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An homotopical way to compare 2-cats & double categories

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STRUCTURE

- Preliminaries
- Motivation
- First attempt.
- Modification
- Generalization

PRELIMINARIES

DEF. A 2-category $\mathcal{A}$ consists of

- objects A, B, C
- morphisms $f : A \to B$ between objects
- 2-morphisms or 2-cells between 1-morphisms

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \alpha"] & B \end{tikzcd}$$

- we also have 1-morphisms identities $id_A : A \to A$, 2-cell identities $id_f : f \Rightarrow f$, together with composition of morphisms, and (vertical & horizontal) of 2-cells, that verify axioms of unitality, associativity and interchange.

Ex. Cat :
- categories
- functors
- natural transformations.

IMPORTANT! Any 2-category $\mathcal{A}$ can be seen as a double category.

- objects: same as $\mathcal{A}$
- hor. mor.: 1-morphism of $\mathcal{A}$
- ver. mor.: only identities.
- squares: one square

$$\begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "g"'] & \cdot
\end{tikzcd}$$

for each 2-cell $\mathcal{A}$

$$\begin{tikzcd}
\cdot \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \cdot
\end{tikzcd}$$

horizontal embedding

This defines a functor $\mathbb{H}: 2Cat \longrightarrow DblCat$, which is part of an adjunction

$$\begin{tikzcd}
DblCat \arrow[r, bend right, "\mathbf{H}"'] \arrow[r, phantom, "\perp"] & 2Cat \arrow[l, bend right, "\mathbb{H}"']
\end{tikzcd}$$

underlying hor. cat.

Model structures

IDEA: We want to work with a category $\mathcal{C}$ and a class of morphisms W that we think morally as isomorphisms, and its "homotopy category"

ex. quasi-isos of $Ch(R\text{-Mod})$ and its derived cat.

Sol: Model str. and take $\mathcal{C}[W^{-1}]$

slogan: Model structures provide a framework to do homotopy theory.

DEF. A model category consists of a category $\mathcal{C}$, with finite co(limits), together with three distinguished classes of morphisms:

1. weak equivalences
2. fibrations
3. cofibrations

satisfying certain axioms.

Arrows between them?

↗ preserves w.e. & cofibs

$$\begin{tikzcd} M \arrow[r, bend left] \arrow[r, phantom, "\perp"] & N \arrow[l, bend left] \end{tikzcd}$$

Quillen pair.

↳ preserves w.e. and fibs.

MOTIVATION

Disclaimer: There exist in the literature several other model structures on DblCat.

- Fiore-Paoli-Pronk: Model structures on the category of small double cats
- Fiore-Paoli: A Thomason model structure on the cat. of small n-fold cats

So why another one?

None of those are compatible with 2-categories

↳ with Lack's model str. on 2Cat.

(And it has proven useful to study 2-Cat inside DblCat)

1.

To be able to study 2Cat homotopically by looking at it inside DblCat through the functor

$\mathbb{H}: 2\mathrm{Cat} \longrightarrow \mathrm{DblCat}$

2.

We want this MS to be compatible with Moser's nerve into double $(\infty,1)$-categories

Homotopy theory for 2-categories

(Lack's model structure)

Consensus for the right notion of sameness for 2-categories $\rightsquigarrow$ biequivalence

DEF. A 2-functor is a biequivalence if:

- it's surjective on objects up to equivalence → inside the 2-category
- full on morphisms up to invertible 2-cell
- fully faithful on 2-cells

THM. (Lack) There exists a model structure on 2Cat, whose weak equivalences are the biequivalences !

(some) Properties:

- It's cofibrantly generated
- Every 2-category is fibrant ($X \to 1$ is a fibration)
- A 2-functor is trivial fibration iff it is
-
  - ~~it~~ surjective on objects
  - full on 1-morphisms
  - ff. on 2-cells.

FIRST ATTEMPT

How? Transferring

THM. [Garner, Hess, Kędziorek, Riehl, Shipley]

let M be a model category, with an adjunction

$$\begin{tikzcd} 2Cat\ \cancel{M} \arrow[r, bend left, "\cancel{L}\ H"] \arrow[r, phantom, "\perp"] & \cancel{N}\ DblCat \arrow[l, bend left, "\cancel{R}\ \mathbb{H}"] \end{tikzcd}$$

If every object in M is fibrant, + hyp., then there exists a model structure on N where F is a w.e. iff. RF is so in M.

Problem: don't see anything in the vertical direction.

Maybe we could take analogous $\mathbb{V}: \mathrm{DblCat} \longrightarrow 2\mathrm{Cat}$

$$\begin{tikzcd}
2\mathrm{Cat} \times 2\mathrm{Cat} \arrow[r, bend left, "{\overset{L}{(\mathbb{H}, \cancel{\mathbb{V}})}}"] & \mathrm{DblCat} \arrow[l, bend left, "{(\mathbb{H}, \cancel{\mathbb{V}})2}"]
\end{tikzcd} \qquad F \qquad ⊛$$

Problem: We don't see all the squares

$$\begin{tikzcd}
\cdot \arrow[r, "|"] \arrow[d, equal] & \cdot \arrow[d, equal] \\
\cdot \arrow[r] & \cdot
\end{tikzcd} + \begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r, equal] & \cdot
\end{tikzcd}$$

Solution: Make a shift, and consider a functor. $\mathrm{DblCat} \xrightarrow{\check{\mathbb{V}}} 2\mathrm{Cat}$, that takes as objects the vertical morphism and as 1-morphisms all the squares

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THM [MSV] The adjunction ⊛, where each copy of 2Cat is endowed with Lack's model str. induces a model str. on DblCat. We call the w.e. double biequivalences.

Prop. A double functor $F: \mathbb{A} \longrightarrow \mathbb{B}$ is a double biequivalence iff both $\mathbb{H}F: \mathbb{H}\mathbb{A} \rightarrow \mathbb{H}\mathbb{B}$ and $\mathbb{V}F: \mathbb{V}\mathbb{A} \longrightarrow \mathbb{V}\mathbb{B}$ are biequivalences in 2Cat.

(These can be unpacked to obtain an explicit description of double biequivalence)

Properties

- It's cofibrantly generated.
- Every double category is fibrant
- A map is a trivial fibration iff it is
  - surjective on objects
  - full on hor. morphism
  - surjective — ver. —
  - ff. on squares.

Coming back to our first objective:
behavior w.r.t. $\mathbb{H} : 2Cat \longrightarrow DblCat$

Prop. [MSV] The adjunction $$\begin{tikzcd} 2Cat \arrow[r, bend left, "\mathbb{H}"] \arrow[r, phantom, "\perp"] & DblCat \arrow[l, bend left, "\mathbf{H}"] \end{tikzcd}$$ is a Quillen pair

Moreover, the derived functor $\mathbb{H} : Ho(2Cat) \longrightarrow Ho(DblCat)$ is fully faithful.

Even better!

LEFT

Lack's MS is left-induced

$$\begin{tikzcd} 2\text{-}Cat \arrow[r, bend left, "\mathbb{H}"] \arrow[r, phantom, "\perp"] & DblCat \arrow[l, bend left, "\mathbf{H}"] \end{tikzcd}$$

RIGHT

Lack's MS is also right induced.

$$\begin{tikzcd} DblCat \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & 2Cat \arrow[l, bend left, "\mathbb{H}"] \end{tikzcd}$$

Going back to double biequivalences:
are they good?

- **In Cat**, a functor $F : C \longrightarrow D$ is an equivalence iff there exists a functor $G : D \to C$ and natural isos $1 \cong GF$ and $FG \cong 1$

- In 2-Cat, a 2functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a biequivalence iff there exists a pseudo-functor $G : \mathcal{B} \to \mathcal{A}$ and pseudo-natural equivalences $1 \simeq GF$ and $FG \simeq 1$.

In DblCat?

weakly horizontally invariant

⚠ **THM.** A double functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a double biequiv. iff there exists a double pseudo-functor $G : \mathbb{B} \to \mathbb{A}$ and (hor.) pseudo-natural equivs. $1 \simeq FG$ and $GF \simeq 1$.

NOT QUITE

DEF. A double category $\mathbb{A}$ is weakly horizontally invariant if,

$$\begin{tikzcd}
A \arrow[r, "a", "\simeq"'] \arrow[d, "u"'] \arrow[dr, phantom, "\cong"] & C \arrow[d, "w"] \\
A' \arrow[r, "\simeq", "a'"'] & C'
\end{tikzcd}$$

there is $u : A \longrightarrow A'$

What about the second objective? We need another MS.

∞-categorical reason: The previous one is not good for the nerve.

2-dimensional reason: fix the assymetries we saw.

First step: add a generating cofibration:

2-dim. adding $\mathbb{1} \sqcup \mathbb{1} \longrightarrow \nabla \mathbb{2}$ would fix the assymetry on the trivial fibrations.

∞-dim: makes sense, because cofibs. in the ∞-part are monos

Try to keep the same weak equivalences.

PROBLEM: the trivial cofibs. are not stable under pushout anymore.

SOL: Try to keep the w.e. between the fibrant objects.

Challenge: Identify the fibrant objects.

2-dim: we have a class of double cats. where the double bieq. behave really well: w.h.i.

∞-dim:

THM (Lyne Moser) $N\mathbb{A}$ is fibrant iff $\mathbb{A}$ is w.h.i.

THM [MSV] Campbell

There is a model structure on DblCat whose fibrant objects are the w.h.i categories, and whose trivial fibrations are the double functors that are surjective on objects, full on both hor. and ver. morphisms, and fully faithful on squares.

Prop. Double biequiv. are weak equivalences

⟶ What about objective 1. now?

THM. [MSV] There is a Quillen pair

$$\begin{tikzcd} \mathrm{DblCat}_{whi} \arrow[r, bend right, "id"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend right, "id"'] \arrow[r, bend right, "\mathbf{H}"'] \arrow[r, phantom, "\perp"] & 2\mathrm{Cat} \arrow[l, bend right, "\mathbb{H}"'] \end{tikzcd}$$

Moreover, the derived counit is levelwise a w.e.

⊕ $\mathbb{H} \dashv \mathbf{H}$ is a Quillen pair.
Moreover $\mathbb{H}$ is htpically fully faithful and Lack's MS. is left induced via this.

GENERALIZATION

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What we want:

Start w/ fibrant objects & cofibrations

& w.e. btwn fibrant objects and fibration btwn fibrant objects

and complete this info to a MS.

We ask for a w.f.s. (An, NFib) generated by a set J

anodyne extensions

nerve fibrations

DEF. $X \in \mathcal{C}$ is naive fibrant if $X \longrightarrow 1$ is in NFib

DEF. $X \in \mathcal{C}$, a naive fibrant replacement of X is a map $i : X \longrightarrow X'$ with i anodyne extension & X' naive fibrant

For a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$, a naive fibrant replacement of f is a comm. square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X"'] & Y \arrow[d, "i_Y"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

⇝ We use this to construct w.e.

DEF. $f : X \longrightarrow Y$ in $\mathcal{C}$ is a w.e. if there exists a naive fibrant replacement of f s.t. it is in W_f.

THM. [Guetta-MSV] $\mathcal{C}$ locally presentable, $I \subset \mathcal{C}$ set of morphisms, $(An, NFib)$ s.t. $An \subseteq cof(I)$

+ W_f class of maps btw naive fibrant obj. s.t.

verifying certain technical conditions.

$\Rightarrow$ there exists a MS. on $\mathcal{C}$ with

cofibrations $\longrightarrow$ cof(I)

fibrant obj. $\rightsquigarrow$ naive fibrant obj.

w.e. $\rightsquigarrow$ maps s.t. its naive fibrant replacement is W_f.