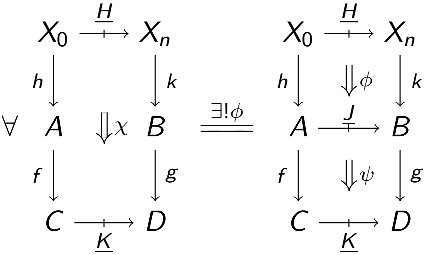
- ModRel of *modular* relations $J\colon A \nrightarrow B$ between partial orders
$(a_1 \leq a_2, \quad a_2 J b_1, \quad b_1 \leq b_2 \quad \Rightarrow \quad a_1 J b_2)$

Examples (cont'd)

- ModRel($\mathcal{E}$) of internal modular relations in a finitely complete category $\mathcal{E}$
- dFib($\mathcal{C}$) of internal discrete two-sided fibrations in a finitely complete 2-category $\mathcal{C}$
- $(\mathcal{V}, \mathcal{V}')$-Prof of $\mathcal{V}$-profunctors between $\mathcal{V}'$-categories ($\mathcal{V} \subseteq \mathcal{V}'$ a universe enlargement)
- MonProf of lax monoidal Set-profunctors between locally large monoidal categories

Restrictions

Restrictions of horizontal morphisms are defined by cartesian cells: ψ below with $|\underline{J}| \leq 1$ is *cartesian* if



If $|\underline{J}| = 1$ then $\underline{K}(f, g) := J$ is the *restriction of* $\underline{K}$ *along* f *and* g.

Examples in Prof:

- $K(f, g) := \left[A^{\mathrm{op}} \times B \xrightarrow{f^{\mathrm{op}} \times g} C^{\mathrm{op}} \times D \xrightarrow{K} \mathrm{Set}\right]$
- $C(f, g) := \left[A^{\mathrm{op}} \times B \xrightarrow{f^{\mathrm{op}} \times g} C^{\mathrm{op}} \times C \xrightarrow{\hom_C} \mathrm{Set}\right]$
  (only exists if all $C(fa, gb)$ are small)
- $f_* := C(f, \mathrm{id})$ (exists if f is *locally small*: all $C(fa, c)$ are small)

Restrictions

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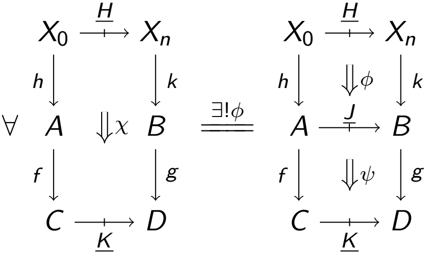


If $|\underline{J}| = 1$ then $\underline{K}(f, g) := J$ is the *restriction of* $\underline{K}$ *along* f *and* g.

In general:

- $f : A \to C$ is *fully faithful* if id_f is cartesian
- f is *admissible* if $f_* := C(f, \mathrm{id})$ exists (*companion* of f)
- A is *admissible* if $I_A := A(\mathrm{id}, \mathrm{id})$ exists (*horizontal unit*)

Cocartesian cells

(Mostly) vertically dual: ϕ below is *weakly unary-cocartesian* if

$$\forall \quad \begin{array}{ccc} X_0 & \overset{H}{\longrightarrow} & X_n \\ h\downarrow & & \downarrow k \\ A & \Downarrow\chi & B \\ f\downarrow & & \downarrow g \\ C & \underset{K}{\longrightarrow} & D \end{array} \quad \overset{\exists!\psi}{=\!=\!=} \quad \begin{array}{ccc} X_0 & \overset{H}{\longrightarrow} & X_n \\ h\downarrow & \Downarrow\phi & \downarrow k \\ A & \overset{J}{\longrightarrow} & B \\ f\downarrow & \Downarrow\psi & \downarrow g \\ C & \underset{K}{\longrightarrow} & D \end{array}$$

If $h = \mathrm{id}_{X_0}$ and $k = \mathrm{id}_{X_n}$ then ϕ defines $J := H_1 \odot \cdots \odot H_n$ as the *weak unary-composite* of $(H_1, \ldots, H_n)$.

Formal Yoneda morphisms

A *Yoneda morphism* is a "dense" morphism $\mathrm{y}\colon A \to \widehat{A}$ that satisfies the *Yoneda axiom*:

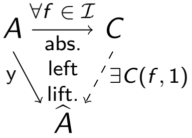
$$\forall J\colon A \nrightarrow B \qquad \exists \quad \begin{array}{ccc} A & \overset{J}{\nrightarrow} & B \\ & {}_{\mathrm{y}}\searrow \quad \text{cart} \quad \swarrow_{J^{\lambda}} & \\ & \widehat{A} & \end{array}$$

This recovers (up to equivalence):

- ModRel: power sets ($\widehat{A} \cong PA$)
- ModRel($\mathcal{E}$): power objects in $\mathcal{E}$
  (ModRel($\mathcal{E}$) has Yoneda morphisms $\iff$ $\mathcal{E}$ is a topos)
- Prof: classical Yoneda embeddings $\mathrm{y}\colon A \to \mathrm{Set}^{A^{\mathrm{op}}}$
- MonProf: "Day-convolution" on $\mathrm{Set}^{A^{\mathrm{op}}}$

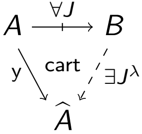
Yoneda structures versus the present Yoneda morphisms

Yoneda structure on a 2-category $\mathcal{C}$ (Street and Walters)

- right ideal $\mathcal{I}$ of admissible morphisms
- 

- besides being $\mathcal{I}$ an ideal, notion of admissibility is "isolated"

Yoneda morphisms in an augmented virtual double category

- *all* $J\colon A \nrightarrow B$ are admissible
  f is admissible $\iff$ f_* exists
- 

- notion of admissibility related to that of fully faithfulness, adjunction and restriction

Example: 2-toposes

Weber shows that a "2-topos" structure on $\mathcal{C}$ induces a Yoneda structure on $\mathcal{C}$. The latter's formal Yoneda embeddings in fact satisfy our notion in $\mathrm{dFib}(\mathcal{C})$.

Right adjoint to left Kan extension

Given a locally small category A denote by $(\mathrm{Set}^{A^{\mathrm{op}}})_{\mathrm{s}}$ the locally small category of small presheaves on A. In "Limits of small functors" by Day and Lack:

Proposition

Given a functor $f\colon A \to C$ consider $f^{\sharp}\colon (\mathrm{Set}^{A^{\mathrm{op}}})_{\mathrm{s}} \to (\mathrm{Set}^{C^{\mathrm{op}}})_{\mathrm{s}}$ given by left Kan extension along f. $f^{\sharp}$ admits a right adjoint if and only if the Set-profunctor $f_{*}\colon A \nrightarrow C$ is a small.

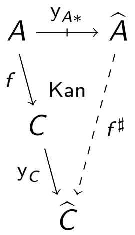
Right adjoint to left Kan extension

Proposition

Given a functor $f \colon A \to C$ consider $f^\sharp \colon (\mathrm{Set}^{A^{\mathrm{op}}})_s \to (\mathrm{Set}^{C^{\mathrm{op}}})_s$ given by left Kan extension along f. $f^\sharp$ admits a right adjoint if and only if the Set-profunctor $f_* \colon A \nrightarrow C$ is a small.

Formalisation

Consider the "pointwise left Kan cell" on the right. Among the following (b) $\Rightarrow$ (a) holds. Under mild conditions (a) $\Leftrightarrow$ (b) $\Leftarrow$ (c). Under stronger conditions (e.g. in sProf) all are equivalent.



(a) $f^\sharp$ admits a right adjoint

(b) $\mathrm{y}_C \circ f$ is admissible

(c) f is admissible

Yoneda embeddings

The definition of Yoneda morphism $\mathrm{y}\colon A \to \widehat{A}$ does *not* imply that y is fully faithful!

Lemma

A is admissible if and only if both the restriction $\widehat{A}(\mathrm{y}, \mathrm{y})$ exists and y is fully faithful.

Cocompleteness of presheaf objects

In "Yoneda structures from 2-toposes" by Weber:

Theorem (for Yoneda structures)

If C is "small" and $\widehat{C}$ *is cocomplete* then $\mathrm{y}\colon C \to \widehat{C}$ defines $\widehat{C}$ as the free cocompletion of C.

We can do better. Using "On the size of categories" by Street and Freyd, for any locally small category B we have:

$$B \text{ is essentially small} \iff \mathsf{Set}^{B^{\mathrm{op}}} \text{ is locally small}$$

$$\iff \forall A \overset{H}{\nrightarrow} B \overset{J}{\nrightarrow} E \text{ the weak unary-composite } H \odot J \text{ exists in } \mathsf{Prof}$$

Definition

- Call an object B *weakly small* if the weak unary-composite $H \odot J$ exists for all $A \overset{H}{\nrightarrow} B \overset{J}{\nrightarrow} E$.
- Call an object M *weakly cocomplete* if the "weak left Kan extensions" of all pairs $M \overset{d}{\leftarrow} B \overset{J}{\nrightarrow} E$, with B weakly small, exist.

Cocompleteness of presheaf objects

Theorem (for Yoneda structures)

If C is "small" and $\widehat{C}$ *is cocomplete* then $\mathrm{y}\colon C \to \widehat{C}$ defines $\widehat{C}$ as the free cocompletion of C.

- Call an object B *weakly small* if the weak unary-composite $H \odot J$ exists for all $A \xrightarrow{H} B \xrightarrow{J} E$.
- Call an object M *weakly cocomplete* if the "weak left Kan extension" of all pairs $M \xleftarrow{d} B \xrightarrow{J} E$, with B weakly small, exists.

Theorem (for augmented virtual double categories)

Let $\mathrm{y}\colon A \to \widehat{A}$ be such that the restrictions $\widehat{A}(\mathrm{y}, f)$ exist for all $f\colon X \to \widehat{A}$. The following are equivalent for objects B and E:

- the weak left Kan extensions of all pairs $\widehat{A} \xleftarrow{d} B \xrightarrow{J} E$ exist
- the weak unary-composite $H \odot J$ exists for all $A \xrightarrow{H} B \xrightarrow{J} E$

In particular $\widehat{A}$ is weakly cocomplete.

Thanks!