Double-Categorical Compositional Rewriting Theory

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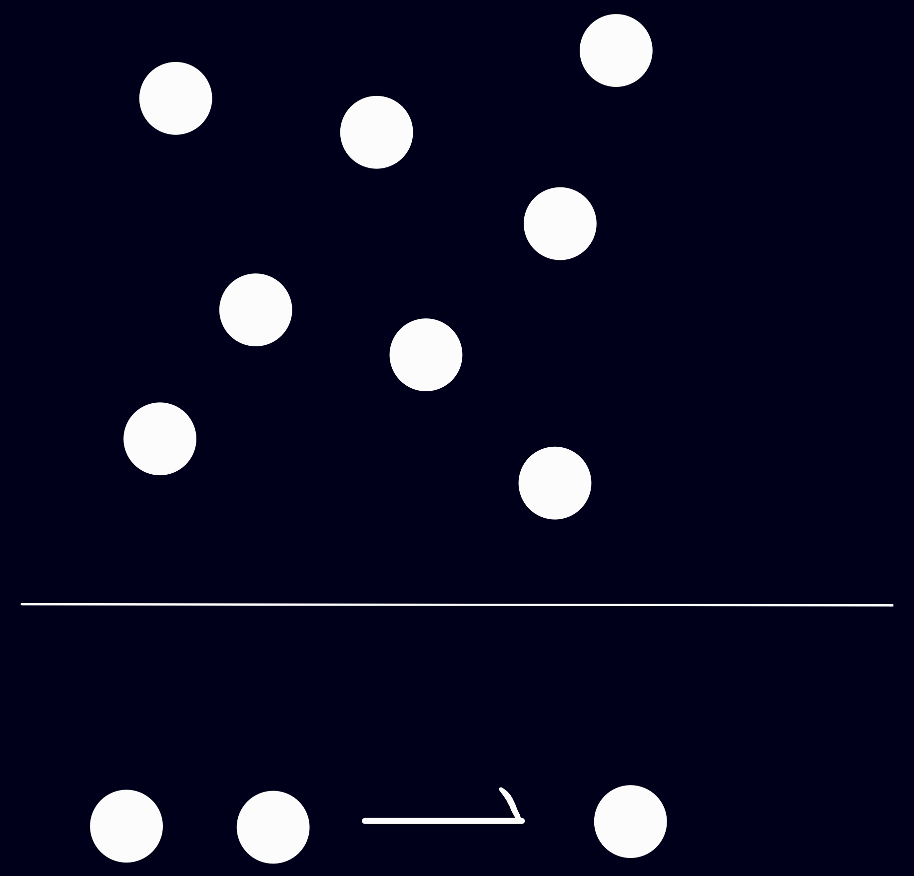
Virtual Double Categories Workshop

November 30, 2022

Based upon joint work with

- P.-A. Melliès & N. Zeilberger
- R. Harmer & J. Krivine (2204.07175)

① MOTIVATION



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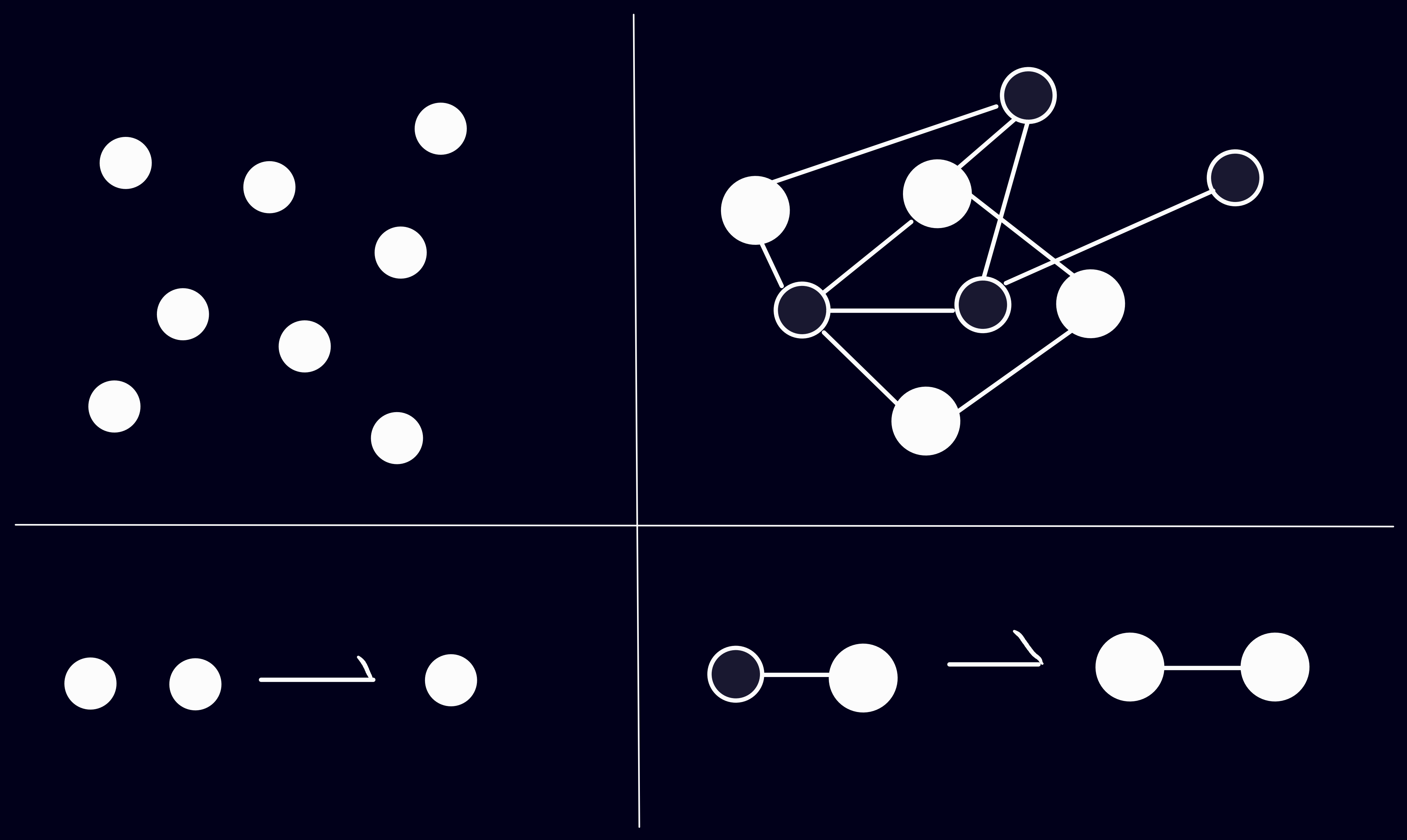

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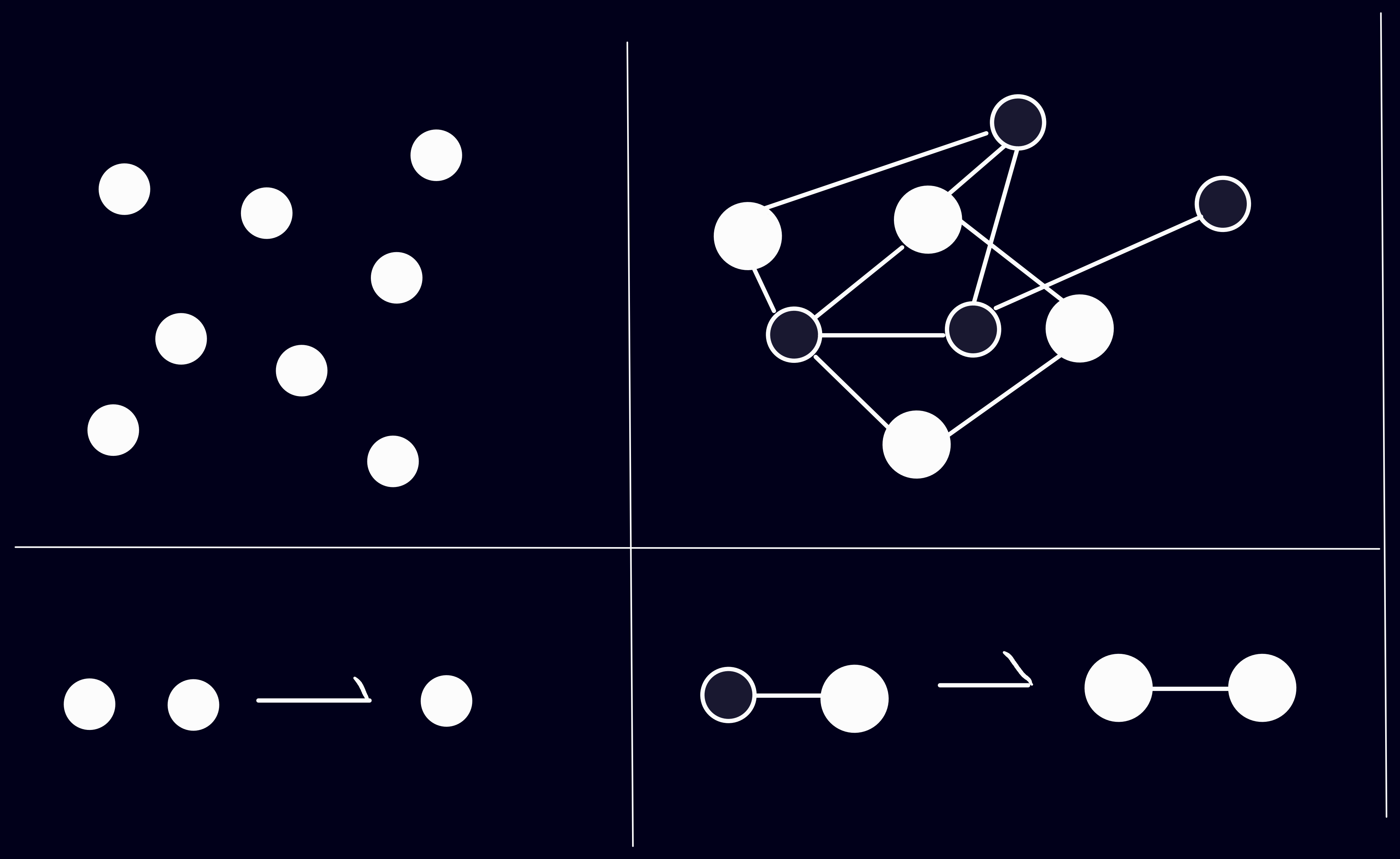
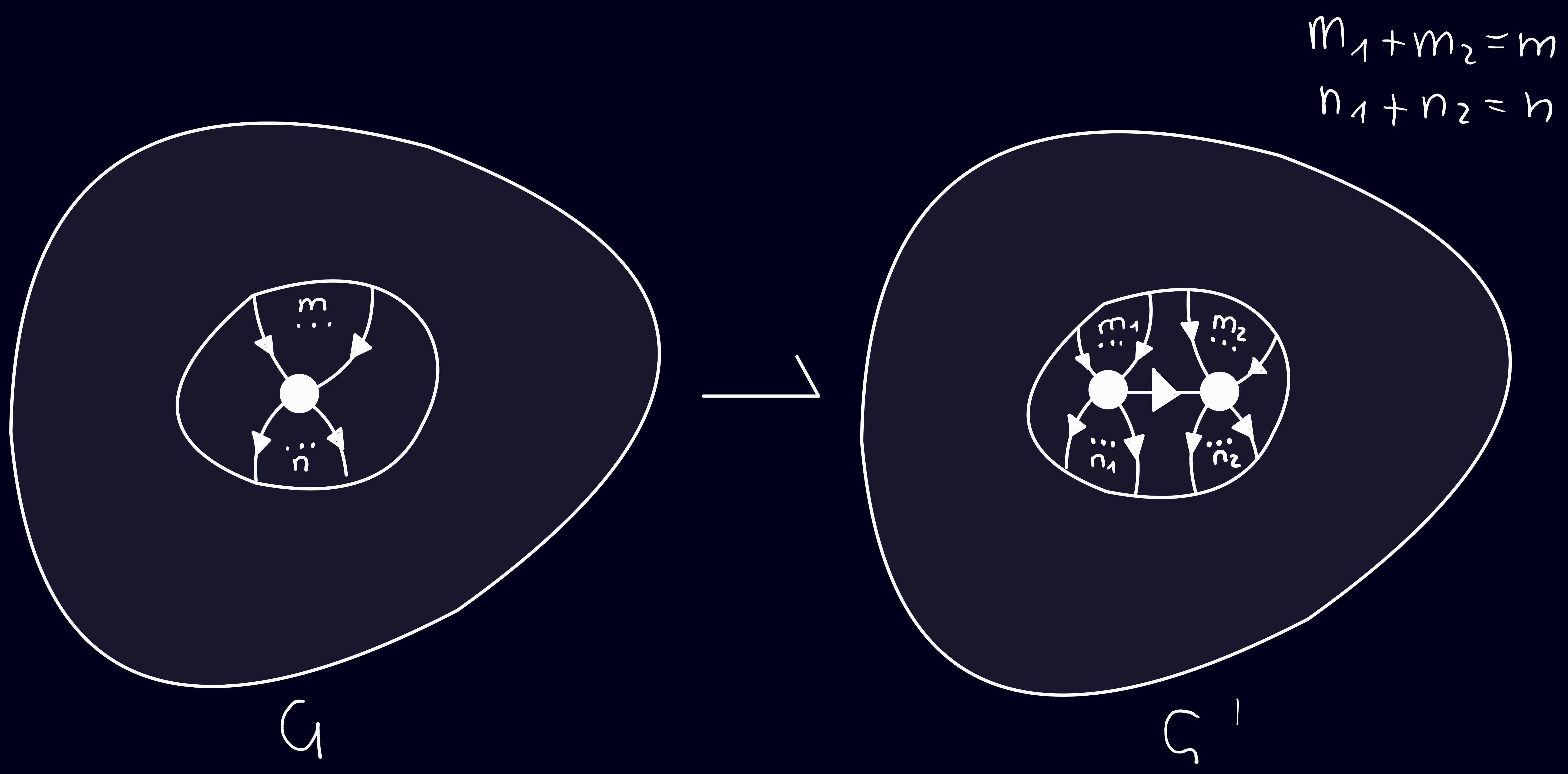
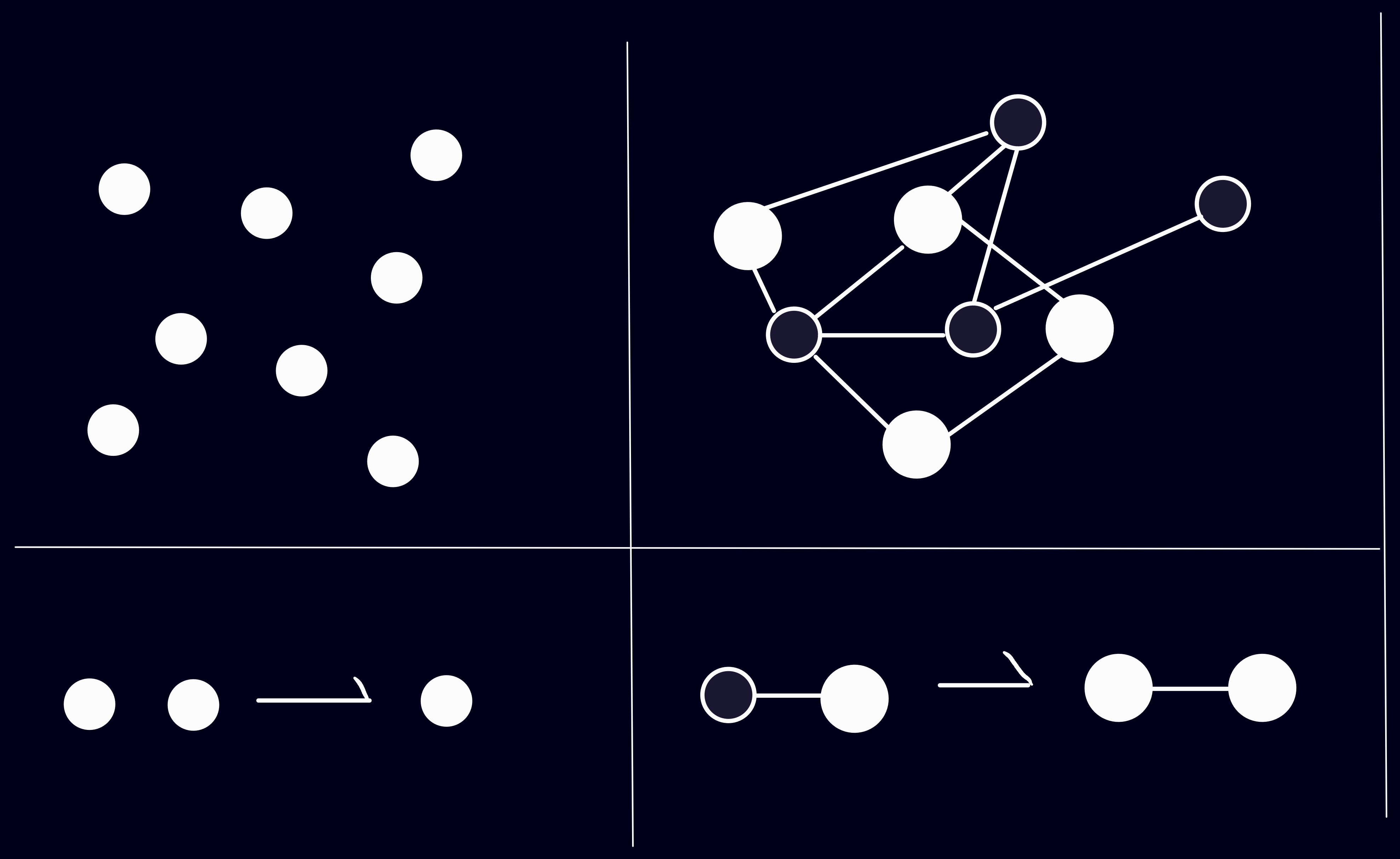
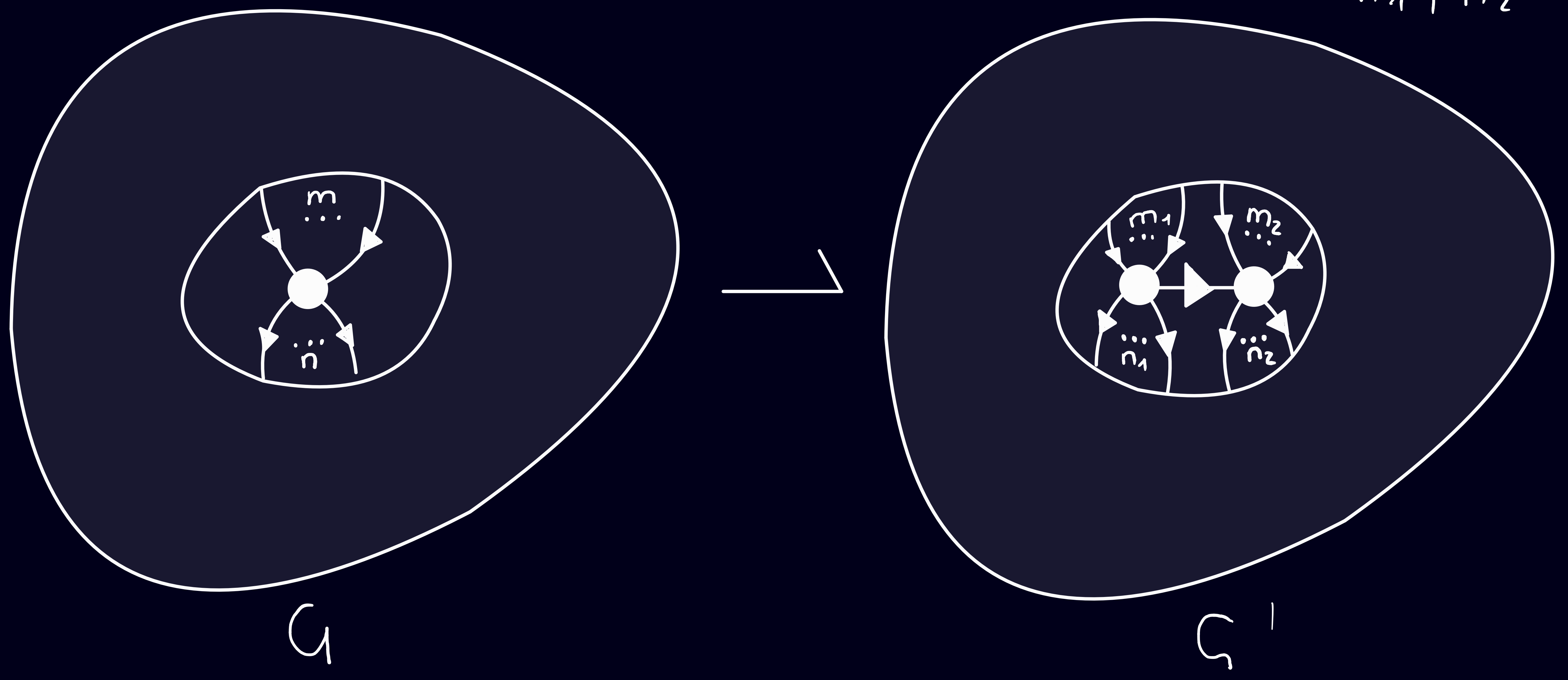




① MOTIVATION

$m_1 + m_2 = m$
$n_1 + n_2 = n$





↪ ALL FORMALIZABLE IN DOUBLE-PUSHOUT (DPO) SEMANTICS:

$$\begin{array}{ccc} O & \xleftarrow{\;r\;} & I \\ {\scriptstyle n}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle m} \\ r_\alpha(X) & \longleftarrow & X \end{array} \;:=\; \begin{array}{ccccc} O & \xleftarrow{\;o_r\;} & K_r & \xrightarrow{\;i_r\;} & I \\ {\scriptstyle n}\downarrow & \text{PO} & \downarrow{\scriptstyle k_\alpha} & \text{PO} & \downarrow{\scriptstyle m} \\ r_\alpha(X) & \xleftarrow[o_\alpha]{} & K_\alpha & \xrightarrow[i_\alpha]{} & X \end{array}$$

PO — Pushout

② MOTIVATION: FORMALIZATION OF REWRITING + COMBINATORICS!

- INSPIRATION FROM COMBINATORIAL SPECIES THEORY:

(i) $\hat{X}(x^n) = x^{n+1}$, $\frac{d}{dx}(x^n) = (n)_1 x^{n-1} \equiv \begin{cases} 0, & n=0 \\ n x^{n-1}, & \text{else} \end{cases}$

↳ (ii) $x^n = \hat{x}^n(1)$

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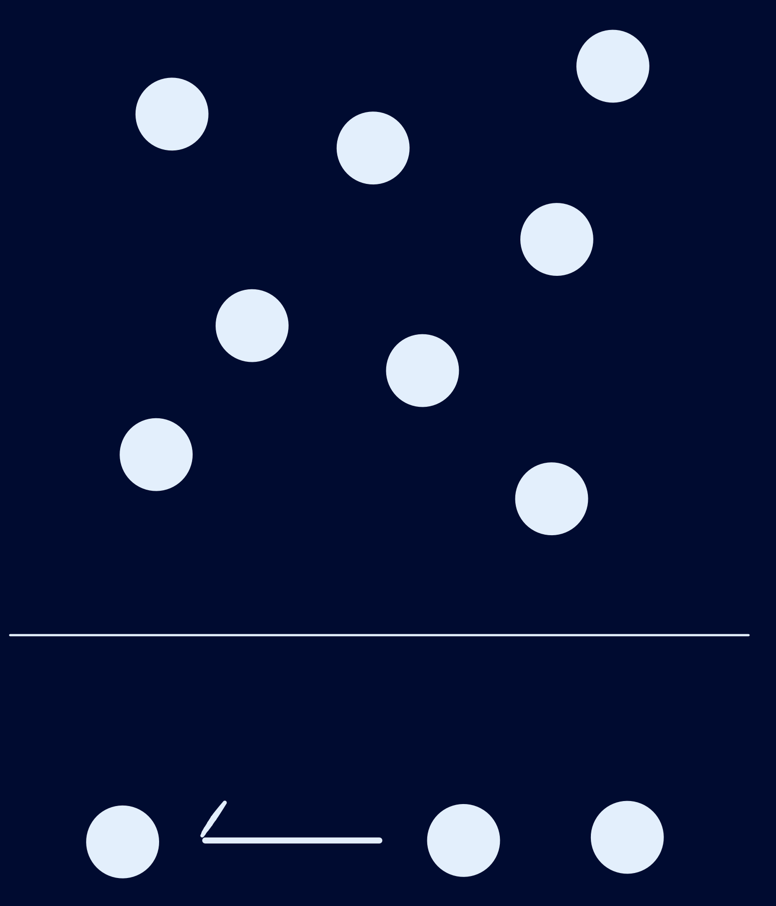
↳ (ii) $x^n = \hat{x}^n(1)$

$$\hat{x}^p \left(\frac{d}{dx}\right)^q (x^n) = (n)_q\, x^{n-q+p}$$

OF WAYS TO REMOVE q ELEMENTS FROM A SET OF n ELEMENTS



$$\hat{x}\left(\frac{d}{dx}\right)^2 x^n = (n)_2\, x^{n-1}$$

② Motivation: Formalization of Rewriting + Combinatorics!

- Inspiration from combinatorial species theory:

(i) $\hat{X}(x^n) = x^{n+1}$, $\frac{d}{dx}(x^n) = (n)_1 x^{n-1} \equiv \begin{cases} 0, & n=0 \\ nx^{n-1}, & \text{else} \end{cases}$

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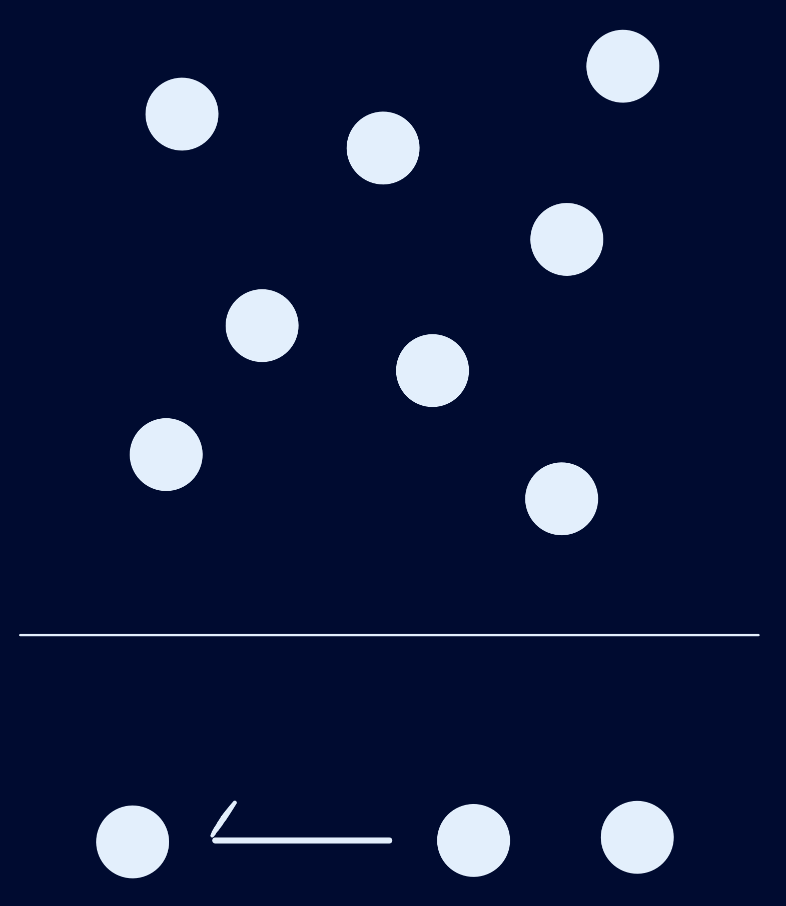
# of ways to remove q elements from a set of n elements

(iii) $\hat{X}^p \left(\frac{d}{dx}\right)^q \hat{X}^r \left(\frac{d}{dx}\right)^s = \sum_{k\geq 0} \underbrace{\binom{q}{k} k! \binom{r}{k}}_{\in \mathbb{Z}_{\geq 0}} \hat{X}^{p+r-k} \left(\frac{d}{dx}\right)^{q+s-k}$



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OF WAYS TO REMOVE q ELEMENTS FROM A SET OF n ELEMENTS

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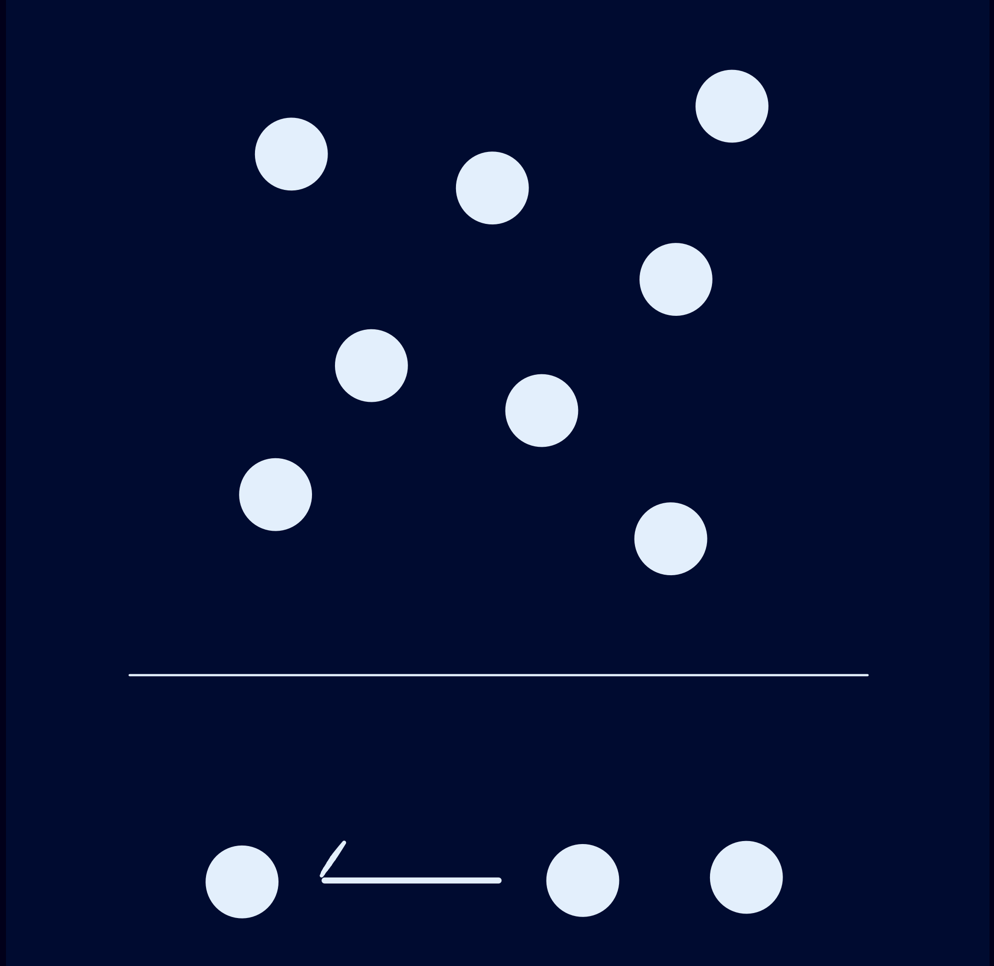


GOAL:

(i) " $\rho(\delta(r))|X\rangle := \sum_\alpha |r_\alpha(X)\rangle = \sum_y \overbrace{\mu_X^y}^{\in \mathbb{Z}_{\geq 0}} |y\rangle$ "

(ii) " $|X\rangle = \rho(\delta(X \leftarrow \emptyset))|\emptyset\rangle$ "

(iii) " $\rho(\delta(r_2))\rho(\delta(r_1)) = \sum_{r_u} \underbrace{\mu_{r_1,r_2}^{r_u}}_{\in \mathbb{Z}_{\geq 0}} \rho(\delta(r_u))$ "

(iv) " $\rho(\delta(r_2))\rho(\delta(r_1)) = \rho(\delta(r_2) \odot \delta(r_1))$ "

$\odot$ – RULE ALGEBRA PRODUCT

③ Conceptual Obstacle: Essential Uniqueness of Universal Constructions

Recap:

$$
\begin{array}{ccc}
O & \xleftarrow{\;r\;} & I \\
{\scriptstyle n}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle m} \\
r_\alpha(X) & \longleftarrow & X
\end{array}
\quad := \quad
\begin{array}{ccccc}
O & \xleftarrow{\;o_r\;} & K_r & \xrightarrow{\;i_r\;} & I \\
{\scriptstyle n}\downarrow & \text{PO} & \downarrow{\scriptstyle k_\alpha} & \text{PO} & \downarrow{\scriptstyle m} \\
r_\alpha(X) & \xleftarrow[o_\alpha]{} & K_\alpha & \xrightarrow[i_\alpha]{} & X
\end{array}
$$

PO — Pushout

Definition:

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
\text{ is a PO } :\Leftrightarrow \forall
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
\text{ with } B \to X,\ C \to X \;:\;
\exists!\, D \dashrightarrow X
$$

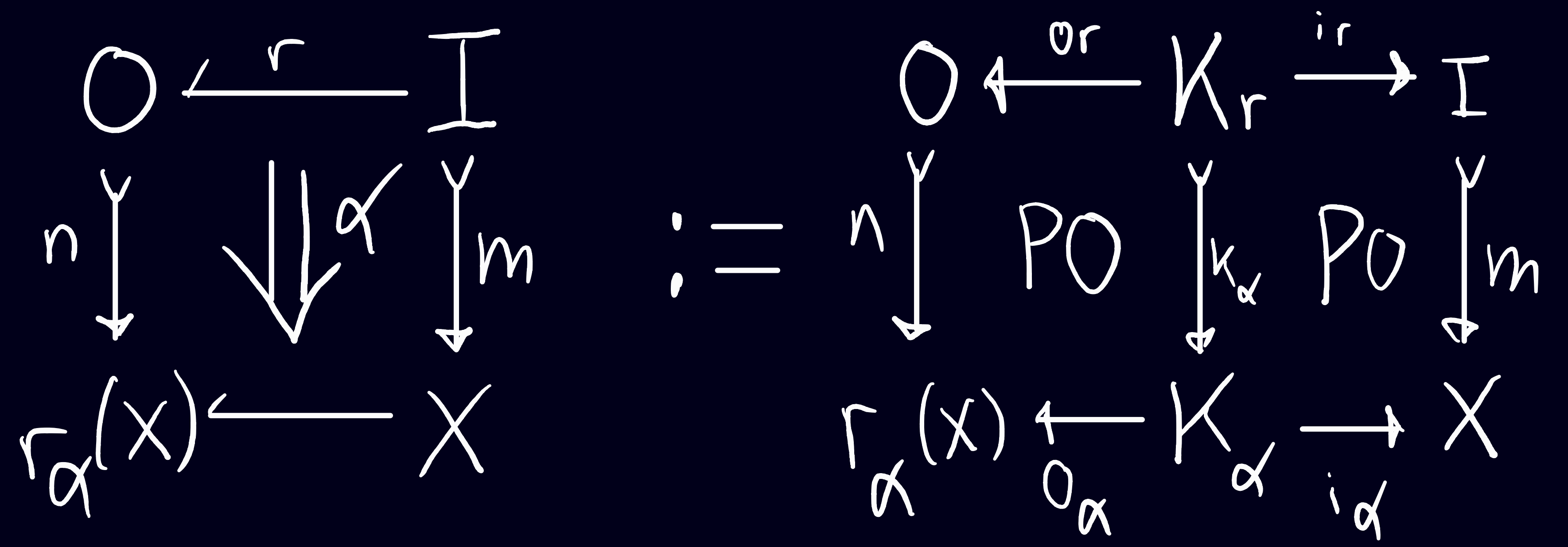
$$
\leadsto
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & \text{PO} & \downarrow \\
C & \longrightarrow & D
\end{array}
\;\wedge\;
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & \text{PO} & \downarrow \\
C & \longrightarrow & D'
\end{array}
\;\Rightarrow\;
\exists!\, D \xrightarrow{\cong} D' \text{ (compatible with } B \to D',\ C \to D')
$$

Example: (in Finset)

$$
\begin{array}{ccc}
I & \hookrightarrow & B \\
\downarrow & \text{PO} & \downarrow \\
C & \hookrightarrow & \text{“}B \cup_I C\text{”}
\end{array}
$$

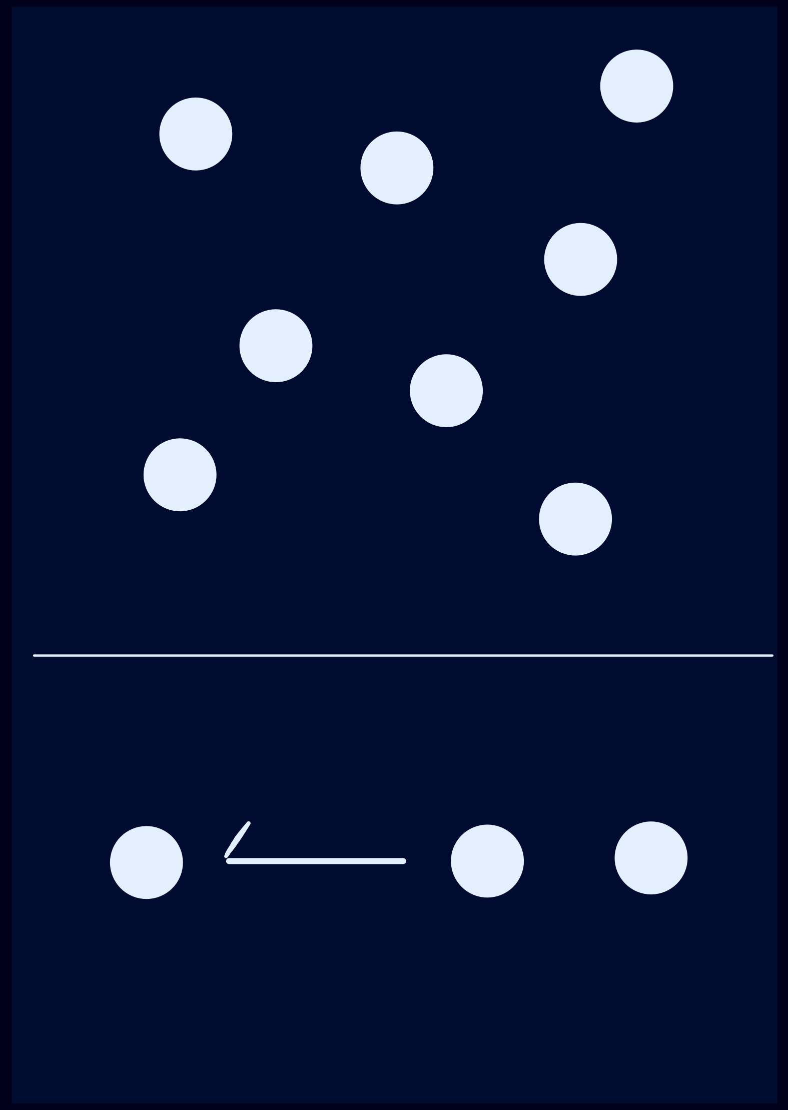
③ Conceptual Obstacle: Essential Uniqueness of Universal Constructions

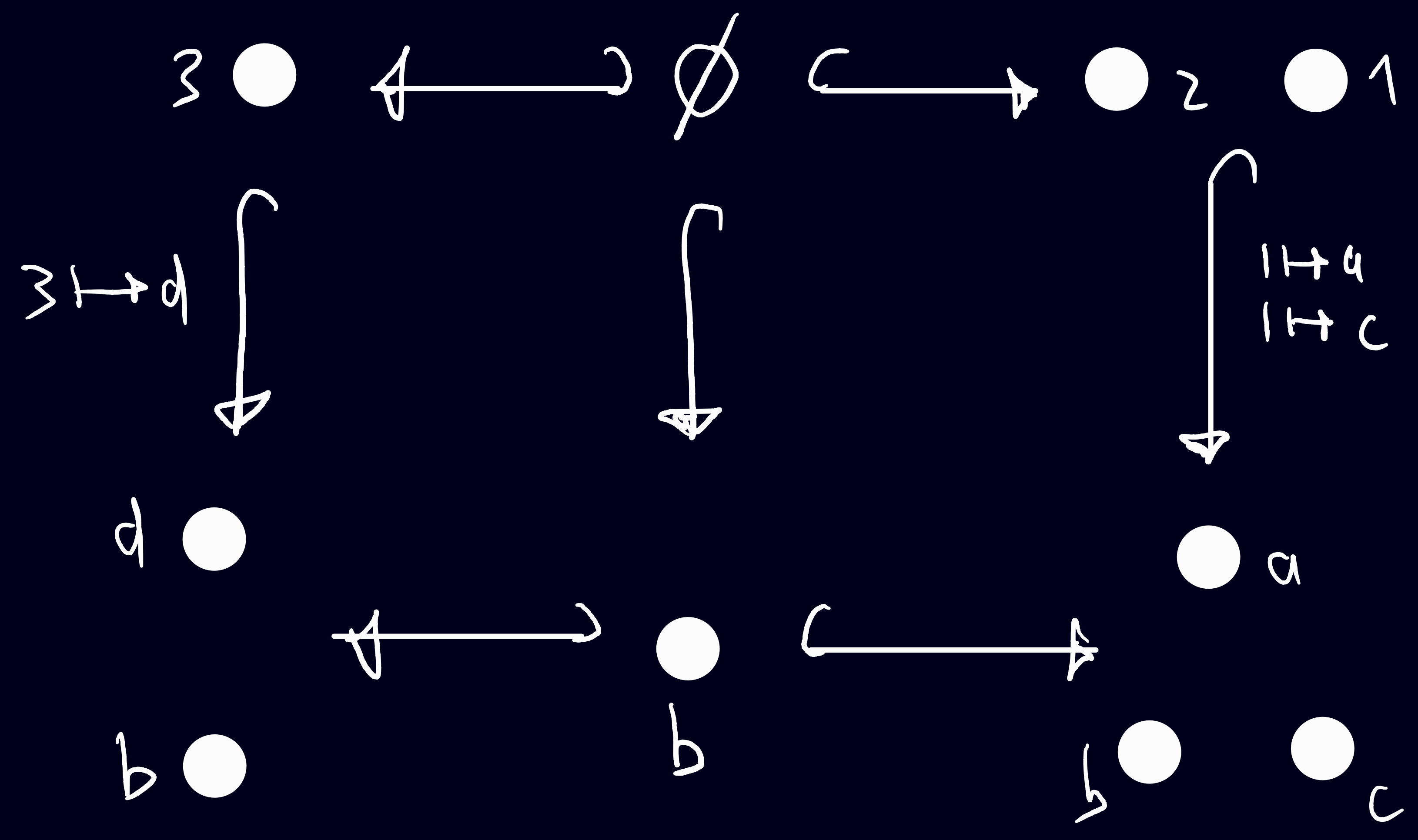
Recap:

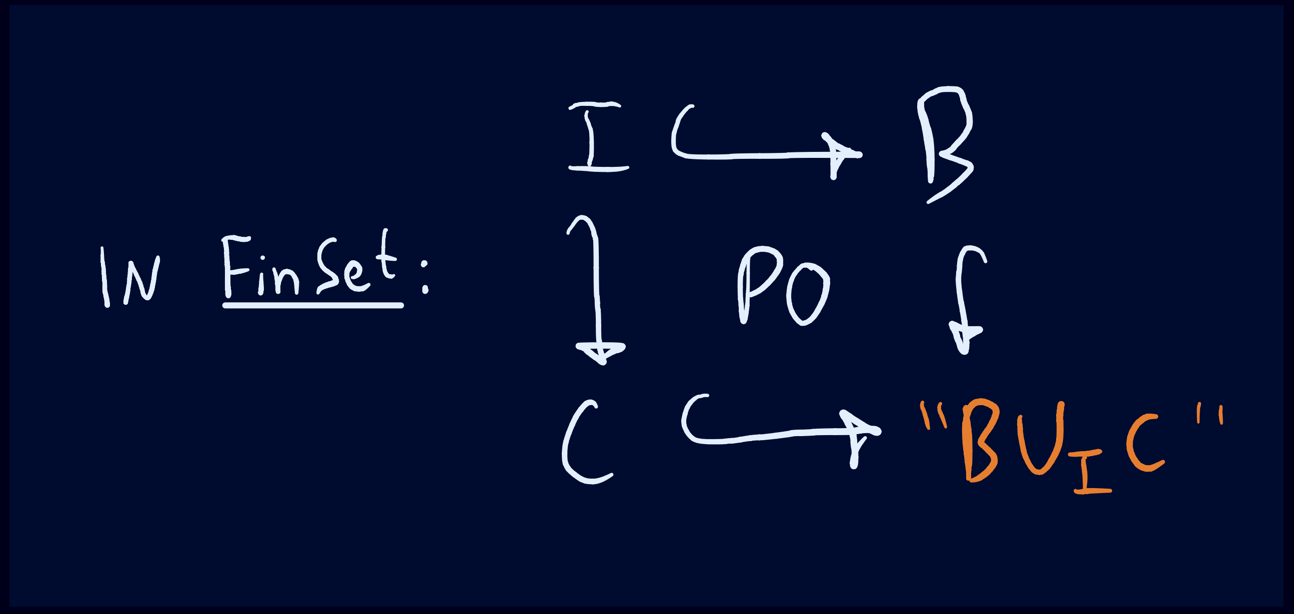


PO — Pushout

Example:







④ ANSATZ: CATEGORIFICATION

GOAL:

(i) "$\rho(\delta(r))|X\rangle := \sum_{\alpha} |r_\alpha(X)\rangle = \sum_{Y} \underbrace{\mu_X^Y}_{\in \mathbb{Z}_{\geq 0}} |Y\rangle$"

(ii) "$|X\rangle = \rho(\delta(X \leftarrow \emptyset))|\emptyset\rangle$"

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(iv) "$\rho(\delta(r_2))\rho(\delta(r_1)) = \rho(\delta(r_2) \odot \delta(r_1))$"

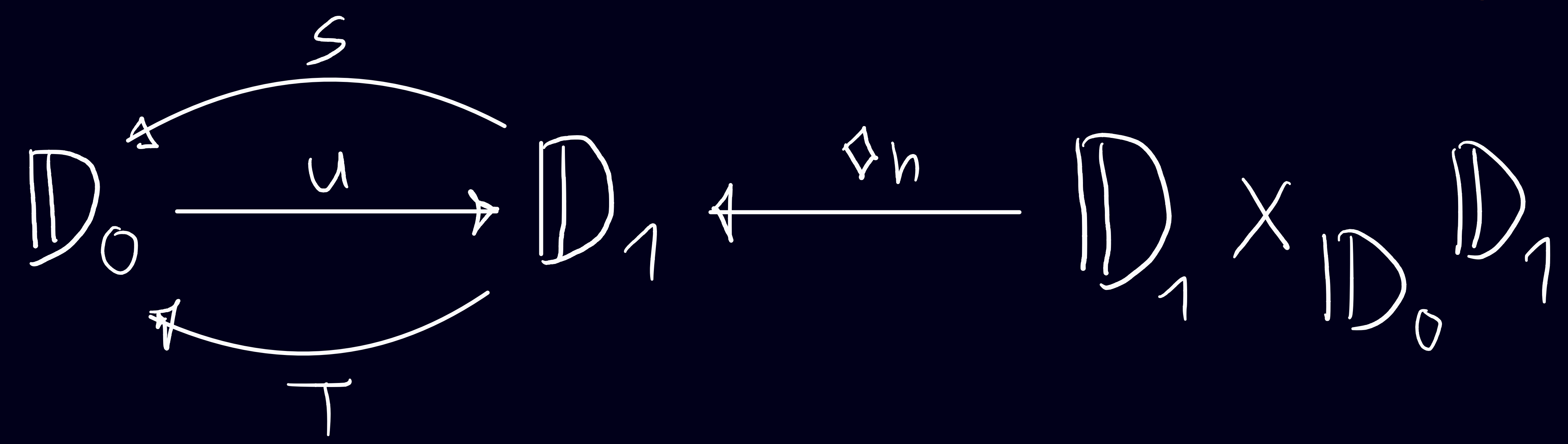
$\odot$ – RULE ALGEBRA PRODUCT

I. FORMALIZE
$$\begin{array}{ccc} O & \xleftarrow{r} & I \\ \downarrow & \Downarrow \alpha & \downarrow \\ r_\alpha(X) & \longleftarrow & X \end{array}$$
AS 2-CELLS IN A DOUBLE CATEGORY

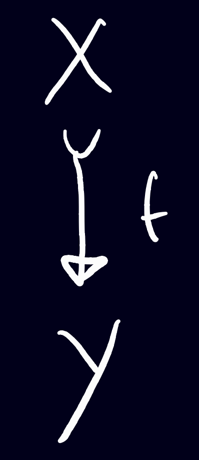
II. FORMALIZE $\mathbb{Z}_{\geq 0}$-COEFFICIENTS AS CARDINALITIES (OF SUITABLE SETS...)

METHODS: DOUBLE CATEGORIES, PRESHEAVES, FIBRATIONS, COENDS, MULTISUMS ...

⑤ DEFINITION: A DOUBLE CATEGORY $\mathbb{D}$ IS A (PSEUDO) INTERNAL CATEGORY IN CAT



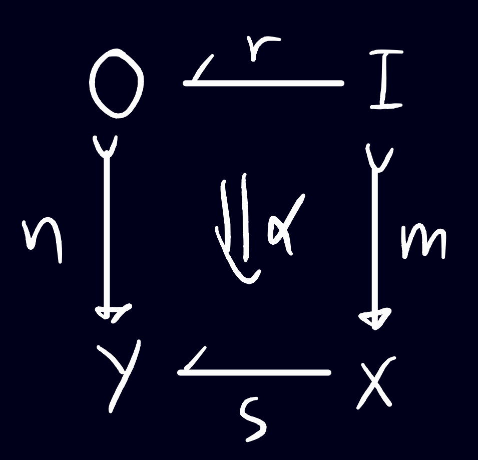
$\mathbb{D}_0$:



"0-cells" – objects of $\mathbb{D}_0$

"vertical morphisms" – morphisms of $\mathbb{D}_0$

$\mathbb{D}_1$:



"horizontal morphisms" – objects of $\mathbb{D}_1$

"2-cells" – morphisms of $\mathbb{D}_1$

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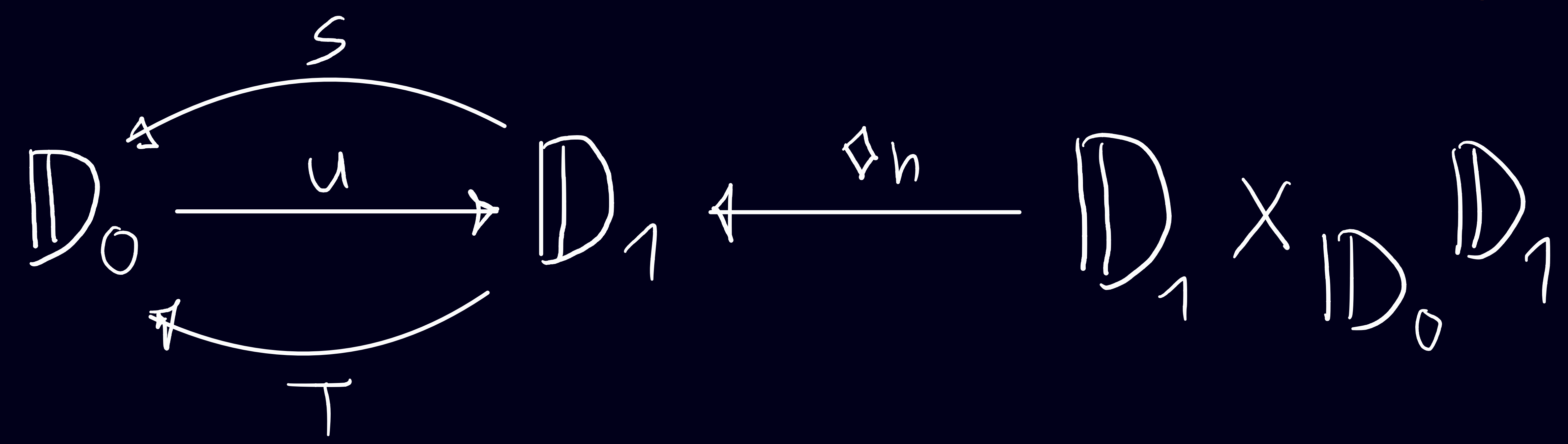

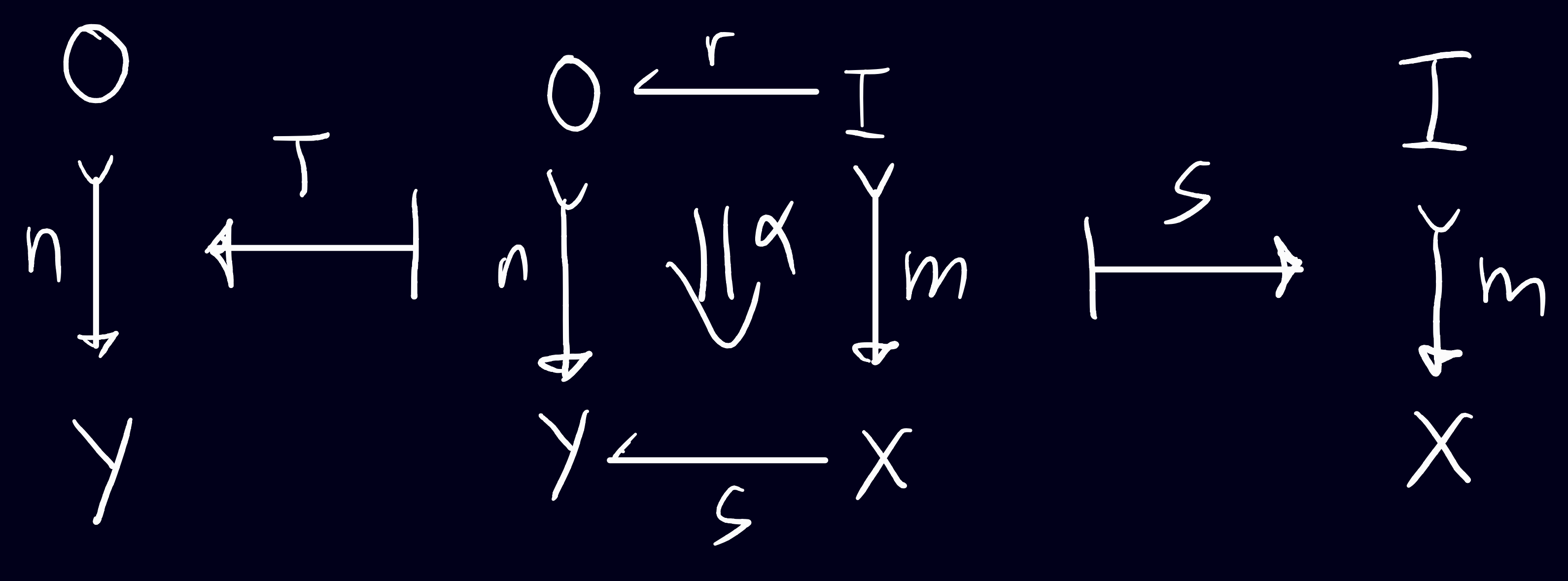

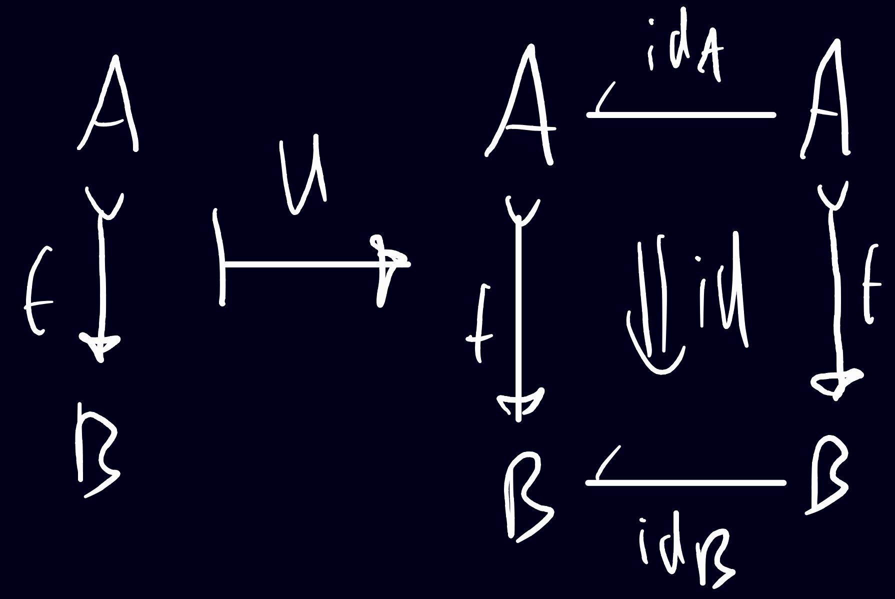

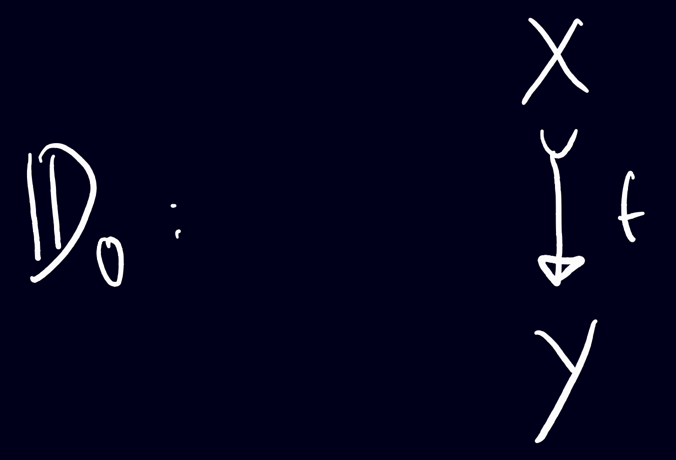

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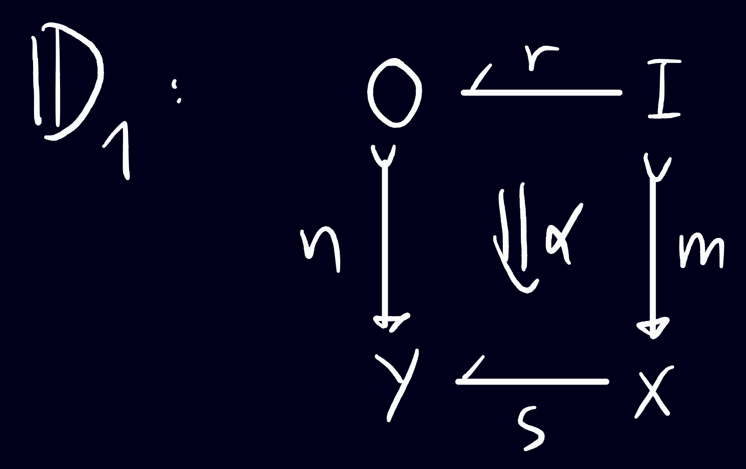

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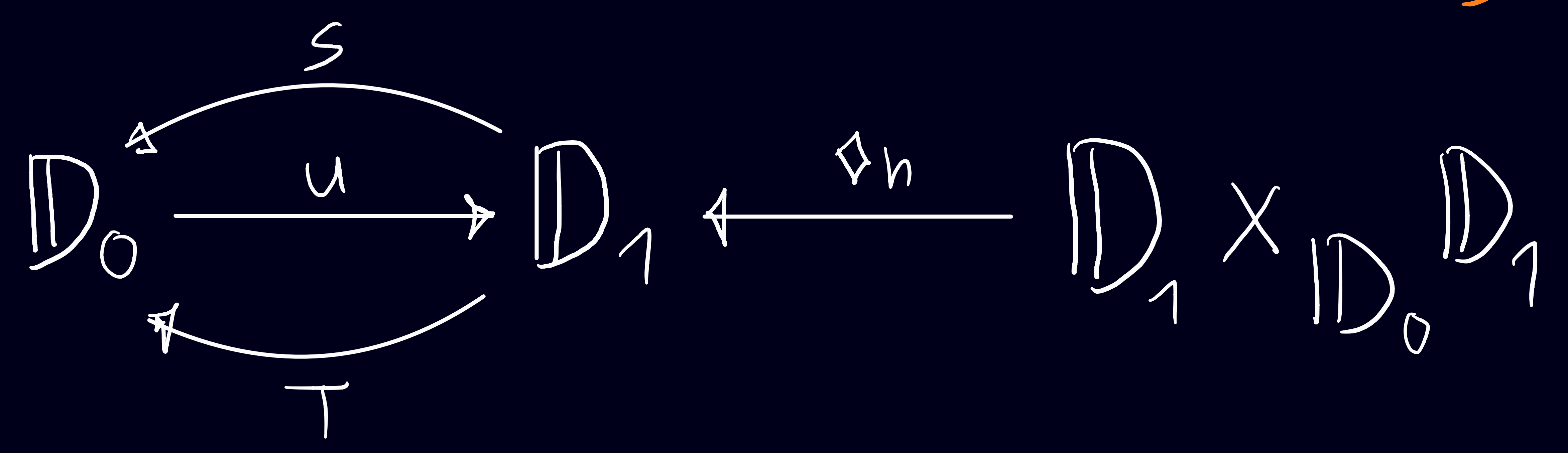

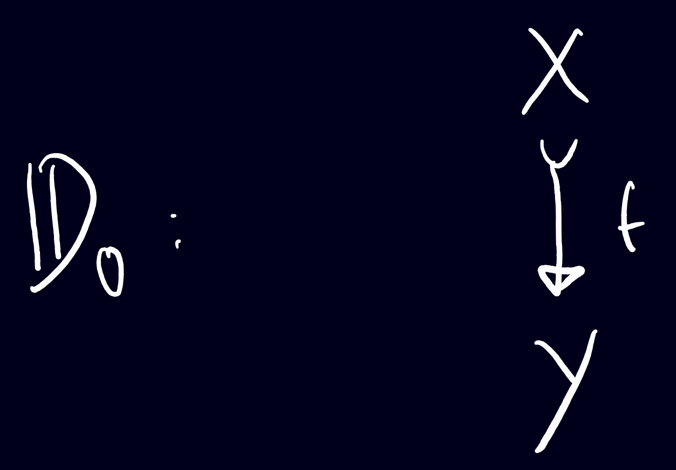

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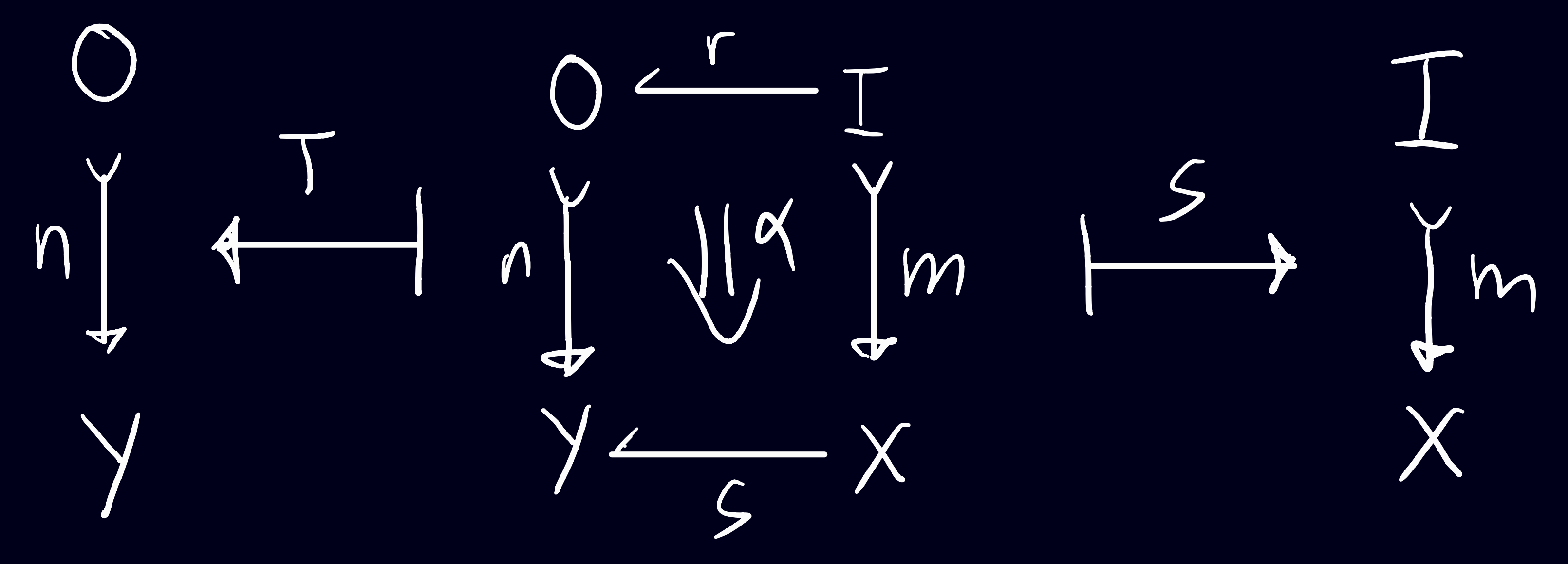

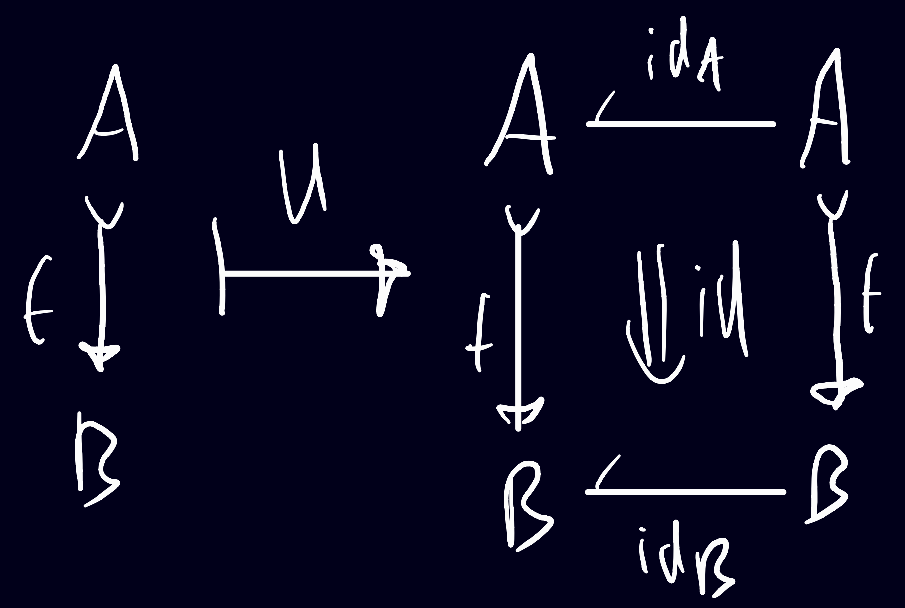

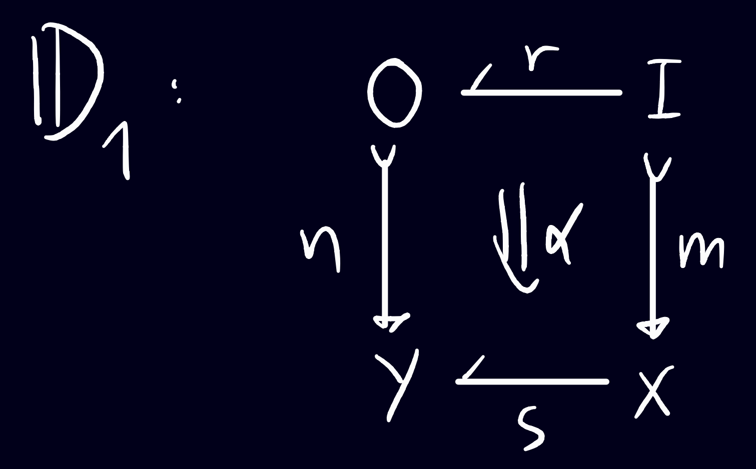

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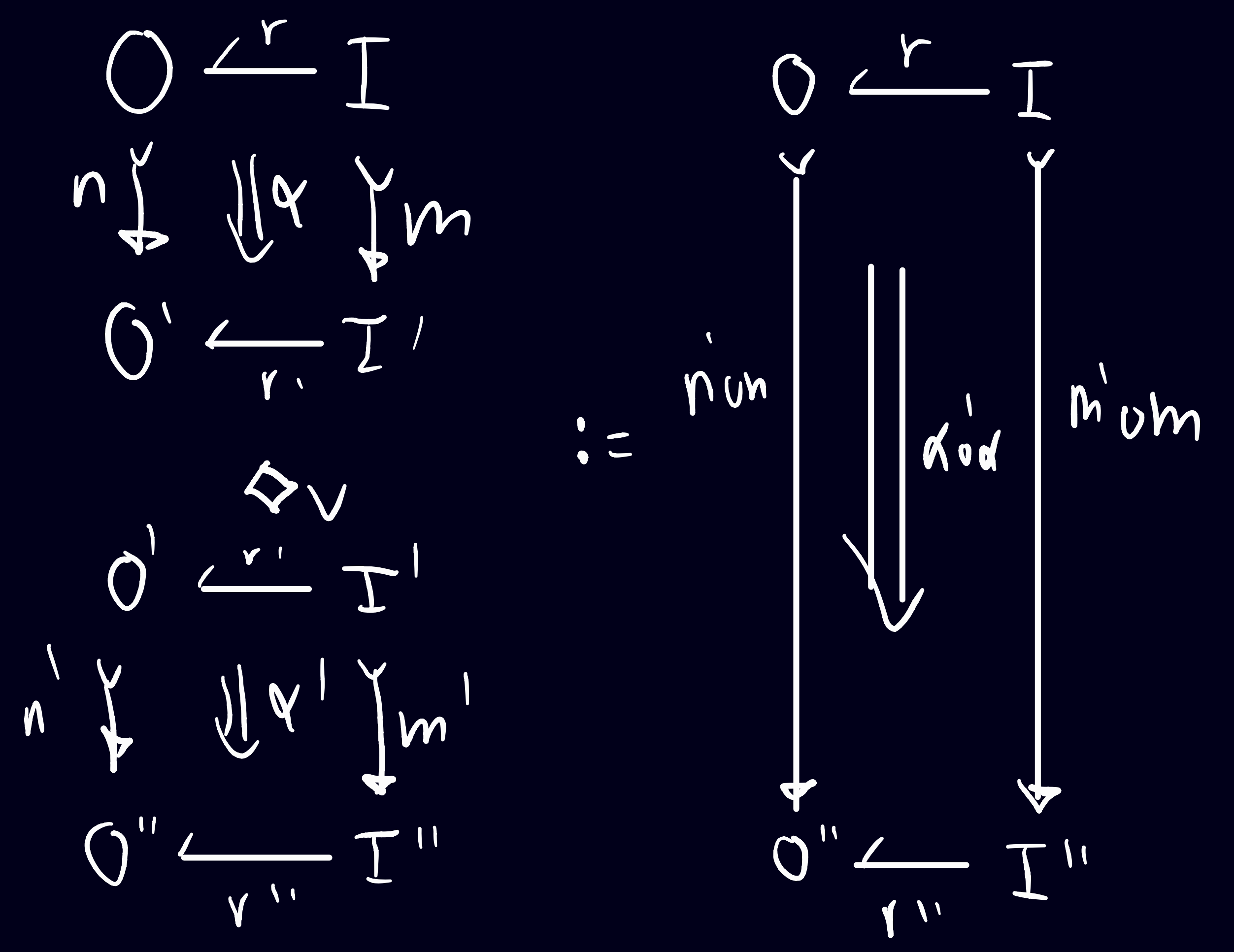

"vertical composition"

– composition in $\mathbb{D}_1$

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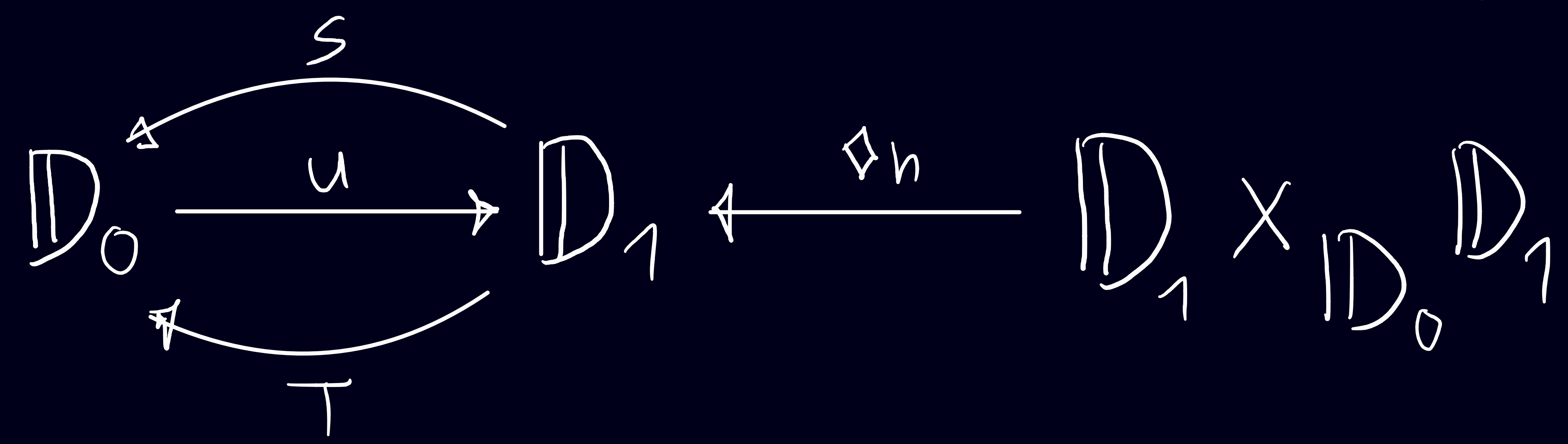
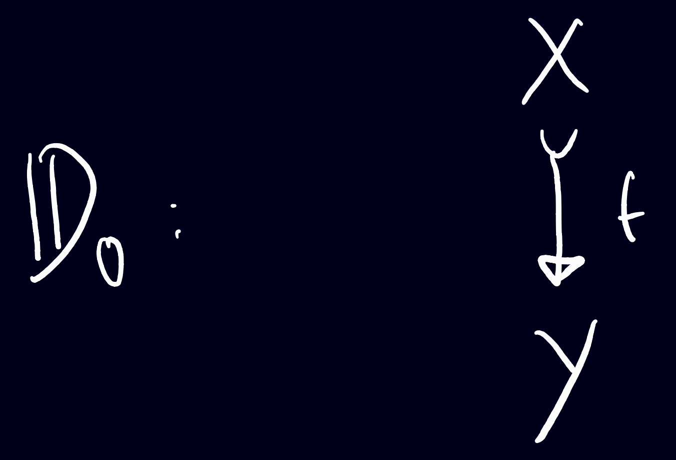


$\mathbb{D}_0$:



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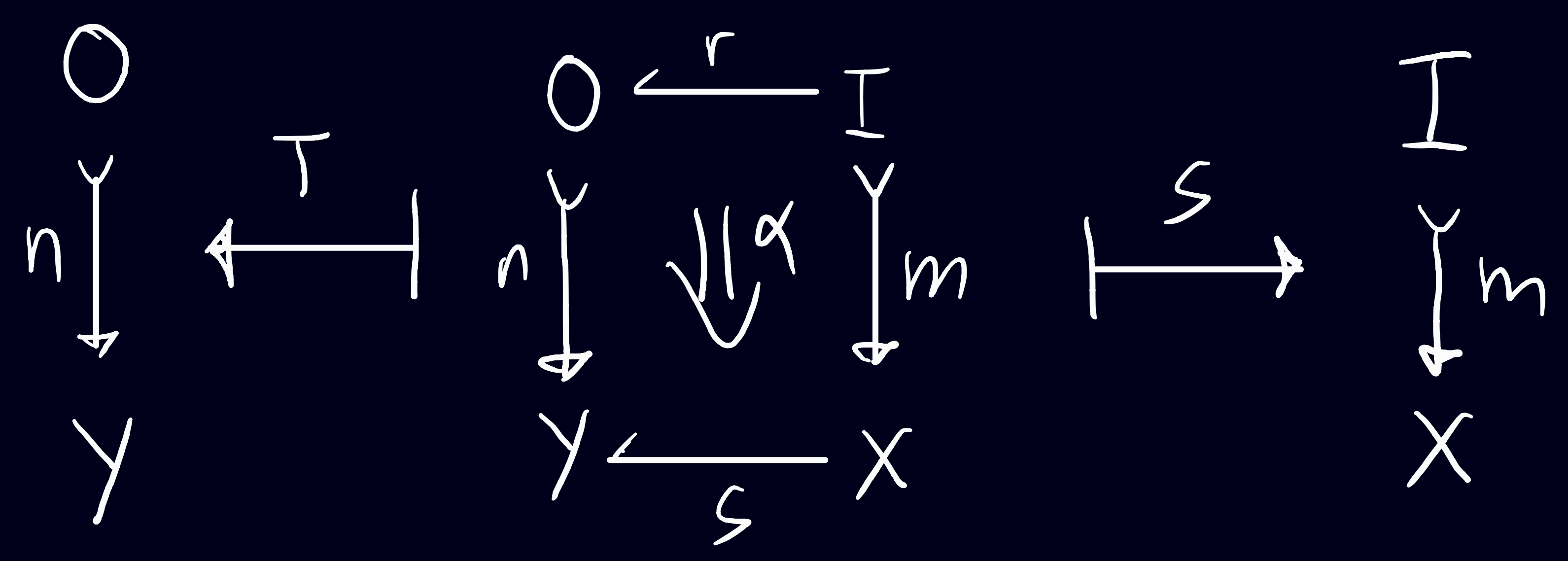

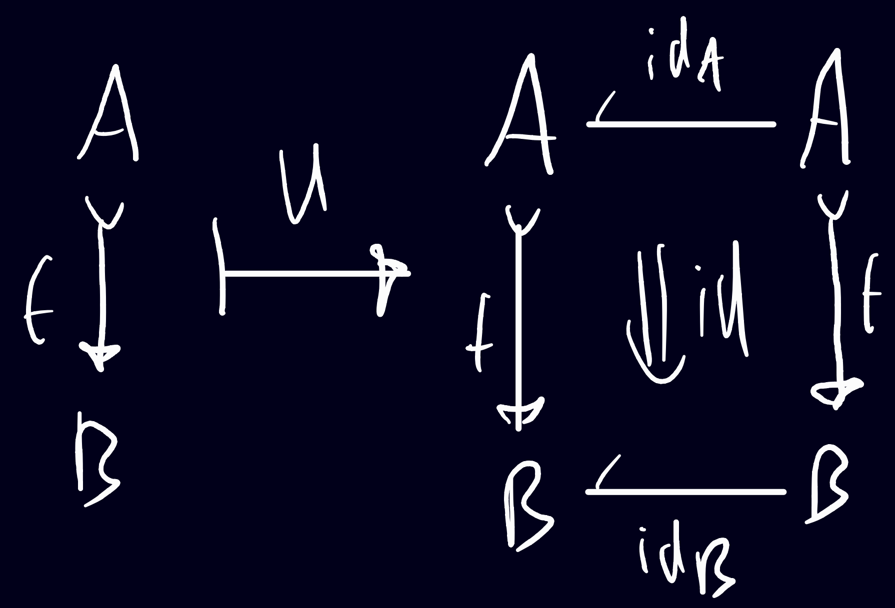

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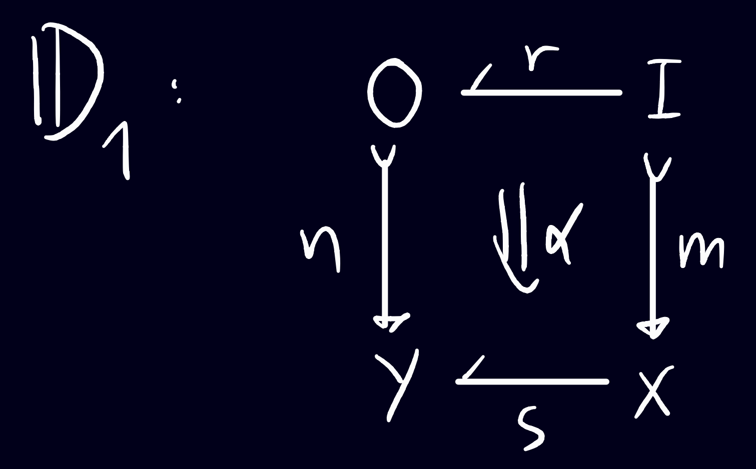

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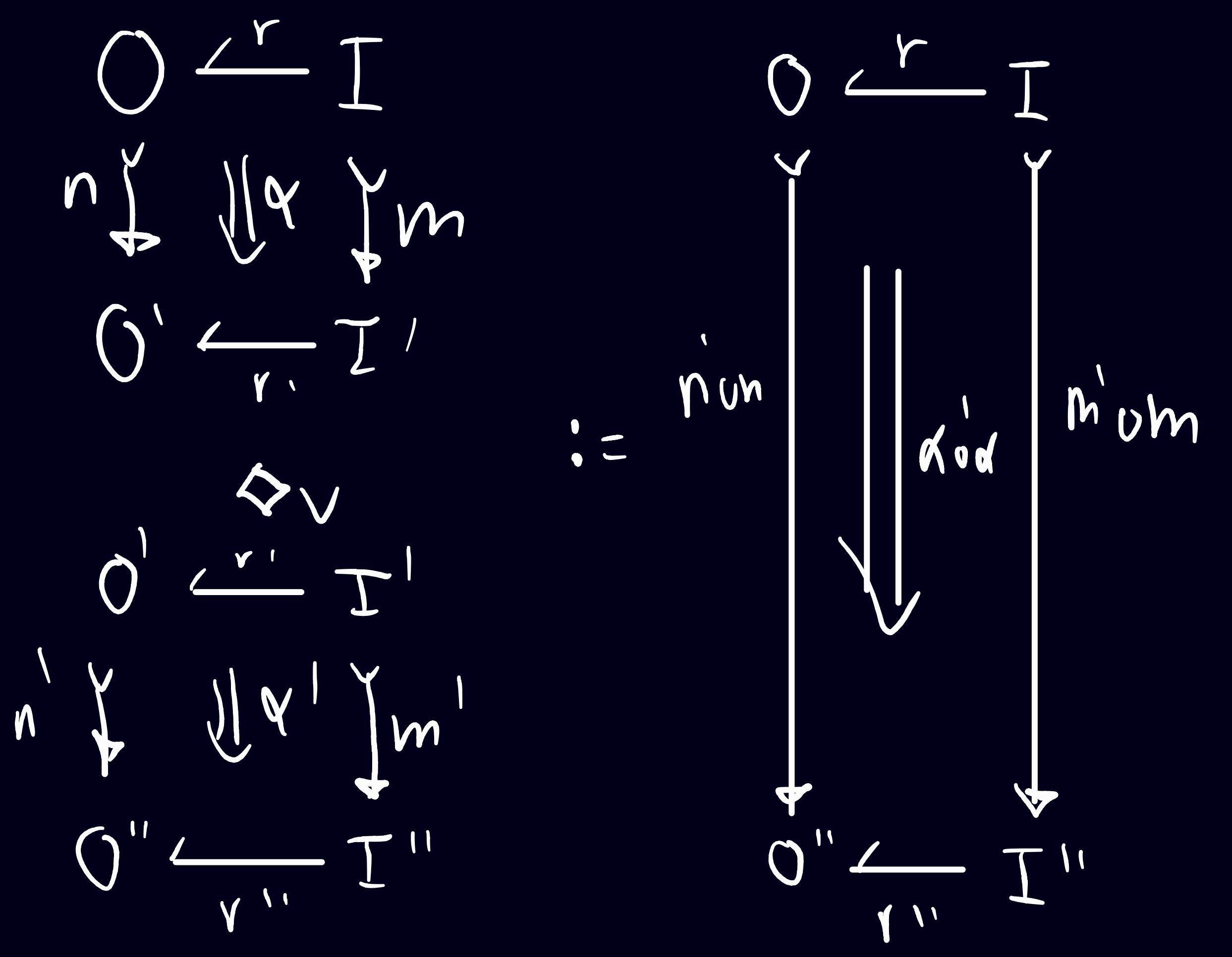
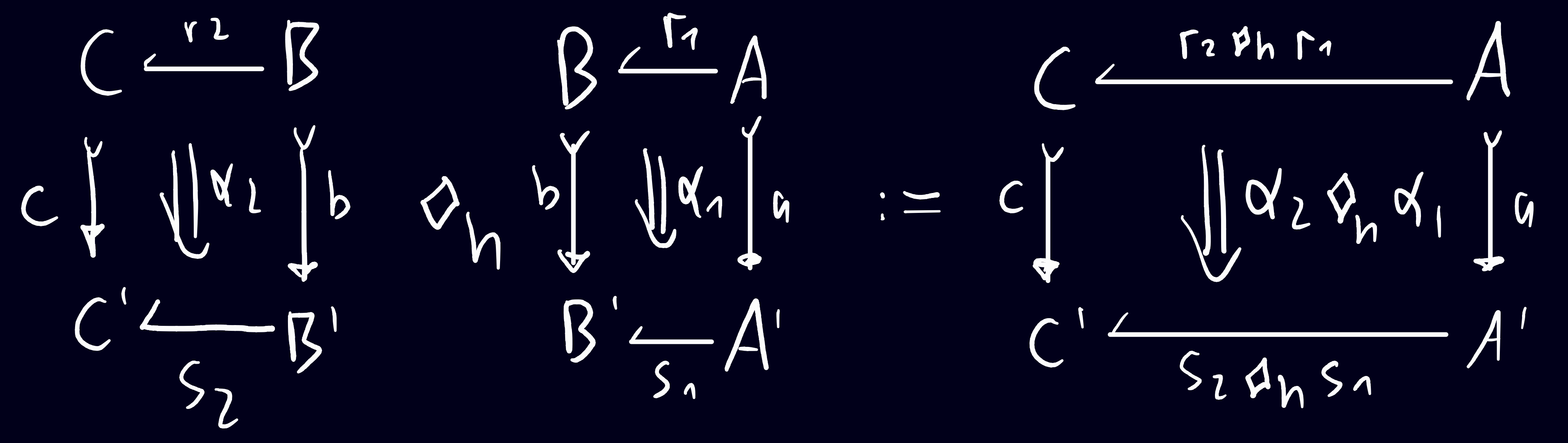


"vertical composition"

— composition in $\mathbb{D}_1$



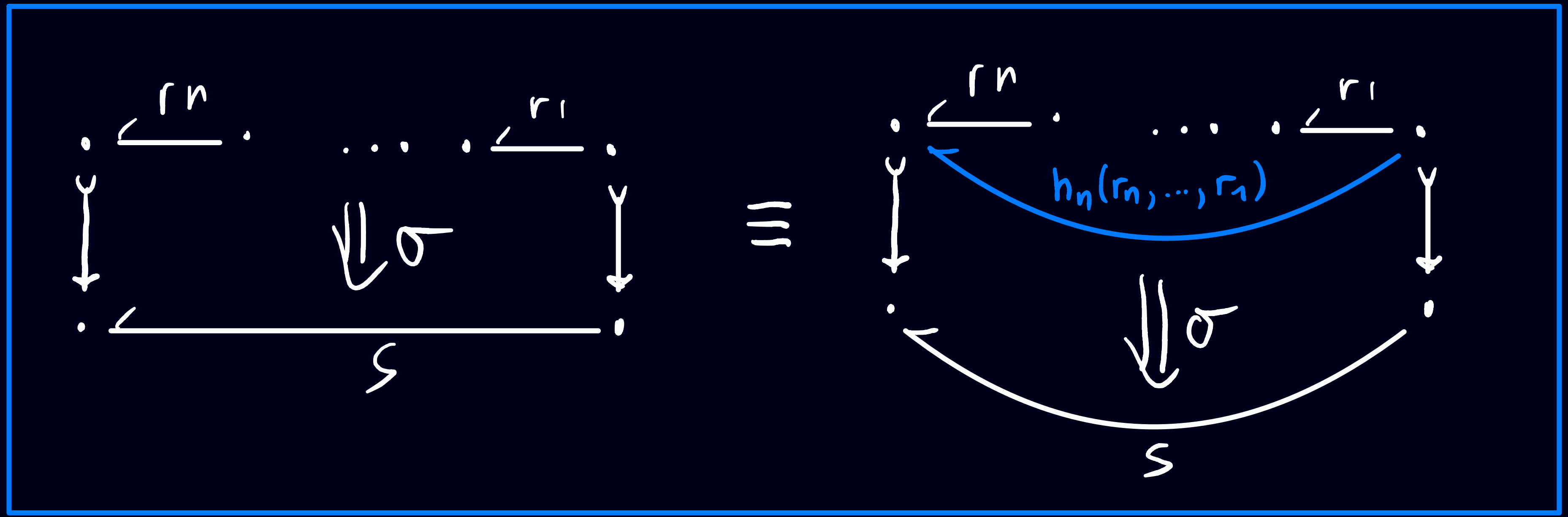
"horizontal composition" ← IN GENERAL ONLY WEAKLY ASSOCIATIVE!

⑥ DEFINITION: A PRESENTATION OF A DOUBLE CATEGORY $\mathbb{D}$ IS A FAMILY $(h_n)_{n\geq 0}$ OF FUNCTORS $h_n : \mathbb{D}_n \to \mathbb{D}_1$, WHERE $\mathbb{D}_n := \underbrace{\mathbb{D}_1 \times_{\mathbb{D}_0} \cdots \times_{\mathbb{D}_0} \mathbb{D}_1}_{n \text{ times}}$,

$$h_0 := U, \quad h_1 := id, \quad h_2(-_2, -_1) := -_2 \odot_h -_1,$$

$$\forall n \geq 2: \; h_{n+1}(-_{n+1}, \ldots, -_1) \cong h_2(-_{n+1}, h_n(-_n, \ldots, -_1))$$

- NOTATIONAL CONVENTION:



⑥ **Definition:** A **presentation** of a double category $\mathbb{D}$ is a family $(h_n)_{n\geq 0}$ of functors $h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1$, where $\mathbb{D}_n := \underbrace{\mathbb{D}_1 \times_{\mathbb{D}_0} \cdots \times_{\mathbb{D}_0} \mathbb{D}_1}_{n \text{ times}}$,

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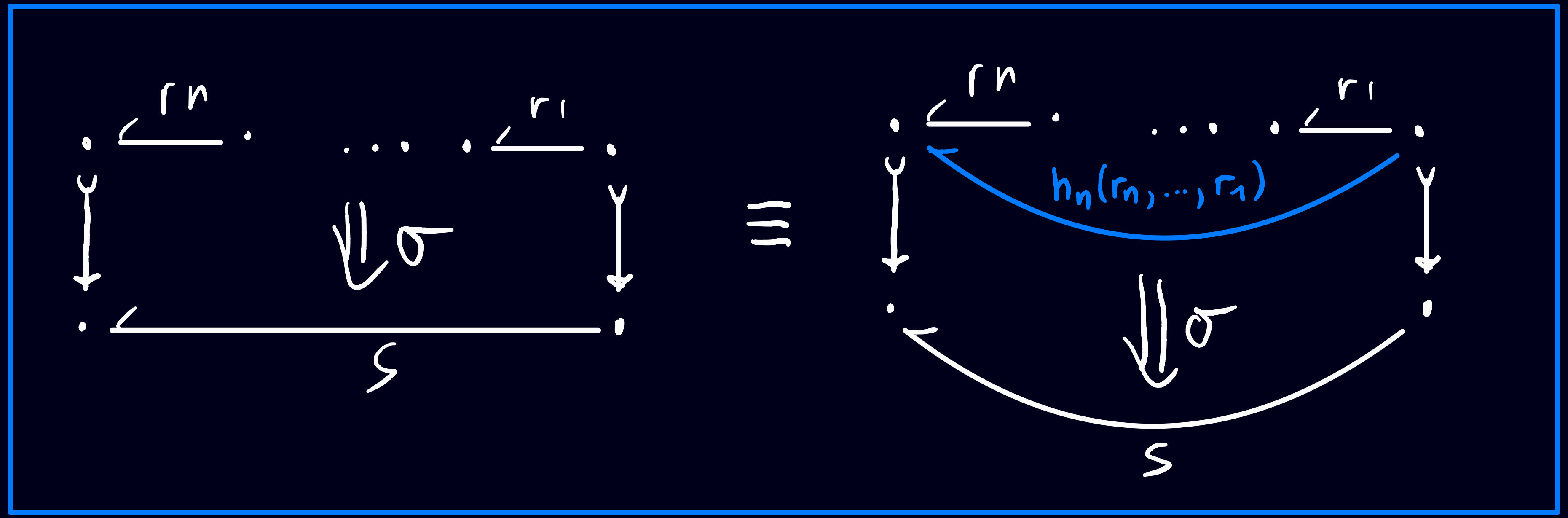
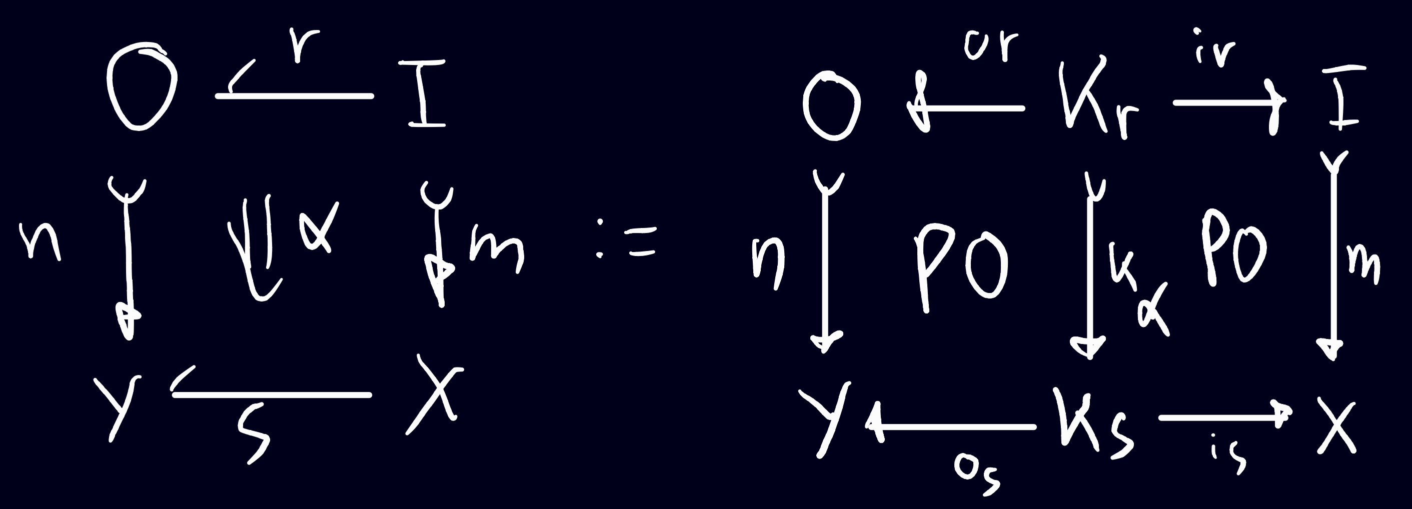
▸ Notational convention:

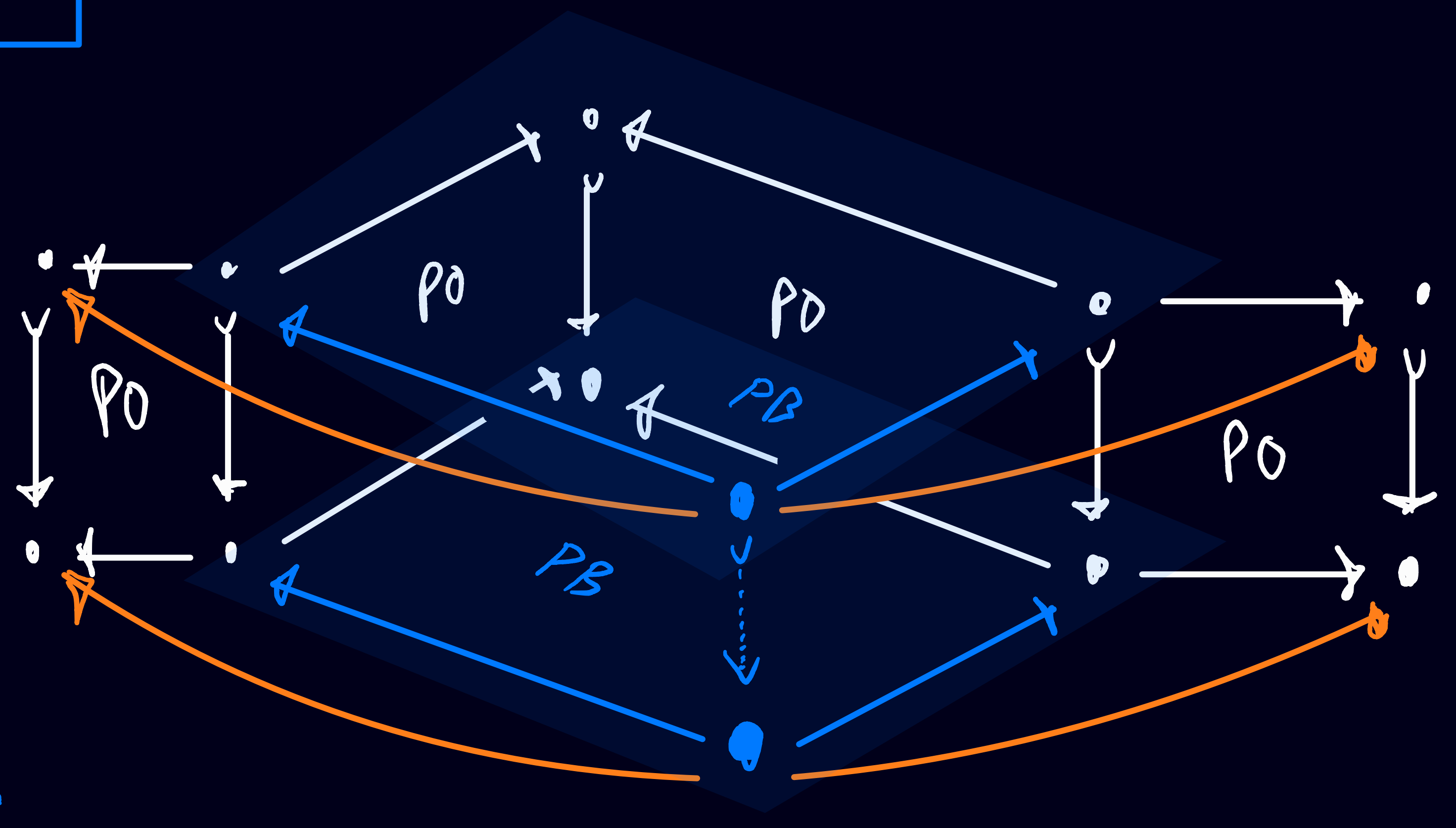


▸ Example:



↗ Horizontal composition:

≙ Choice of pullbacks (PBs)!



⑦ KEY CONCEPT: (COVARIANT) PRESHEAVES $F : \mathbb{D}_1 \to \underline{\mathbf{Set}}$

↳ <u>IDEA</u>: $\forall r \in \mathbb{D}_1 : \hat{\Delta}_r := \mathbb{D}_1(r, -)$

↳ $\left|\hat{\Delta}_r(Y \xleftarrow{s} X)\right| = \left|\left\{\begin{array}{ccc} O & \xleftarrow{r} & I \\ n\downarrow & \Downarrow\alpha & \downarrow m \\ Y & \xleftarrow[s]{} & X \end{array} \in \mathbb{D}_1\right\}\right| \propto$ "# ways to rewrite X into Y along $Y \xleftarrow{s} X$ with rule $O \xleftarrow{r} I$"

▸ BUT: we want "$\rho(\delta(r))|X\rangle = \underbrace{\rho(\delta(r))\,\rho(\delta(\overset{?}{\overbrace{X \leftarrow \emptyset}}))|\overset{?}{\emptyset}\rangle}_{?} = \sum_{\alpha} |r_\alpha(X)\rangle = \sum_{Y} \underbrace{\overbrace{M_{r,X}^{Y}}^{\in \mathbb{Z}_{\geq 0}}}_{?} |Y\rangle$"

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▸ BUT: we want "$\rho(\delta(r))\,|X\rangle = \underbrace{\rho(\delta(r))\,\rho(\delta(\overset{?}{\overbrace{X \leftarrow \emptyset}}))\,|\overset{?}{\emptyset}\rangle}_{?} = \sum_{\alpha} |r_\alpha(X)\rangle = \sum_{Y} \underbrace{\overbrace{M_{r,X}^{Y}}^{\in \mathbb{Z}_{\geq 0}}}_{?}\, |Y\rangle$"

▸ ASSUMPTION: $\mathbb{D}_0$ HAS A STRICT INITIAL OBJECT $\emptyset$ (i.e., $\forall\, X \in \mathbb{D}_0 : \exists!\ \emptyset \rightarrowtail X \ \wedge\ \forall\, X \xrightarrow{f} \emptyset : X = \emptyset$),

AND SUCH THAT (i) $\forall\, X \in \mathbb{D}_0 : \exists!\ (X \leftarrow \emptyset) \in ob(\mathbb{D}_1) \ \wedge\ \exists!\ (\emptyset \leftarrow X) \in ob(\mathbb{D}_1)$

(ii) $\forall\ \begin{array}{c} X \\ {\scriptstyle f}\downarrow \\ Y \end{array} \in \mathbb{D}_0 : \left| \left\{ \begin{array}{ccc} X & \leftarrow & \emptyset \\ {\scriptstyle f}\downarrow & \Downarrow\alpha & \| \\ Y & \leftarrow & \emptyset \end{array} \in \mathbb{D}_1 \right\} \right| \leq 1 \ \wedge\ \left| \left\{ \begin{array}{ccc} \emptyset & \leftarrow & X \\ \| & \Downarrow\beta & \downarrow{\scriptstyle f} \\ \emptyset & \leftarrow & Y \end{array} \in \mathbb{D}_1 \right\} \right| \leq 1$

⑧ DEFINITION: A COEND FOR A FUNCTOR $F: \mathcal{C}^{op} \times \mathcal{C} \to \underline{\mathsf{Set}}$ IS DEFINED AS

$$\int^{C \in \mathcal{C}} F(C,C) = \left(\coprod_{C \in \mathcal{C}} F(C,C) \right) \Big/ \sim$$

with: $(\underset{\in F(C,C)}{(C,x)}) \sim (\underset{\in F(C',C')}{(C',x')}) :\Leftrightarrow \exists\, C \xrightarrow{\gamma} C',\ y \in F(C',C) :\ x = F(\gamma, id)y \ \wedge\ x' = F(id, \gamma)y$

8 DEFINITION: A COEND FOR A FUNCTOR $F : \mathcal{C}^{op} \times \mathcal{C} \to \underline{Set}$

IS DEFINED AS $\displaystyle\int^{C \in \mathcal{C}} F(C,C) = \left(\coprod_{C \in \mathcal{C}} F(C,C) \right) \Big/ \sim$

with: $\underset{\in F(C,C)}{(C,x)} \sim \underset{\in F(C',C')}{(C',x')} :\Longleftrightarrow \exists\, C \xrightarrow{\gamma} C',\ y \in F(C',C) : x = F(\gamma, id)y \ \wedge\ x' = F(id,\gamma)y$

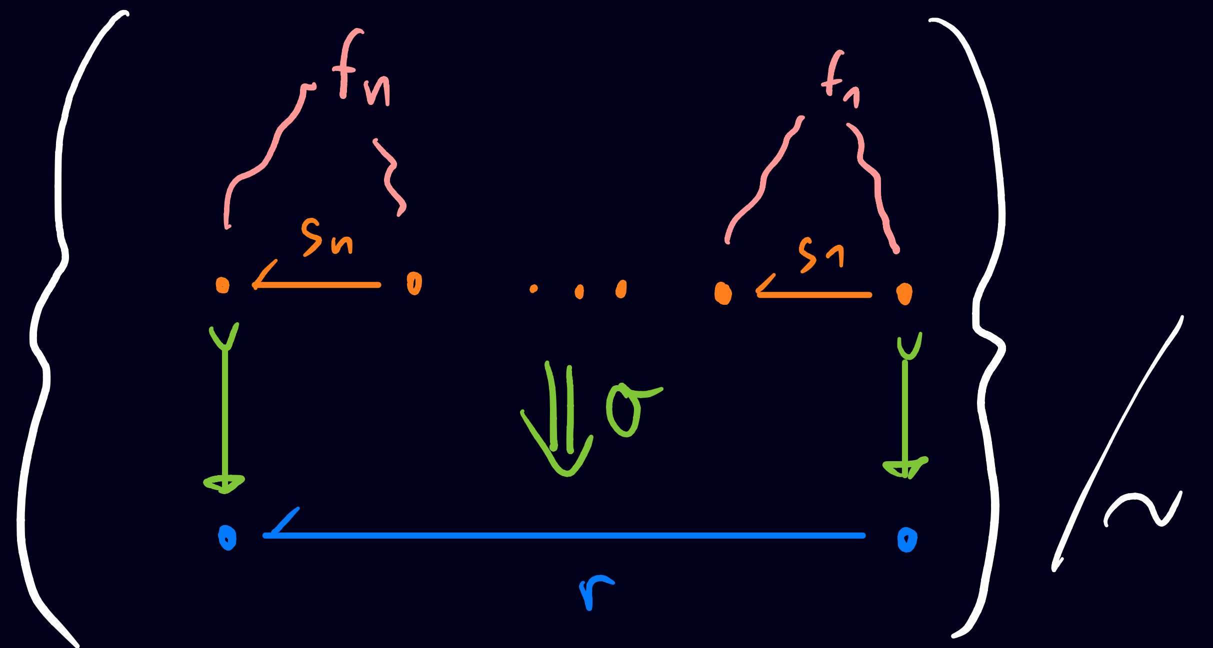
KEY CONCEPT: CONVOLUTION PRODUCTS OF PRESHEAVES $F_n, \dots, F_1 : \mathbb{D}_1 \to \underline{Set}$

$$(F_n * \cdots * F_1) := r \mapsto \int^{S = (S_n, \dots, S_1) \in \mathbb{D}_n} \mathbb{D}_1(h_n(S), r) \times \mathbb{F}_n(S)$$

$(\cong Lan_{h_n}(\mathbb{F}_n))$

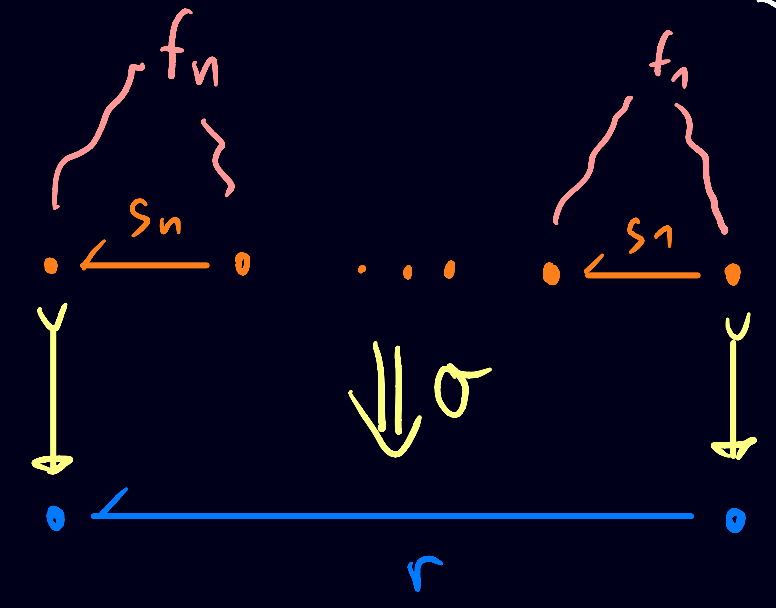
- $\mathbb{D}_1(h_n(-), r) : \mathbb{D}_n^{op} \to \underline{Set}$
- $\mathbb{F}_n := F_n \times \dots \times F_1 : \mathbb{D}_n \to \underline{Set}$

$$= \left\{ (S, (\sigma, f)) \;\middle|\; \begin{array}{l} S \in \mathbb{D}_n \\ \sigma \in \mathbb{D}_1(h_n(S), r) \\ f \in \mathbb{F}_n(S) \end{array} \right\} \Big/ \sim \ \equiv$$



9

$$(F_n * \cdots * F_1)(r) = \left\{ (S,(\sigma,f)) \;\middle|\; \begin{array}{l} S \in \mathbb{D}_n \\ \sigma \in \mathbb{D}_1(h_n(S), r) \\ f \in \mathbb{F}_n(S) \end{array} \right\} \Big/ \sim \;\; \equiv \left\{ \right. $$

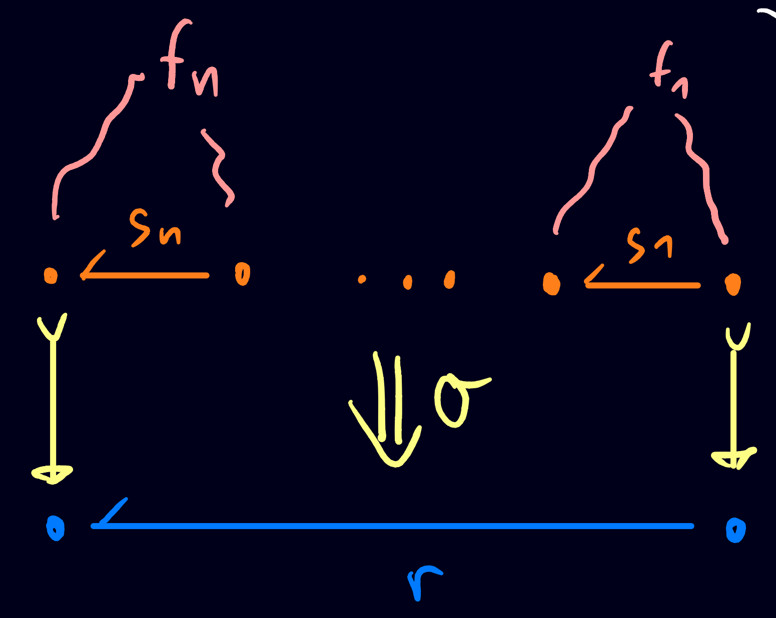


$$\left. \right\} \Big/ \sim$$

- $(S,(\sigma,f)) \sim (S',(\sigma',f')) :\Leftrightarrow \exists\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ (\tau,g) \in \mathbb{D}_1(h_n(S'), r) \times \mathbb{F}_n(S):$

$$(\sigma,f) = (\mathbb{D}_1(h_n(A), r)\tau,\ g) \;\wedge\; (\sigma',f') = (\tau,\ \mathbb{F}_n(A)g)$$

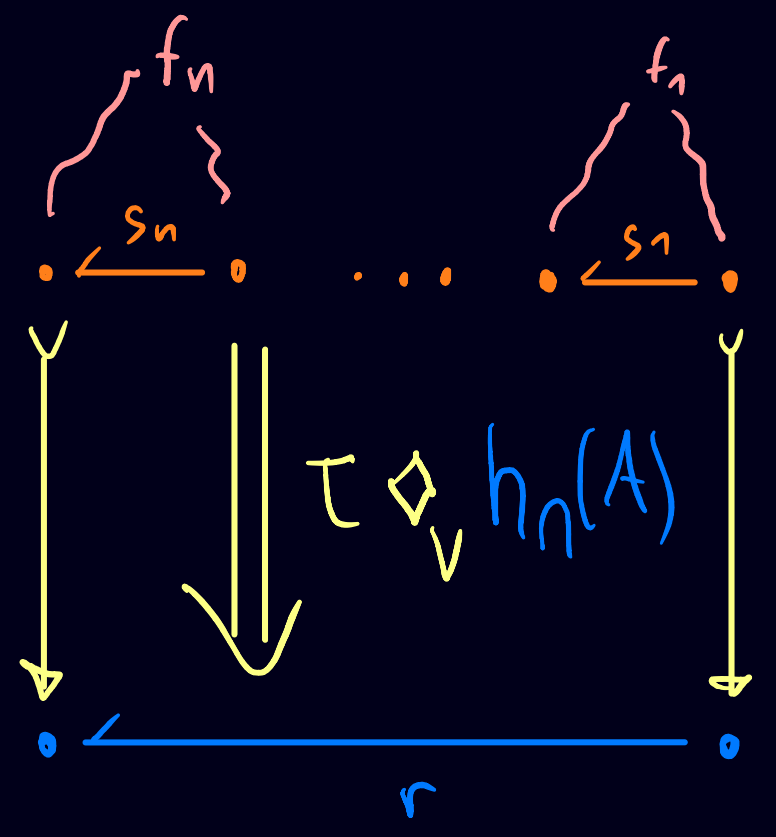
(9)

$$(F_n * \cdots * F_1)(r) = \left\{ (S,(\sigma,f)) \;\middle|\; \begin{array}{l} S \in \mathbb{D}_n \\ \sigma \in \mathbb{D}_1(h_n(S), r) \\ f \in \mathbb{F}_n(S) \end{array} \right\} \Big/ \sim \;\equiv\; \left\{ \right.$$  $$\left. \right\} \Big/ \sim$$

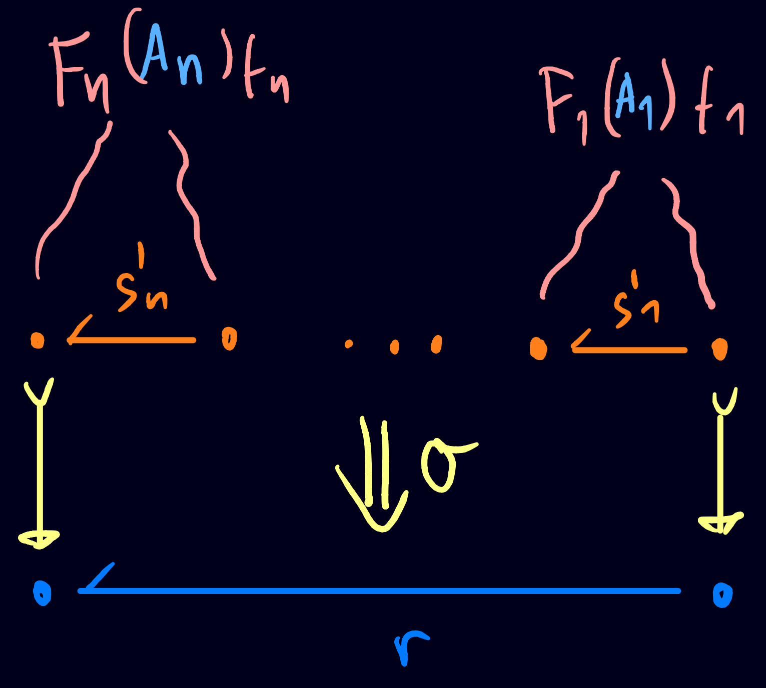
- $(S,(\sigma,f)) \sim (S',(\sigma',f')) :\Leftrightarrow \exists\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ (\tau,g) \in \mathbb{D}_1(h_n(S'), r) \times \mathbb{F}_n(S):$

$$(\sigma,f) = (\mathbb{D}_1(h_n(A), r)\tau,\ g) \;\wedge\; (\sigma',f') = (\tau,\ \mathbb{F}_n(A)g)$$

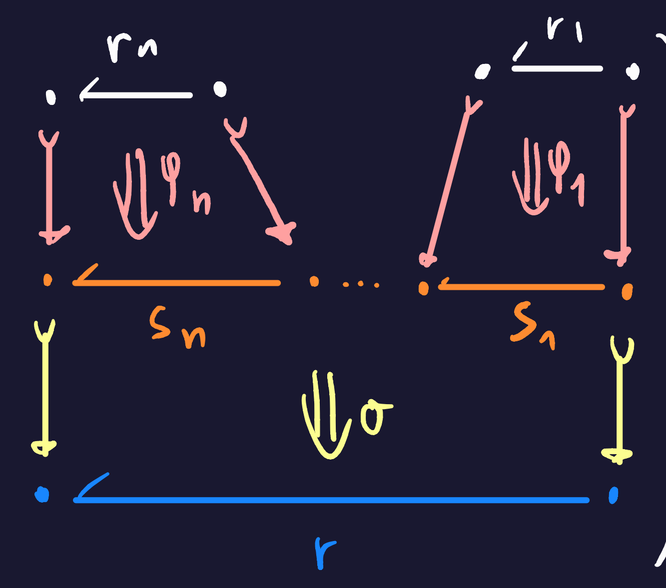
↳ $\forall\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ h_n(S') \xrightarrow{\tau} r \in \mathbb{D}_1$:



$\sim$

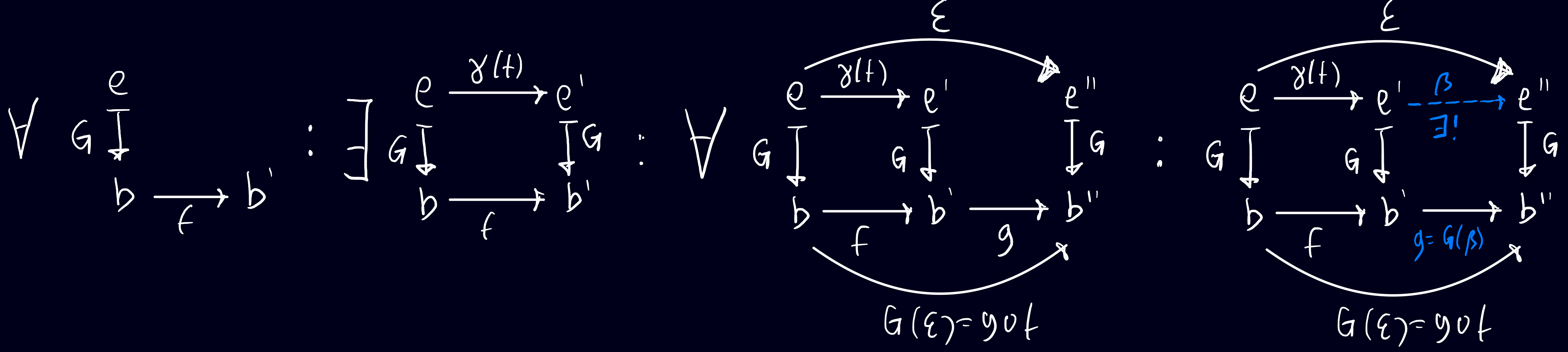


EXAMPLE: $\hat{\Delta}_{r_j} := \mathbb{D}_1(r_j, -) : \mathbb{D}_1 \to \underline{Set}$ $(j=1,\ldots,n)$

↳ $(\hat{\Delta}_{r_n} * \cdots * \hat{\Delta}_{r_1})(r) = \left\{ \right.$  $\left. \right\} \big/ \sim$

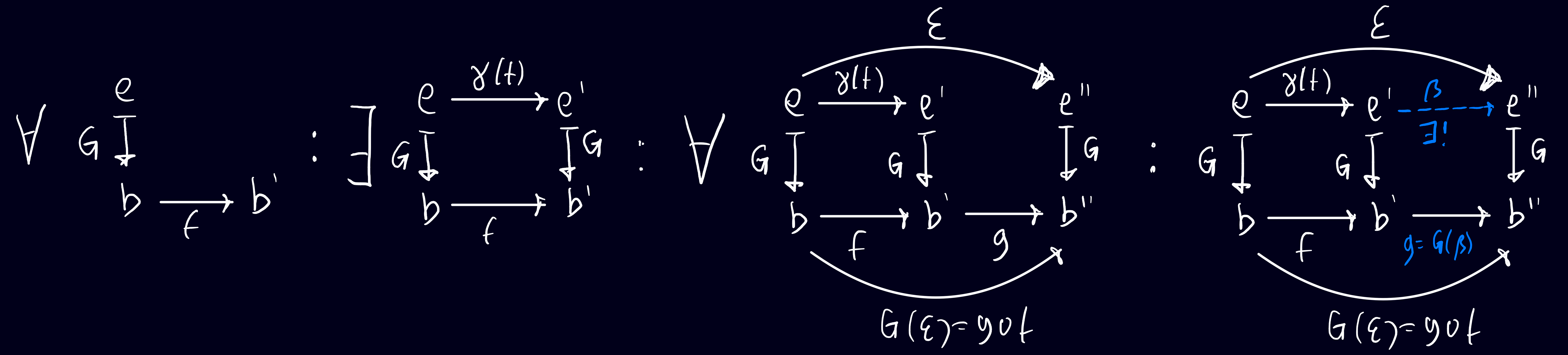
(10) Key concept: Fibrational structures

• Definition: A functor $G : \mathcal{E} \to \mathcal{B}$ is a **Grothendieck opfibration** iff

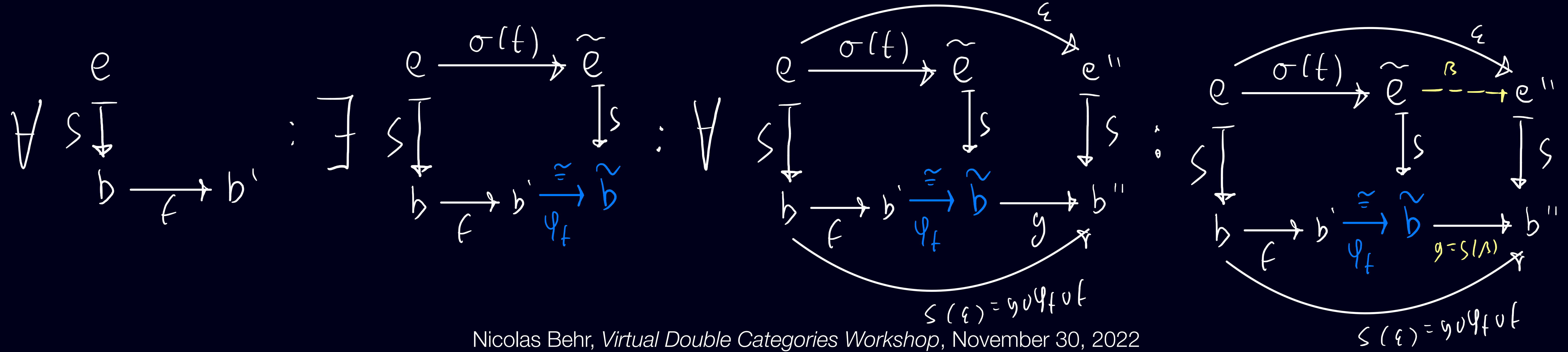


⑩ KEY CONCEPT: FIBRATIONAL STRUCTURES

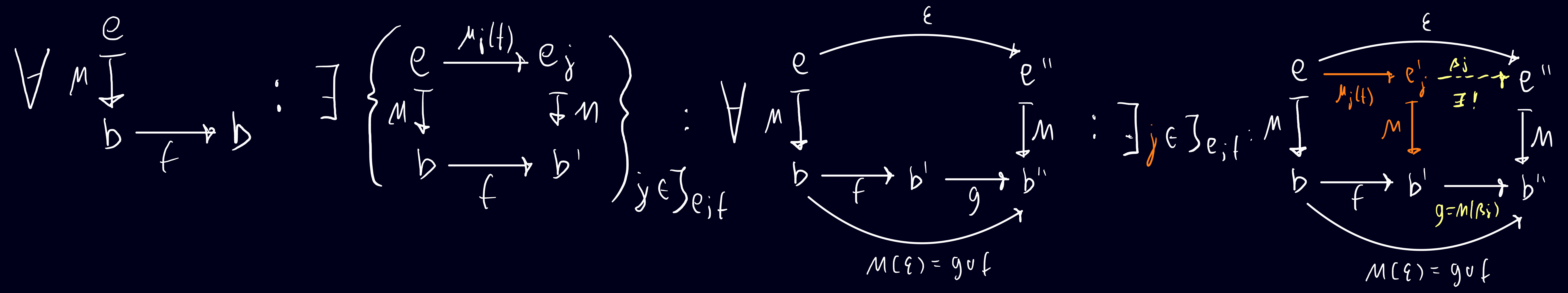
- DEFINITION: A functor $G : \mathcal{E} \to \mathcal{B}$ is a GROTHENDIECK OPFIBRATION iff

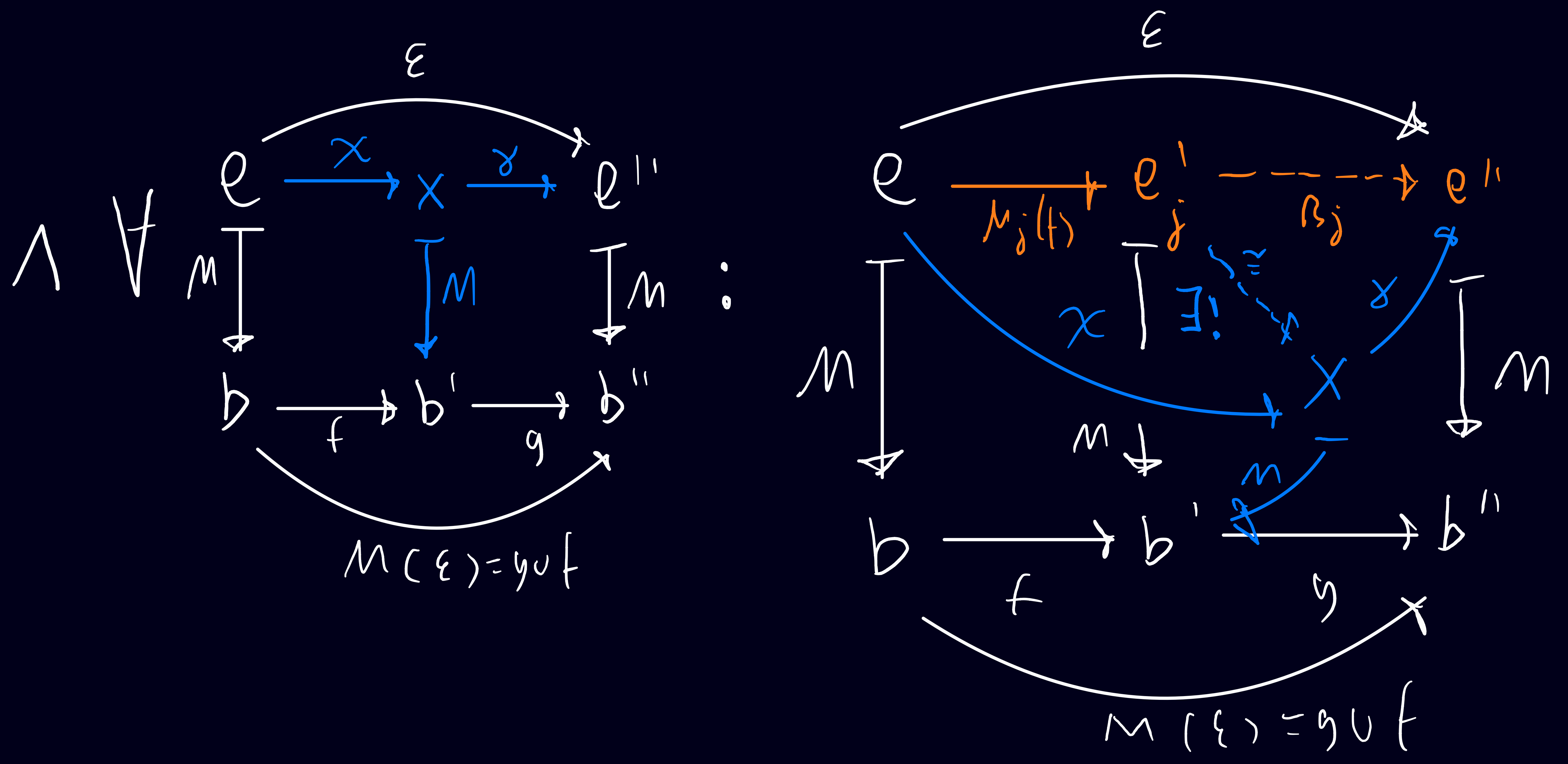


- DEFINITION: A functor $S : \mathcal{E} \to \mathcal{B}$ is a STREET OPFIBRATION iff

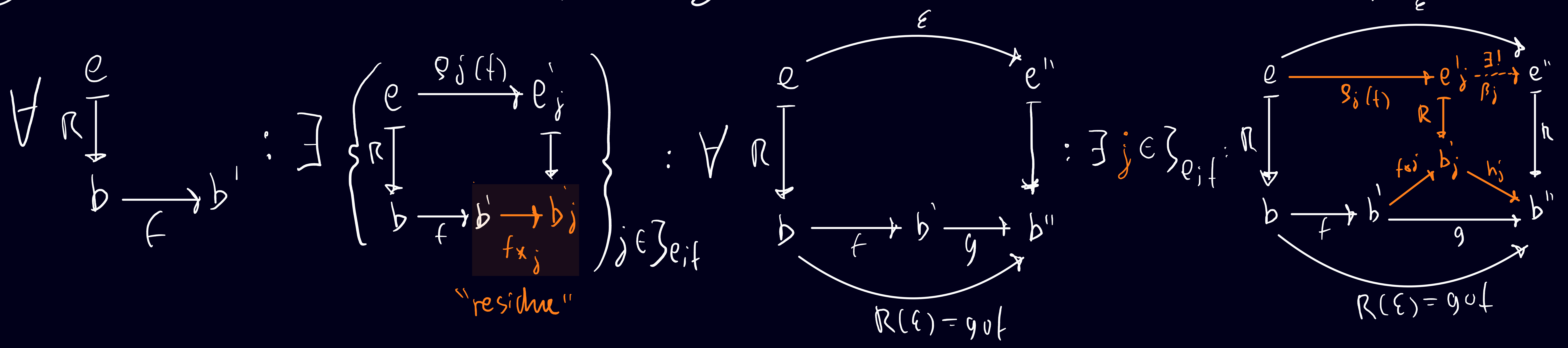


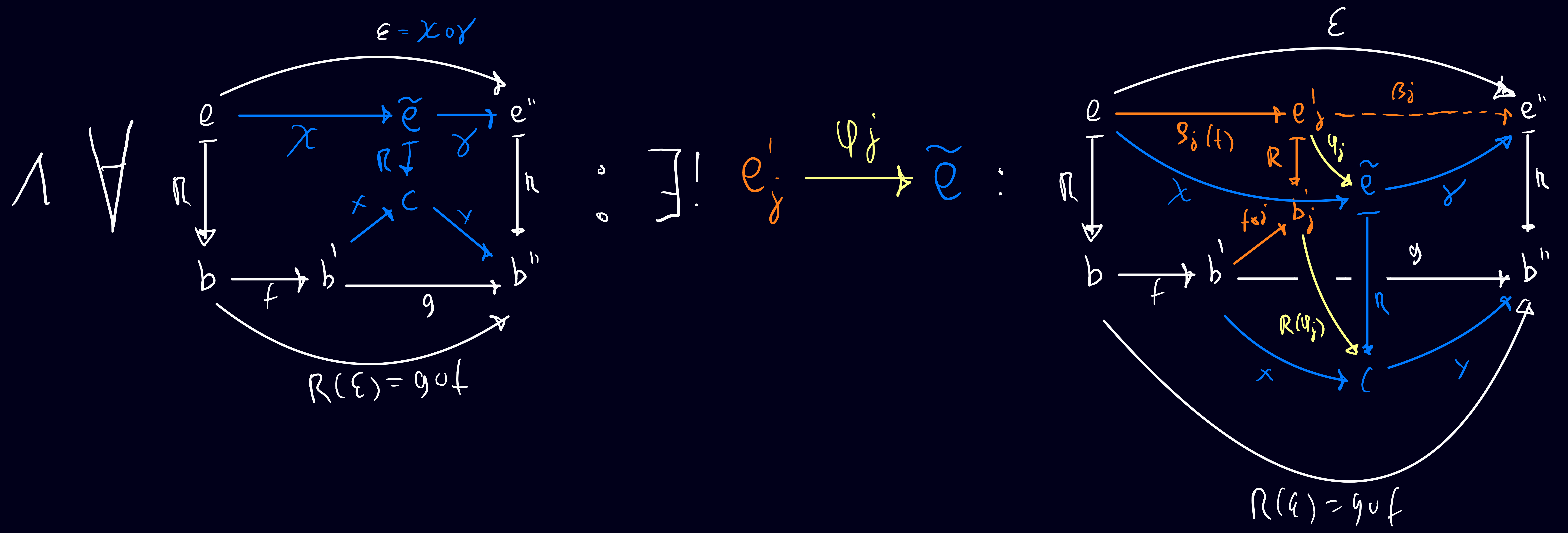
11 DEFINITION: A FUNCTOR $M : \mathcal{E} \to \mathcal{B}$ IS A MULTI-OPFIBRATION IFF





(12) DEFINITION: A FUNCTOR $R : \mathcal{E} \to \mathcal{B}$ IS A RESIDUAL MULTI-OPFIBRATION IFF





(13) DEFINITION: LET $X : \mathcal{E} \to \mathcal{B}$ BE AN X-OPFIBRATION ($X \in \{G, S, M, R\}$).

THEN A CLEAVAGE FOR X IS DEFINED AS A CHOICE OF REPRESENTATIVE FOR EACH X-OPCARTESIAN LIFTING:

$$G^*\left(\begin{array}{c} e \\ G\downarrow \\ b \xrightarrow{f} b' \end{array}\right) := \begin{array}{ccc} e & \xrightarrow{G^*(f)} & b \\ G\downarrow & & \downarrow G \\ b & \xrightarrow{f} & b \end{array}$$

$$S^*\left(\begin{array}{c} e \\ S\downarrow \\ b \xrightarrow{f} b' \end{array}\right) := \begin{array}{cccc} e & & \xrightarrow{\theta^*(f)} & \tilde{e} \\ S\downarrow & & & \downarrow S \\ b & \xrightarrow{f} & b' \xrightarrow[\varphi_f]{\cong} & \tilde{b} \end{array}$$

$$M^*\left(\begin{array}{c} e \\ M\downarrow \\ b \xrightarrow{f} b' \end{array}\right) := \left\{\begin{array}{ccc} e & \xrightarrow{\mu_j^*(f)} & e'_j \\ M\downarrow & & \downarrow m \\ b & \xrightarrow{f} & b' \end{array}\right\}_{j \in \mathcal{Z}^*_{e;f}}$$

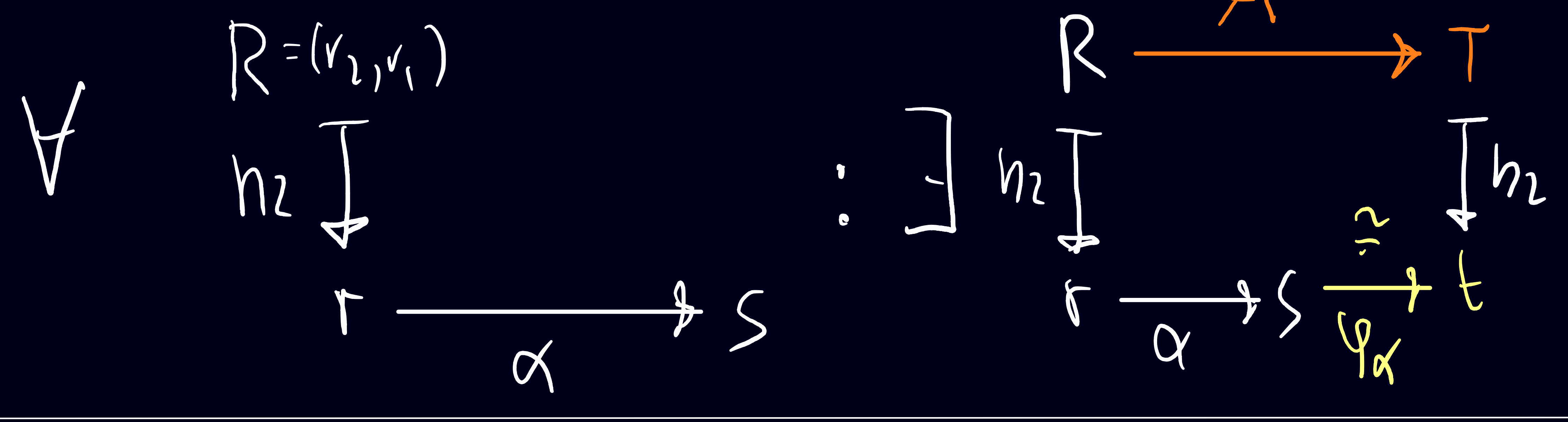
$$R^*\left(\begin{array}{c} e \\ R\downarrow \\ b \xrightarrow{f} b' \end{array}\right) := \left\{\begin{array}{cccc} e & & \xrightarrow{\rho_j^*(f)} & e'_j \\ R\downarrow & & & \downarrow R \\ b & \xrightarrow{f} & b' \xrightarrow{f\times_j} & b'_j \end{array}\right\}_{j \in \mathcal{Z}^*_{e;f}}$$

ONE REPRESENTATIVE PER EQUIVALENCE CLASS IN $\mathcal{Z}_{e;f}$!

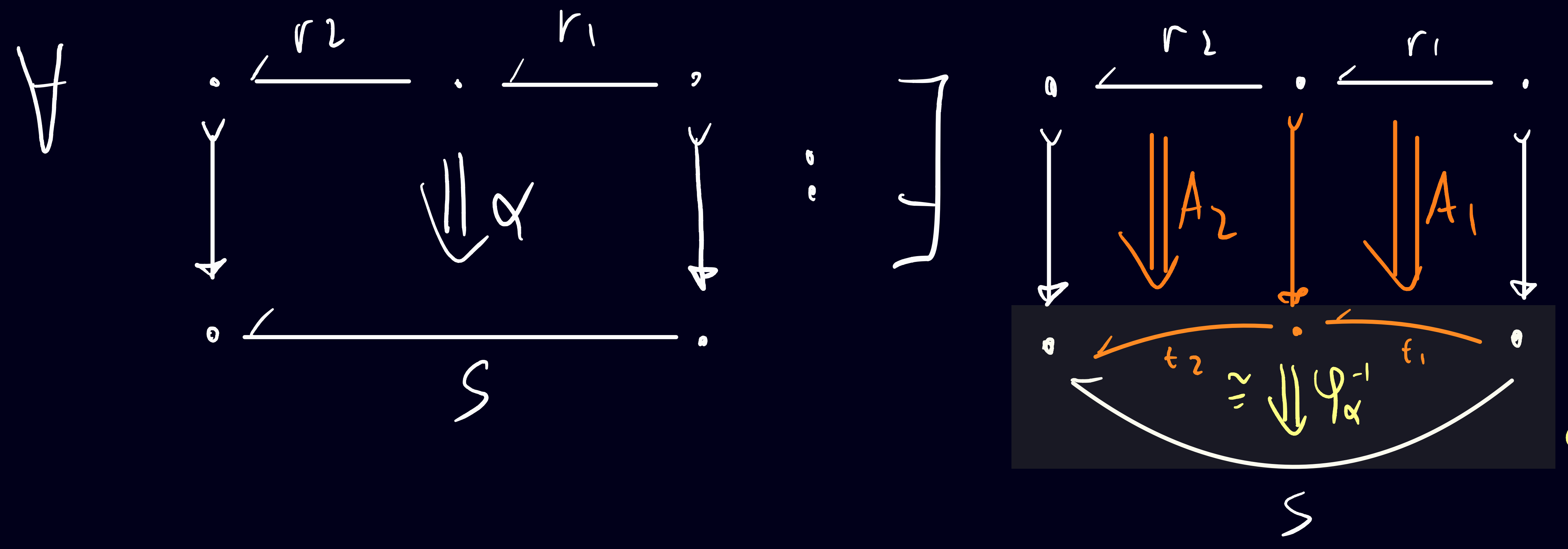
14 EMPIRICAL RESULT: $\mathbb{D}$ FOR COMPOSITIONAL* REWRITING SEMANTICS

* 2204.07175

- $h_2 = \diamond_h : \mathbb{D}_2 \longrightarrow \mathbb{D}_1$ IS A "GLOBULAR" STREET OPFIBRATION, i.e.,



: $S(\varphi_\alpha) = id_{S(s)} \wedge T(\varphi_\alpha) = id_{T(s)} \wedge$ (STREET OPFIBRATION CONDITIONS)



: $\varphi_\alpha^{-1} \diamond_v (A_2 \diamond A_1) = \alpha$

← "GLOBULAR" ISOMORPHISM

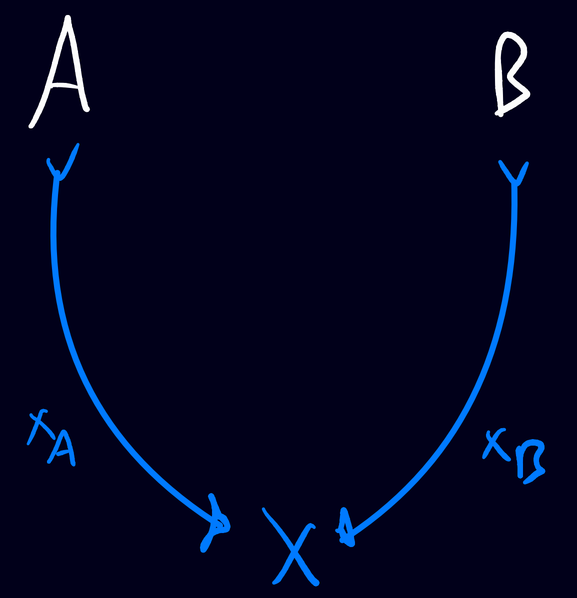
By INDUCTION ON n, ONE FINDS THAT

$\forall n \geq 2 : h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1$ ARE "GLOBULAR" STREET OPFIBRATIONS

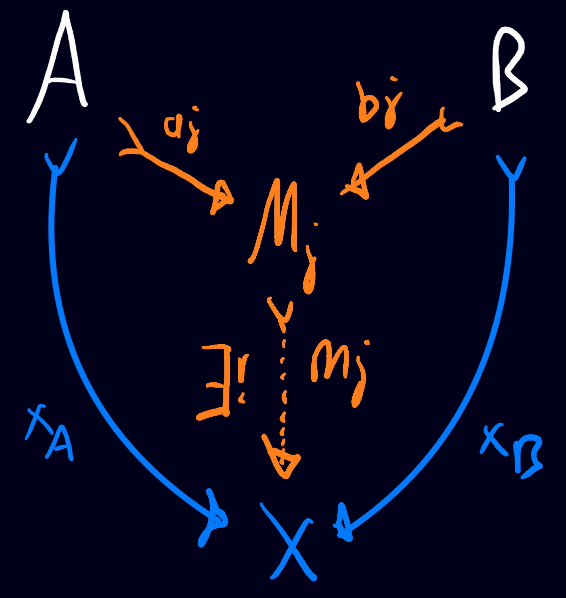
15

$\mathbb{D}_0$ HAS MULTI-SUMS:

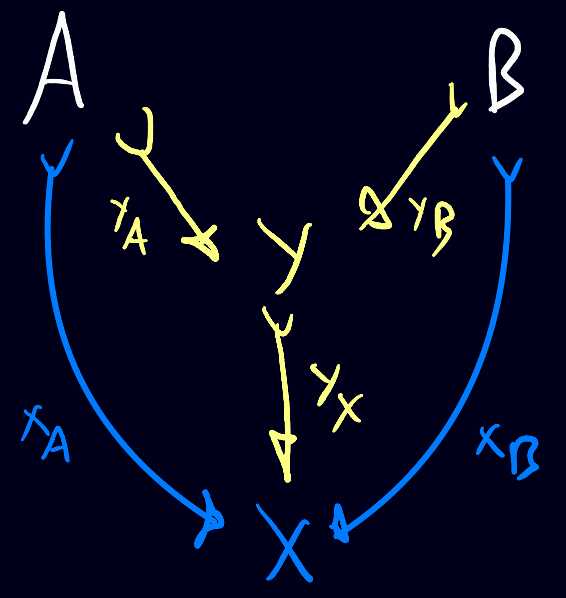
$\forall\, (A,B) \in \mathbb{D}_0 \times \mathbb{D}_0 : \exists \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{j \in \mathcal{J}(A,B)} : \forall$



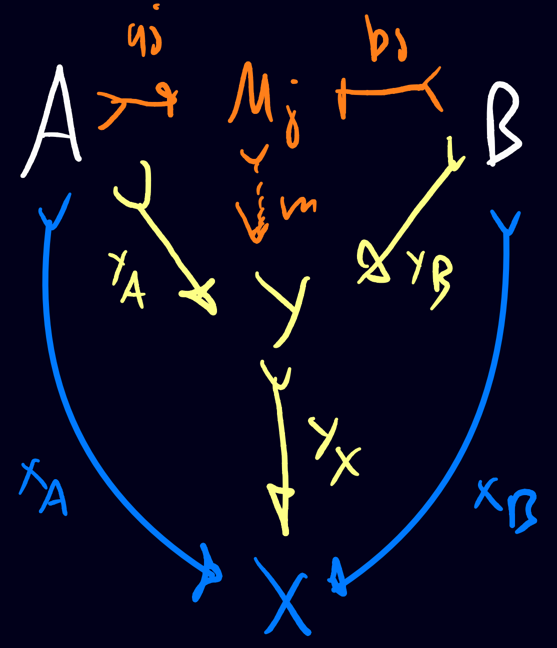
$: \exists j \in \mathcal{J}(A,B) :$



$\wedge\ \forall$



$: \exists!\ M_j \xrightarrow{m} Y :$



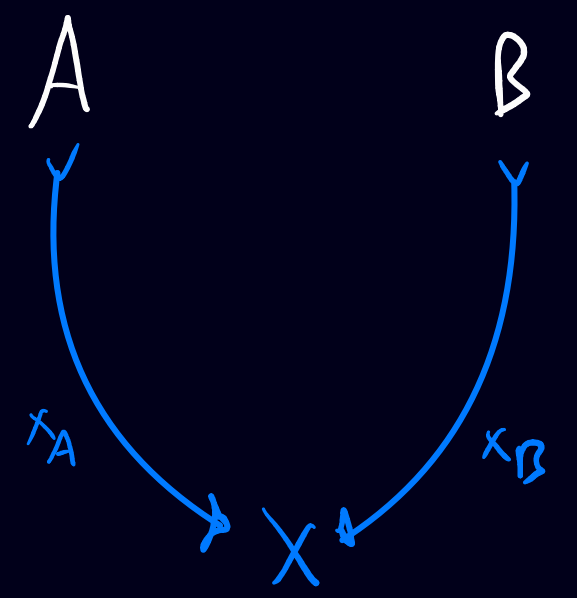
DEFINITION: CLEAVAGE FOR MULTI-SUMS:

$$\forall\, (A,B) \in \mathbb{D}_0 \times \mathbb{D}_0 : \; ms(A,B) = \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{\mathcal{J}^{\circ}(A,B)}$$

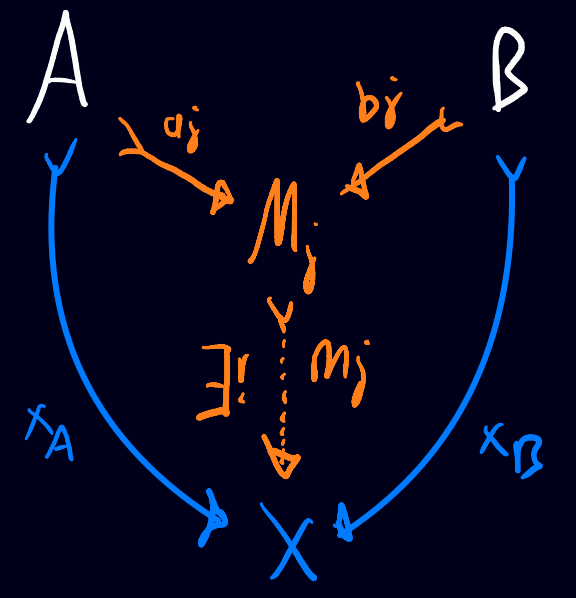
15

$\mathbb{D}_0$ HAS MULTI-SUMS:

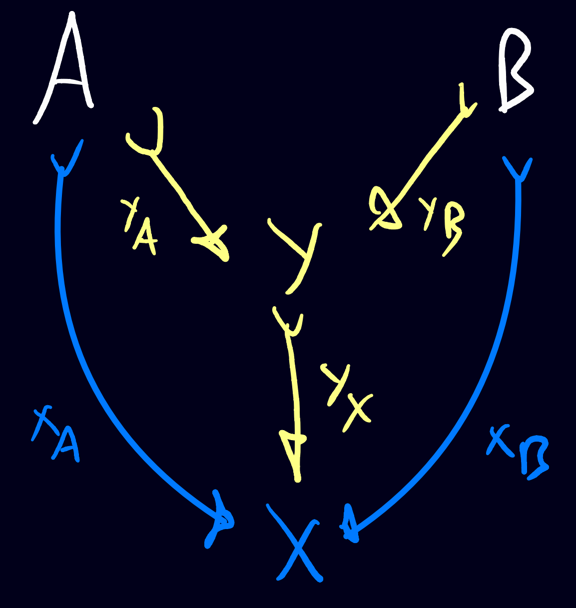
$\forall (A,B) \in \mathbb{D}_0 \times \mathbb{D}_0 : \exists \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{j \in J_{(A,B)}} : \forall$



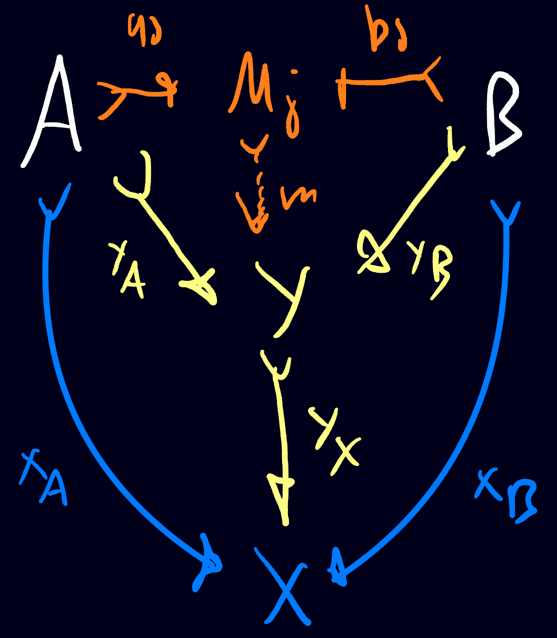
$: \exists j \in J_{(A,B)} :$



$\wedge \; \forall$



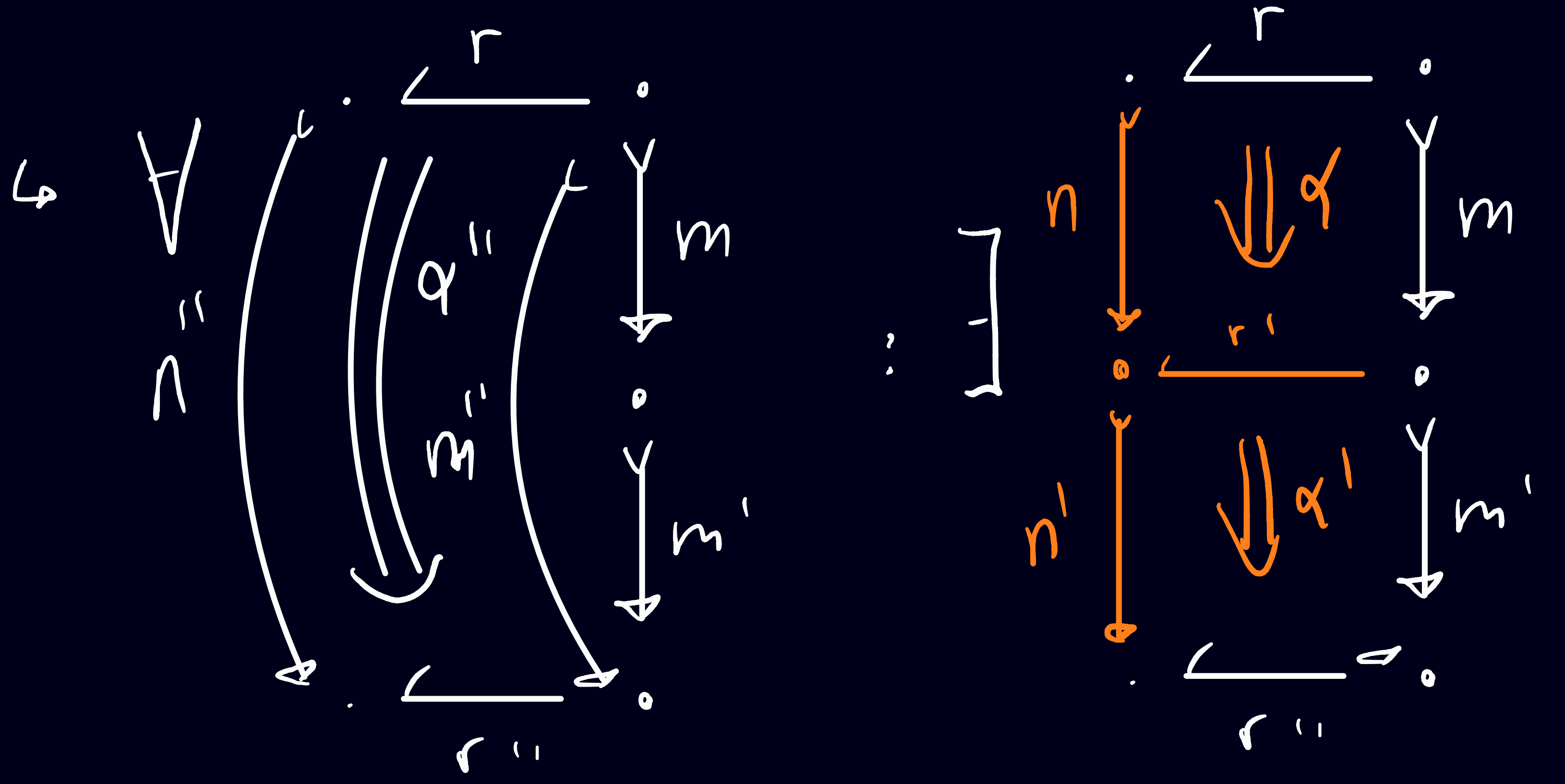
$: \exists! \; M_j \xrightarrow{m} Y :$



DEFINITION: CLEAVAGE FOR MULTI-SUMS:

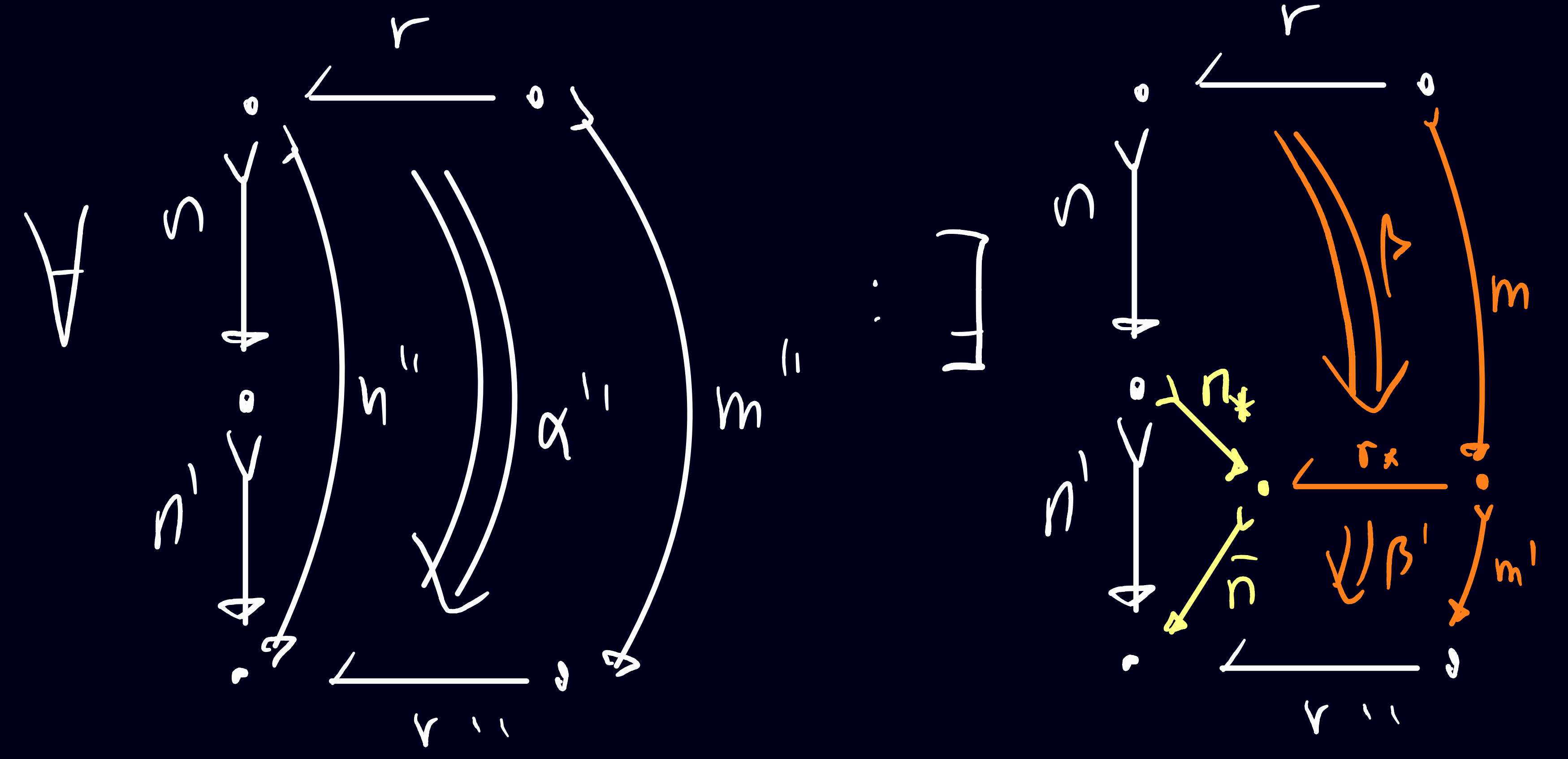
$\forall (A,B) \in \mathbb{D}_0 \times \mathbb{D}_0 : \; ms(A,B) = \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{J^{\circ}_{(A,B)}}$

$S : \mathbb{D}_1 \to \mathbb{D}_0$ IS A MULTI-OPFIBRATION



$: \; n' \circ n = n'' \;\wedge\; \alpha' \circ_v \alpha = \alpha''$

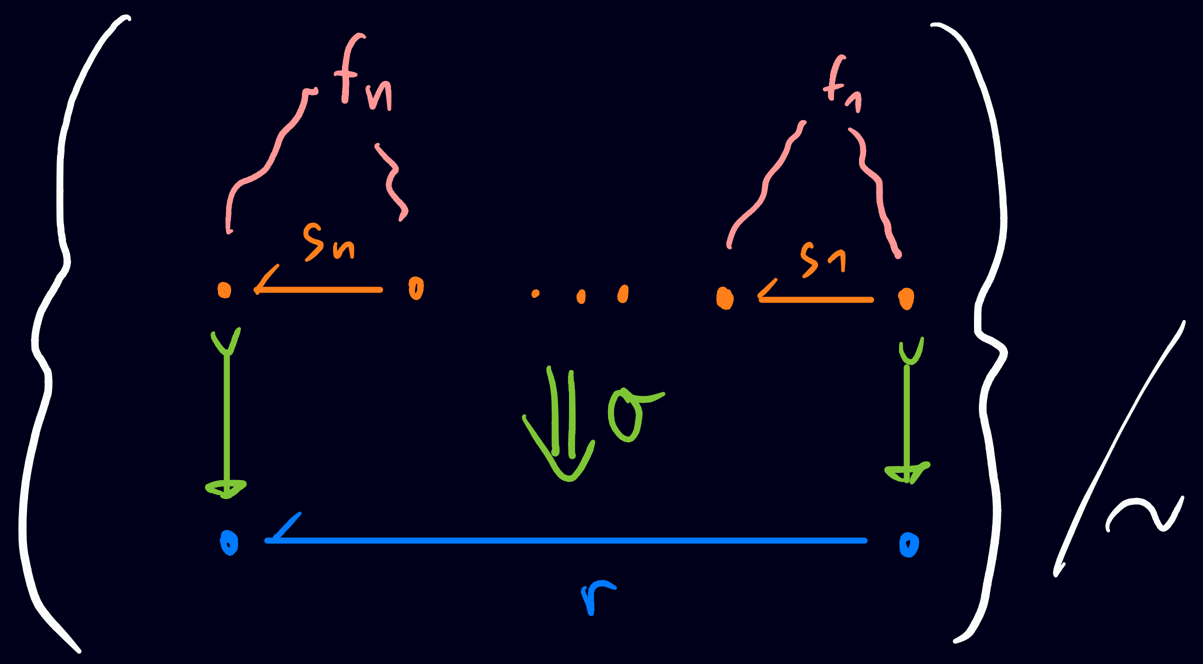
$T : \mathbb{D}_1 \to \mathbb{D}_0$ IS A RESIDUAL MULTI-OPFIBRATION



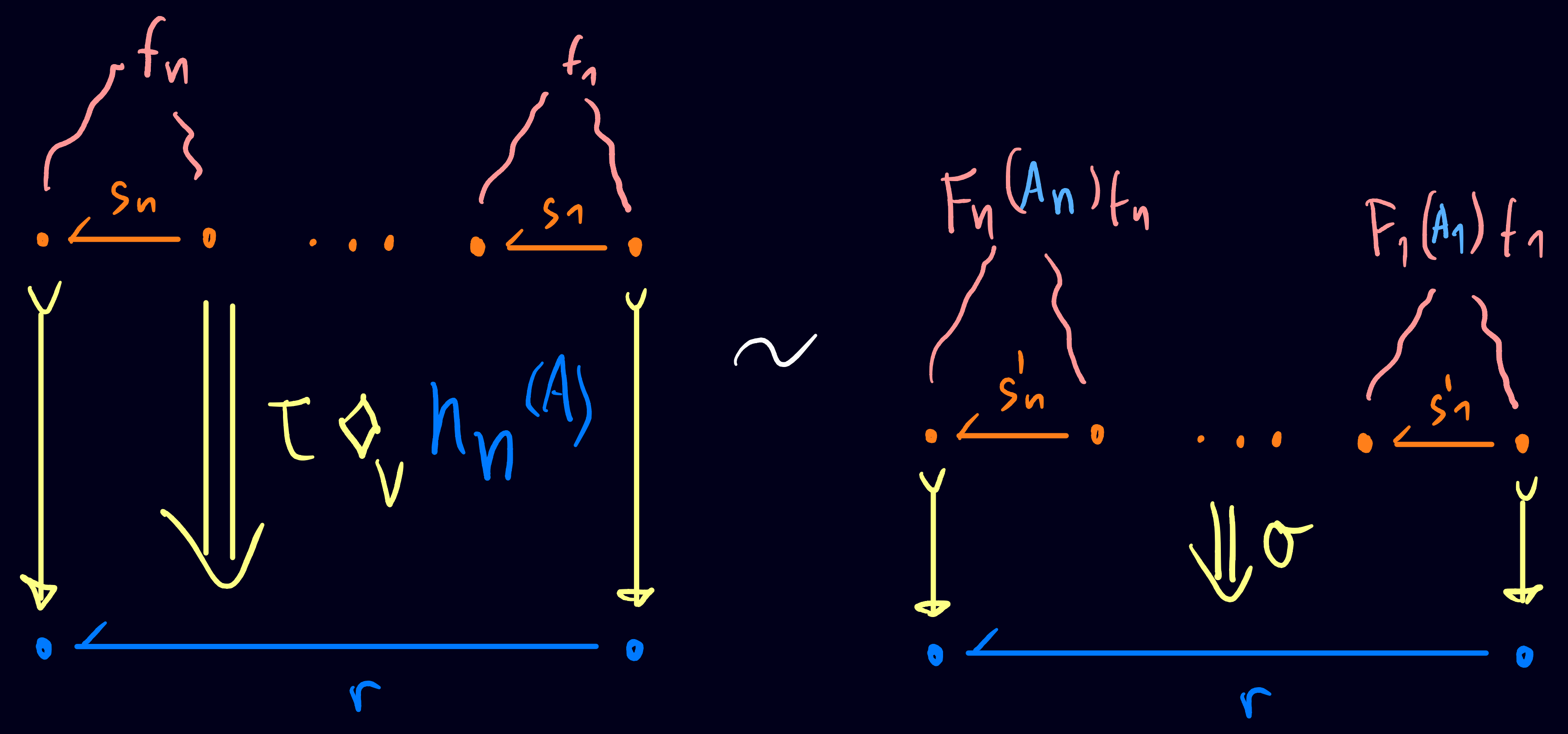
$: \; \bar{n} \circ n_x = n' \;\wedge\; m' \circ m = m'' \;\wedge\; \beta' \circ_v \beta = \beta''$

(16) CONVOLUTION PRODUCTS REVISITED

RECAP: $(F_n * \cdots * F_1)(r) := \int^{S=(s_n,\dots,s_1)\in \mathbb{D}_n} \mathbb{D}_1(h_n(S), r) \times \mathbb{F}_n(S) =$

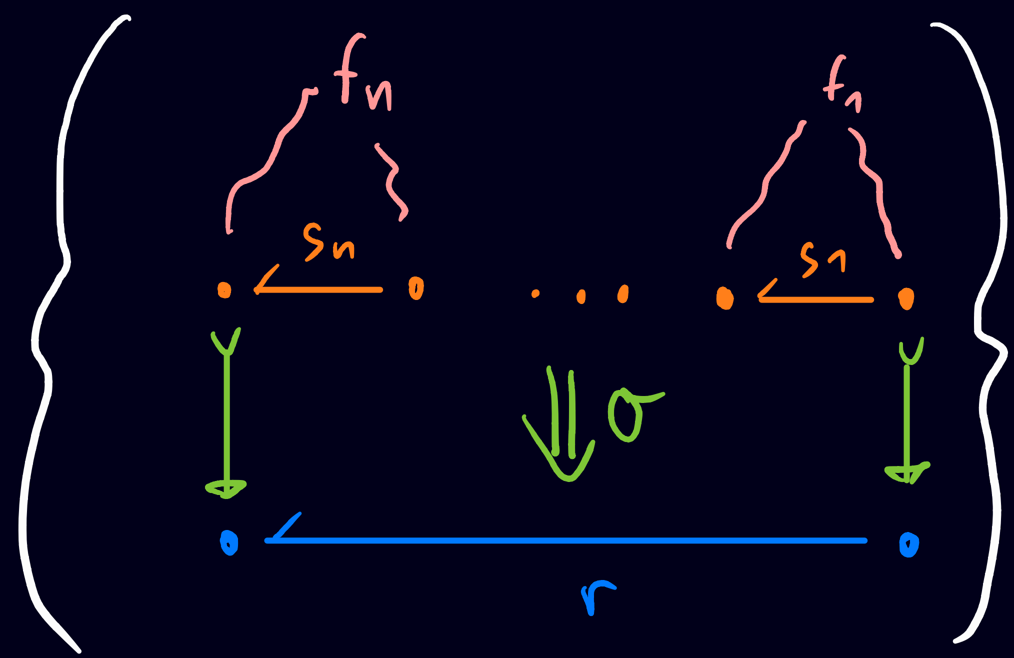


$\forall S \xrightarrow{A} S' \in \mathbb{D}_n,$
$h_n(S') \xrightarrow{\tau} r \in \mathbb{D}_1$:



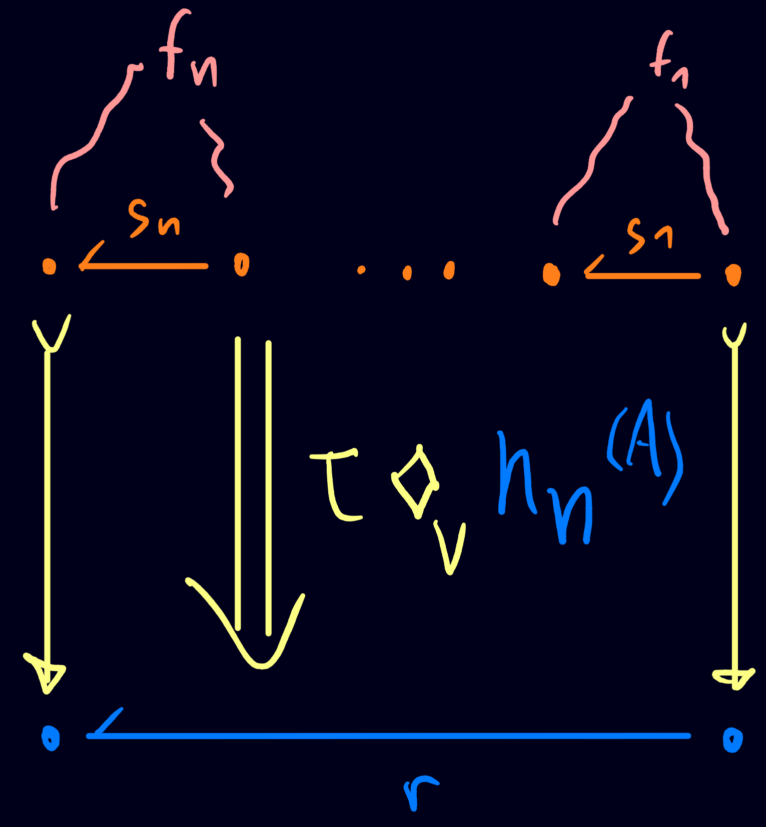
16 CONVOLUTION PRODUCTS REVISITED

RECAP: $(F_n * \cdots * F_1)(r) := \int^{S=(S_n,\ldots,S_1)\in \mathbb{D}_n} \mathbb{D}_1(h_n(S), r) \times \mathbb{F}_n(S) =$



$/\sim$

$\forall\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ h_n(S') \xrightarrow{\tau} r \in \mathbb{D}_1$:



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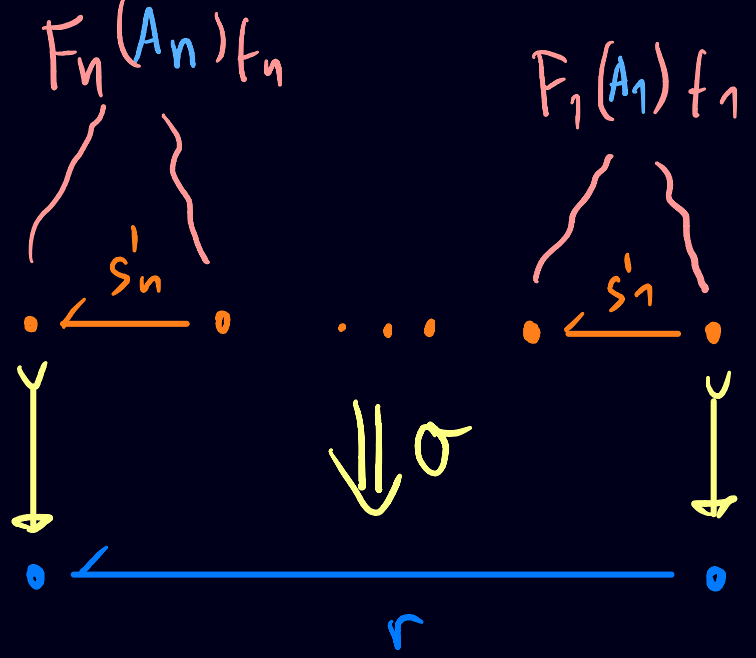
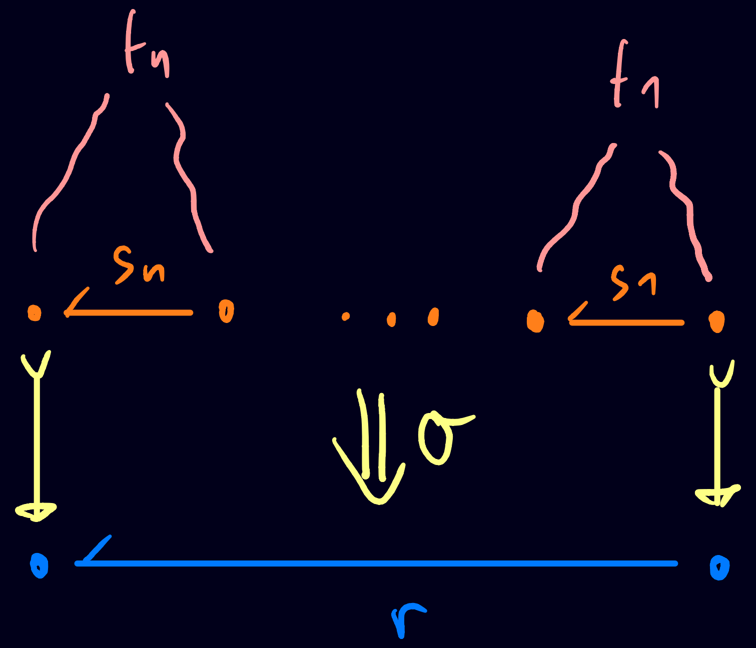
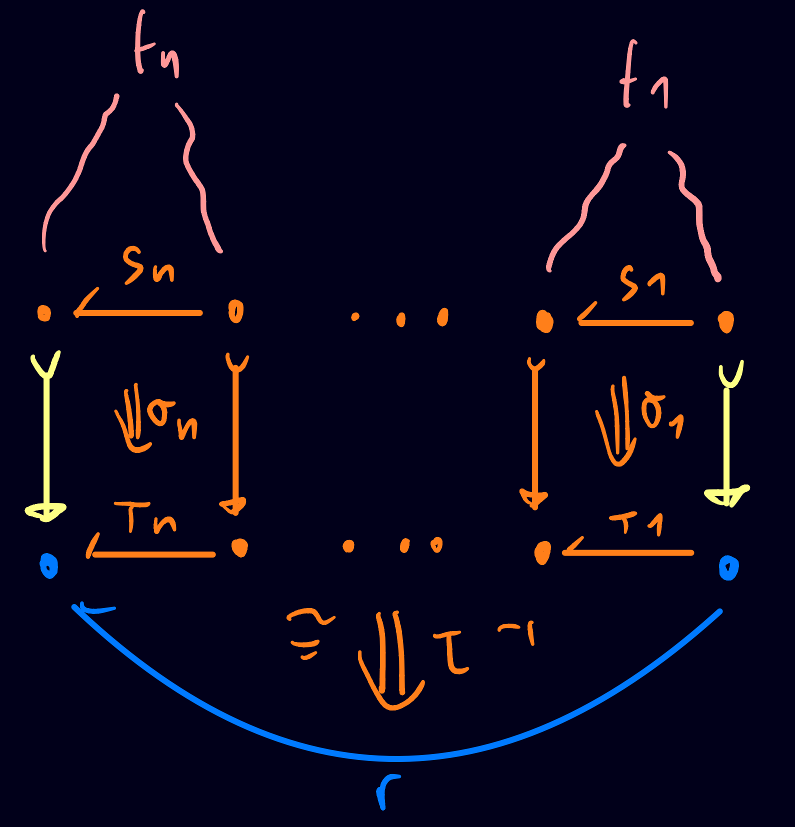


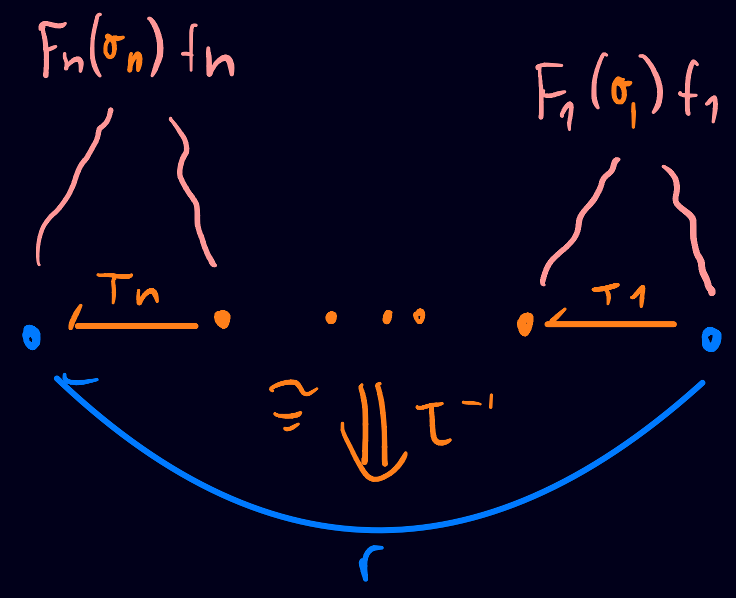
NOW: $h_n : \mathbb{D}_n \to \mathbb{D}_1$ IS A "GLOBULAR" STREET OPFIBRATION $\Rightarrow \forall$



$\exists$:



$\cong$



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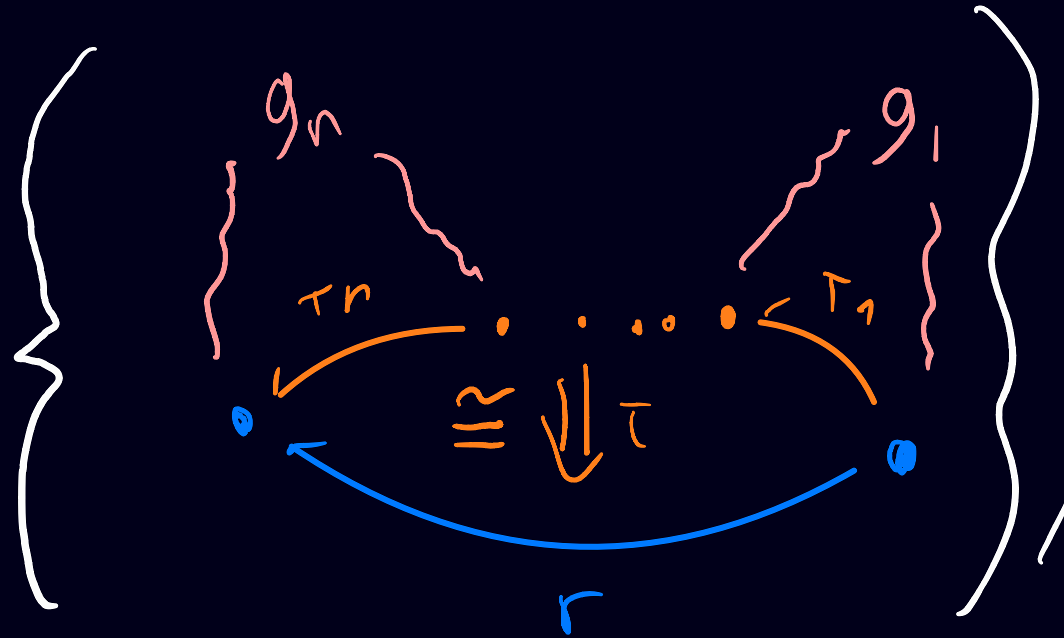
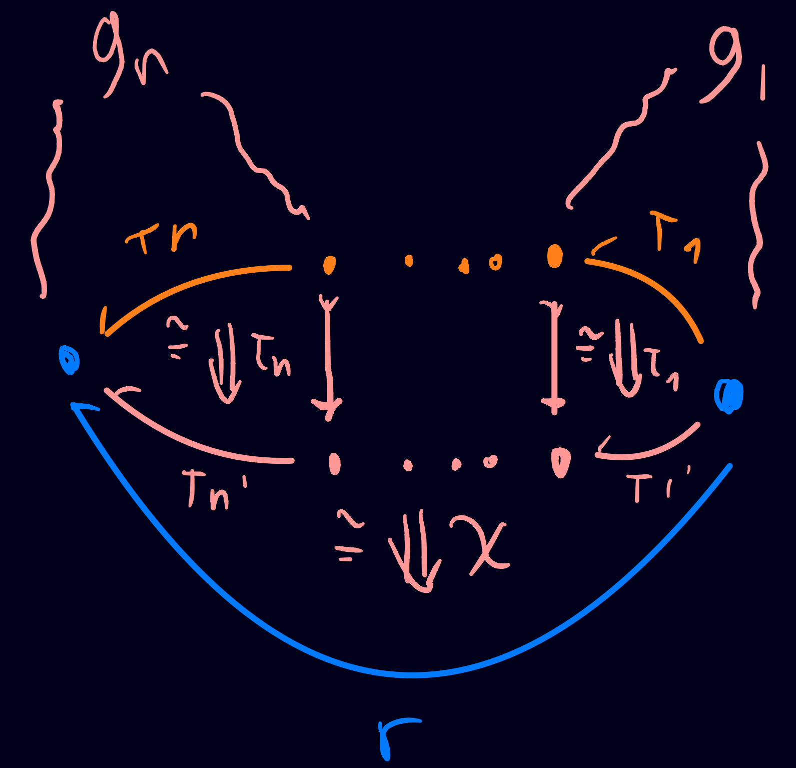
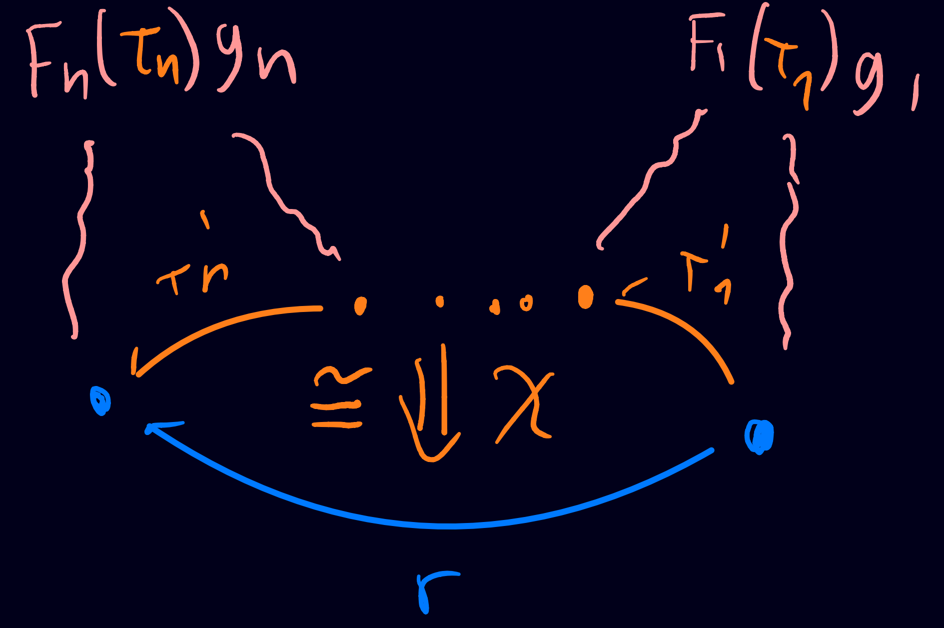
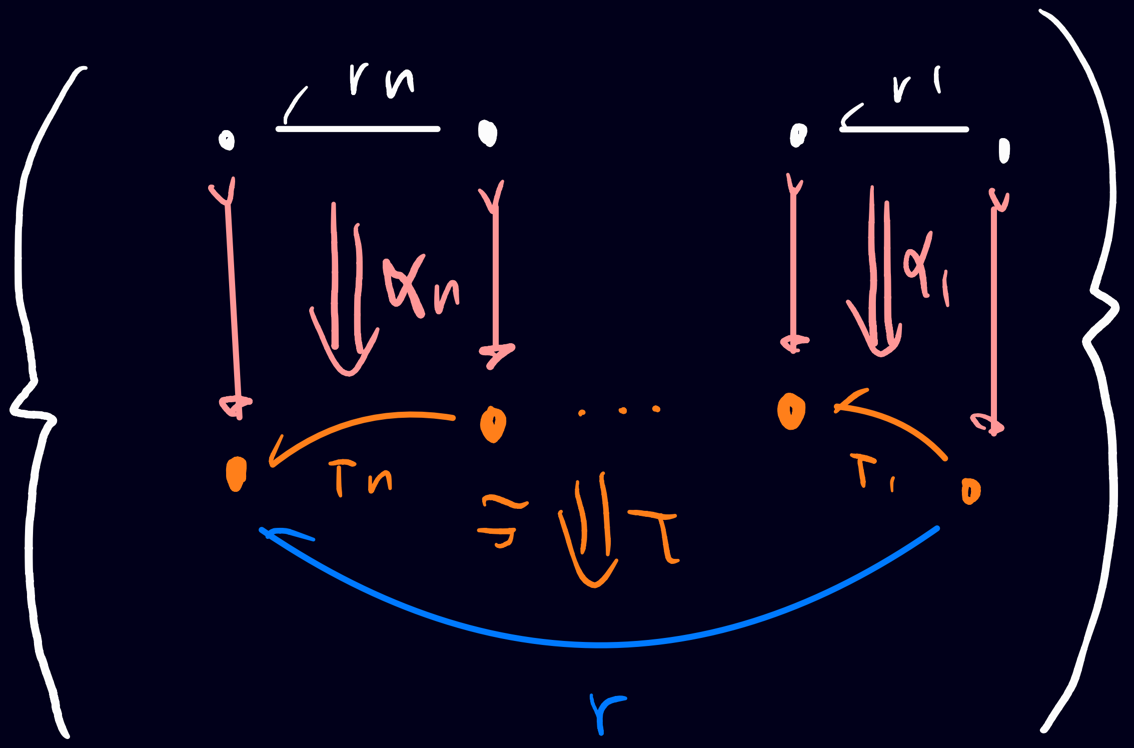
↪ $(F_n * \ldots * F_1)(r) \cong$ 

$/{\cong_g}$

WHERE 

$\cong_g$ 

EXAMPLE: FOR $\hat{\Delta}_{r_j} := \mathbb{D}_1(r_j, -)$ $(j = 1, \ldots, n)$

$(\hat{\Delta}_{r_n} * \ldots * \hat{\Delta}_{r_1})(r) \cong$ 

$/{\cong_g}$

✓

(18) KEY RESULT: WEAK ASSOCIATIVITY OF $*$

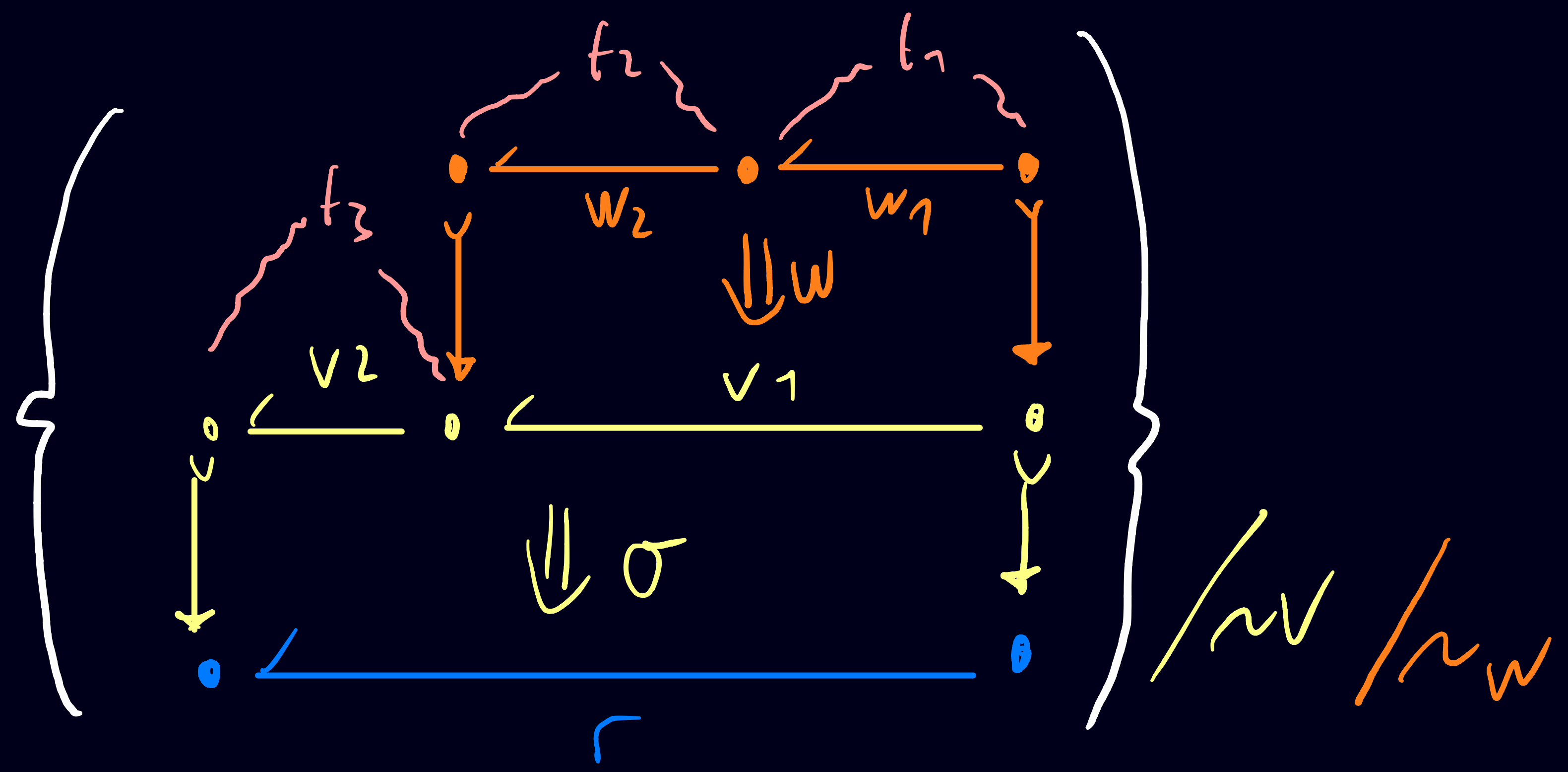
$$\forall F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \underline{Set}\,,\ r \in \mathbb{D}_1 :\ F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

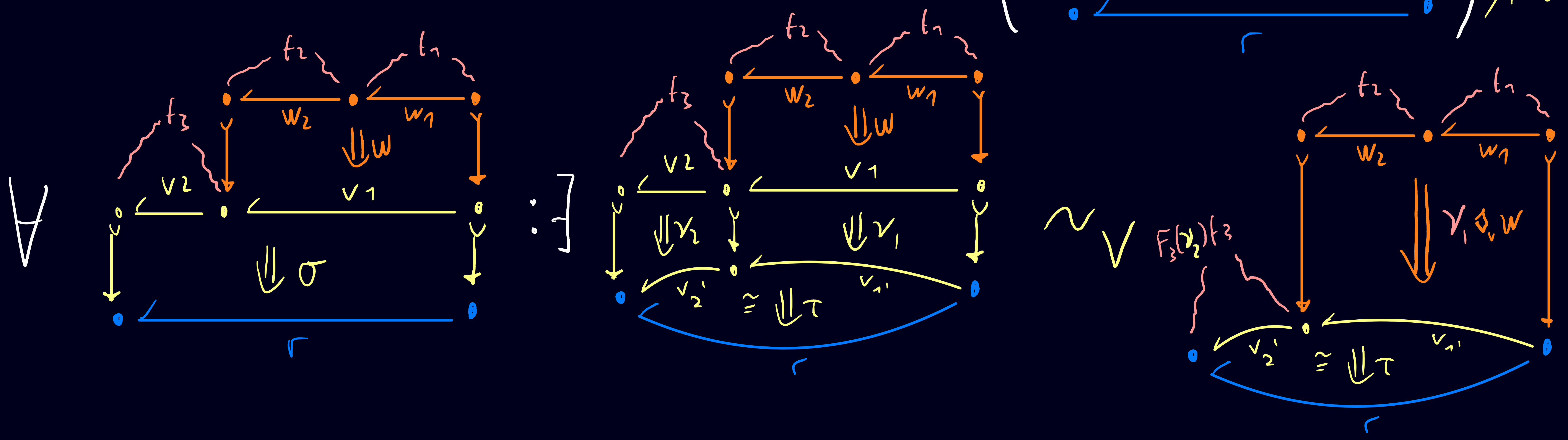
(18) KEY RESULT: WEAK ASSOCIATIVITY OF $*$

$\forall F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \underline{Set}$, $r \in \mathbb{D}_1$: $F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$

PROOF (SKETCH):

$F_3 * (F_2 * F_1)(r) =$



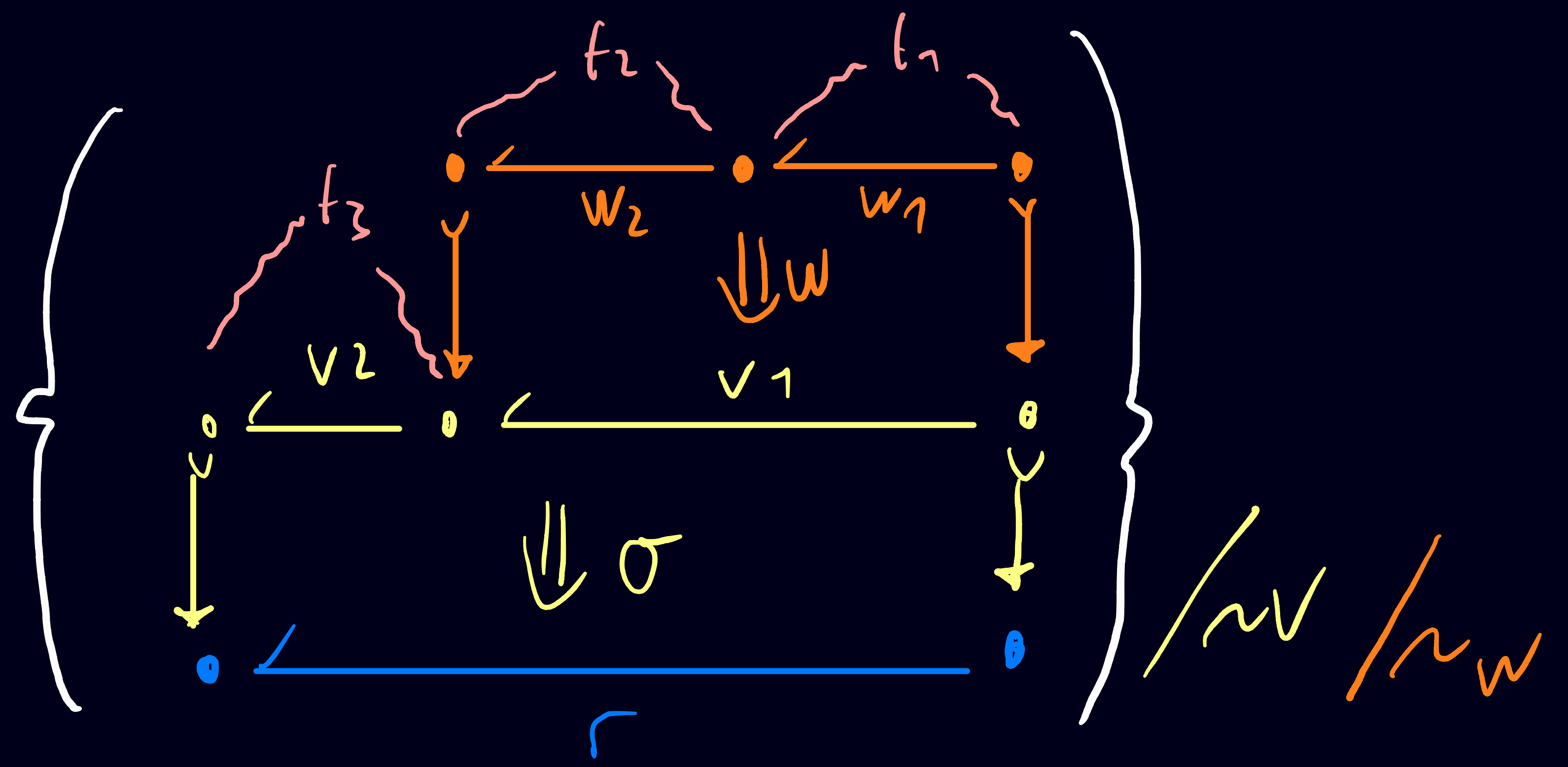


(18) Key Result: Weak Associativity of $*$

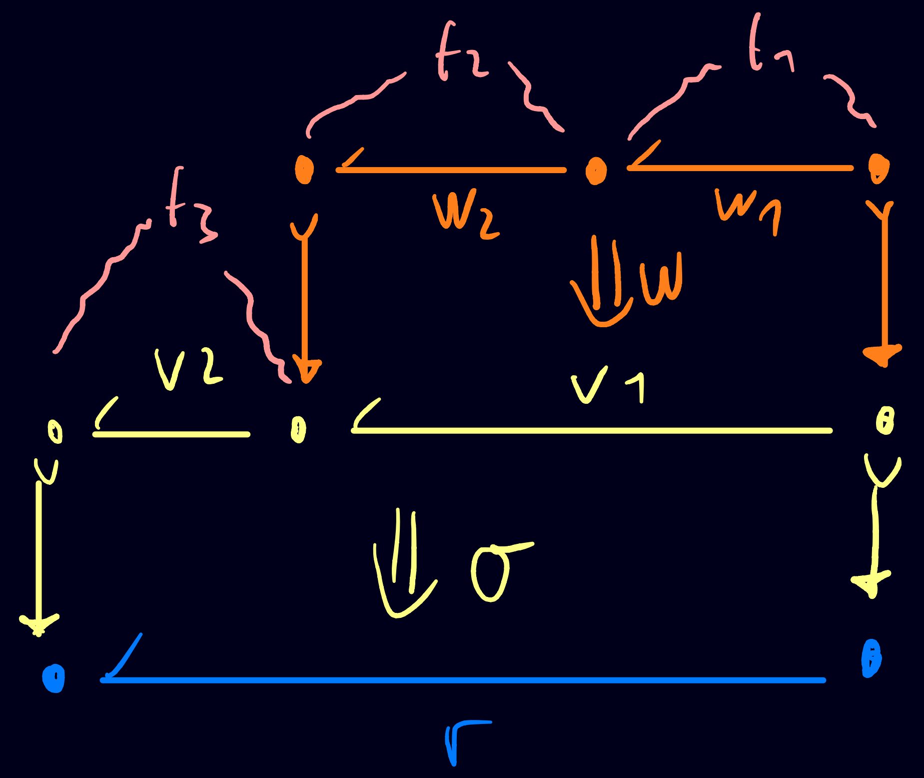
$$\forall\, F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \underline{\mathbf{Set}}\,,\quad r \in \mathbb{D}_1 :\quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

Proof (sketch):

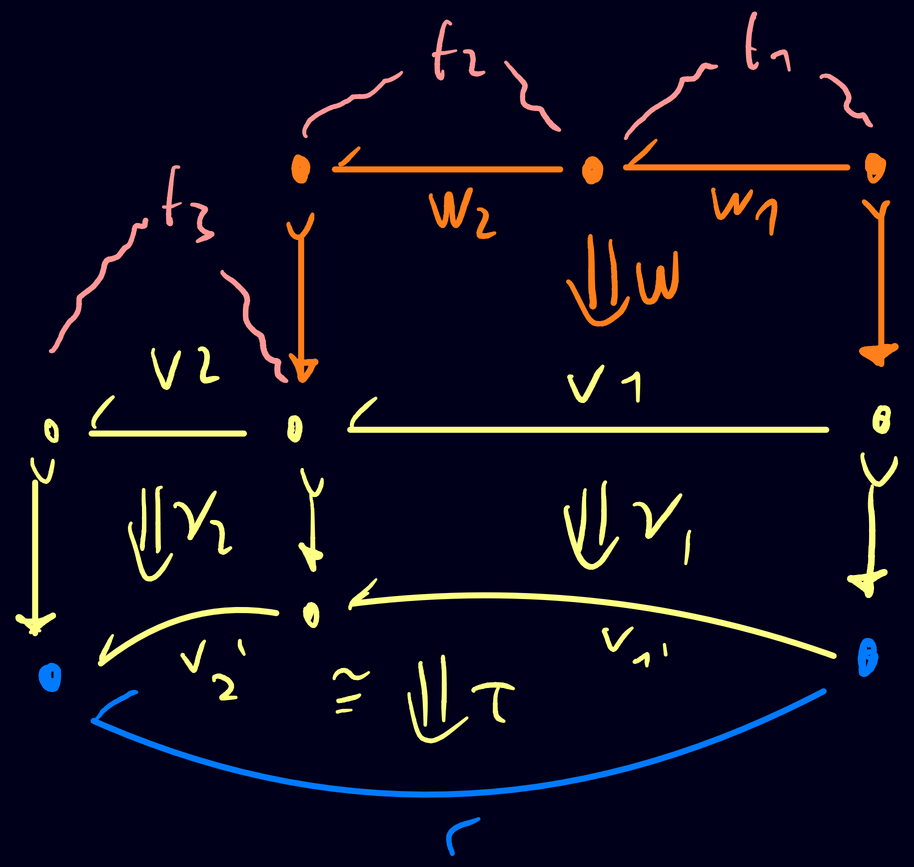
$F_3 * (F_2 * F_1)(r) =$



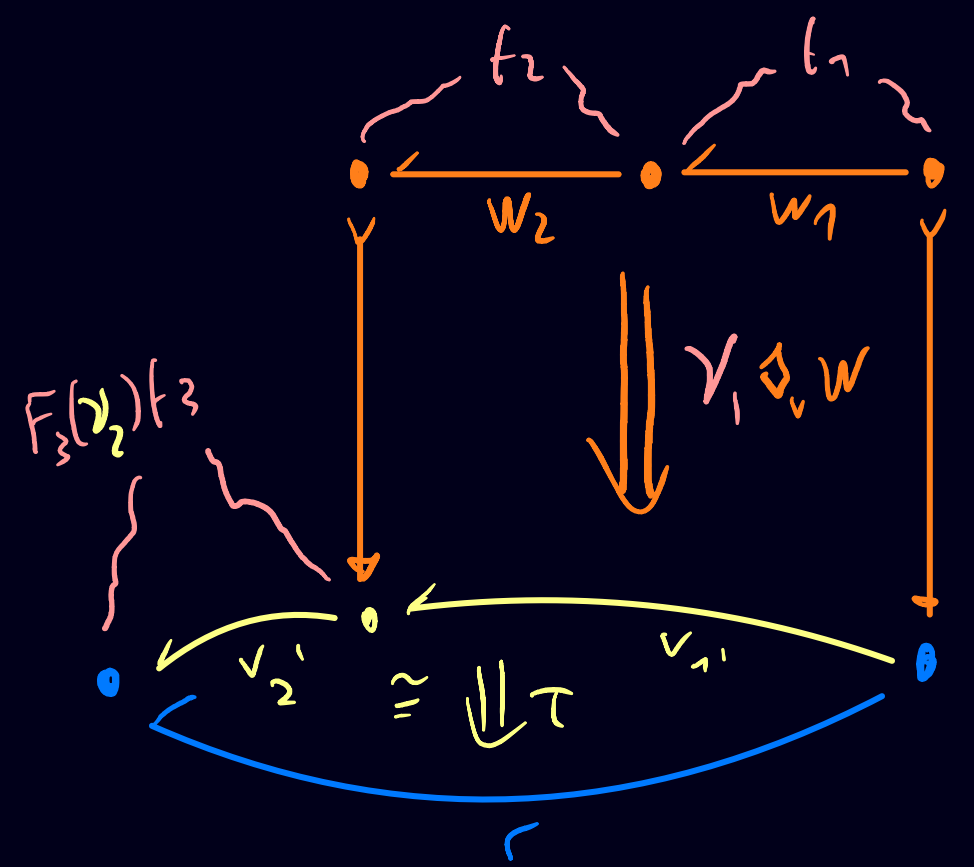
$\forall$



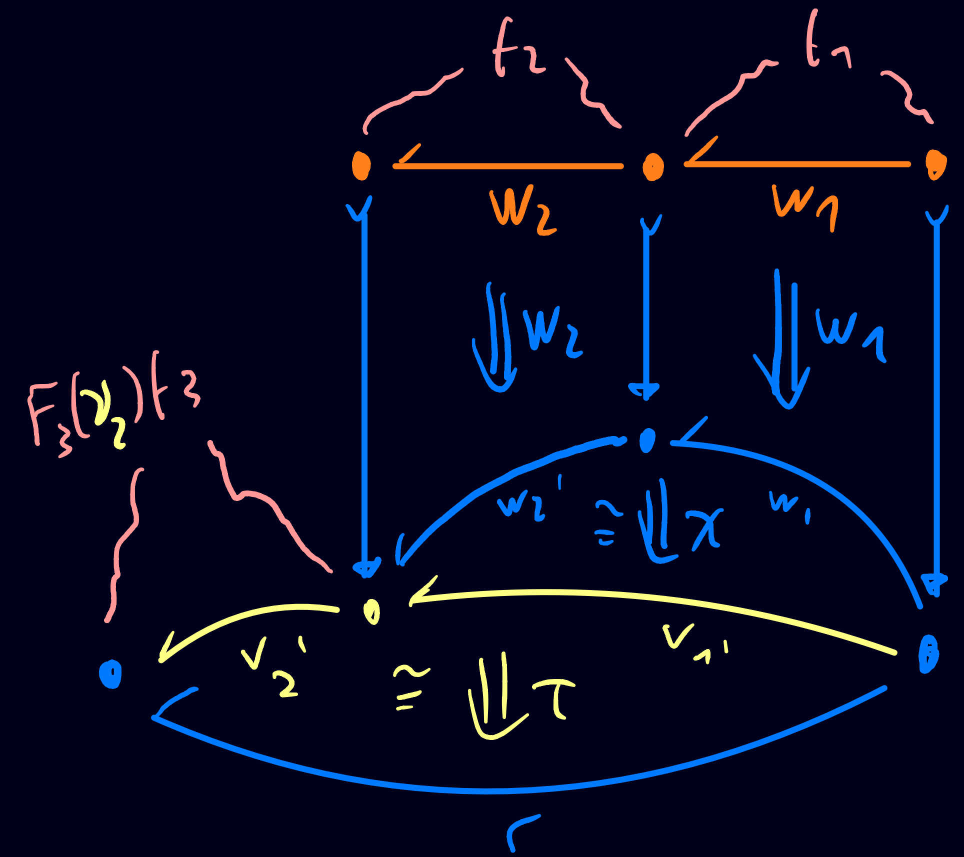
:]



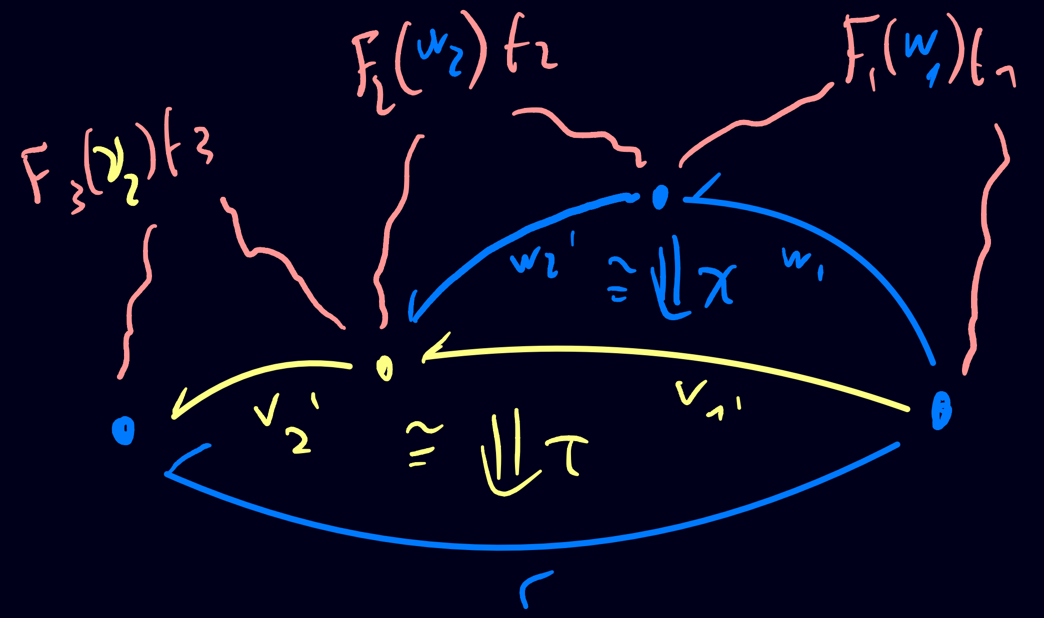
$\sim_V$



:]



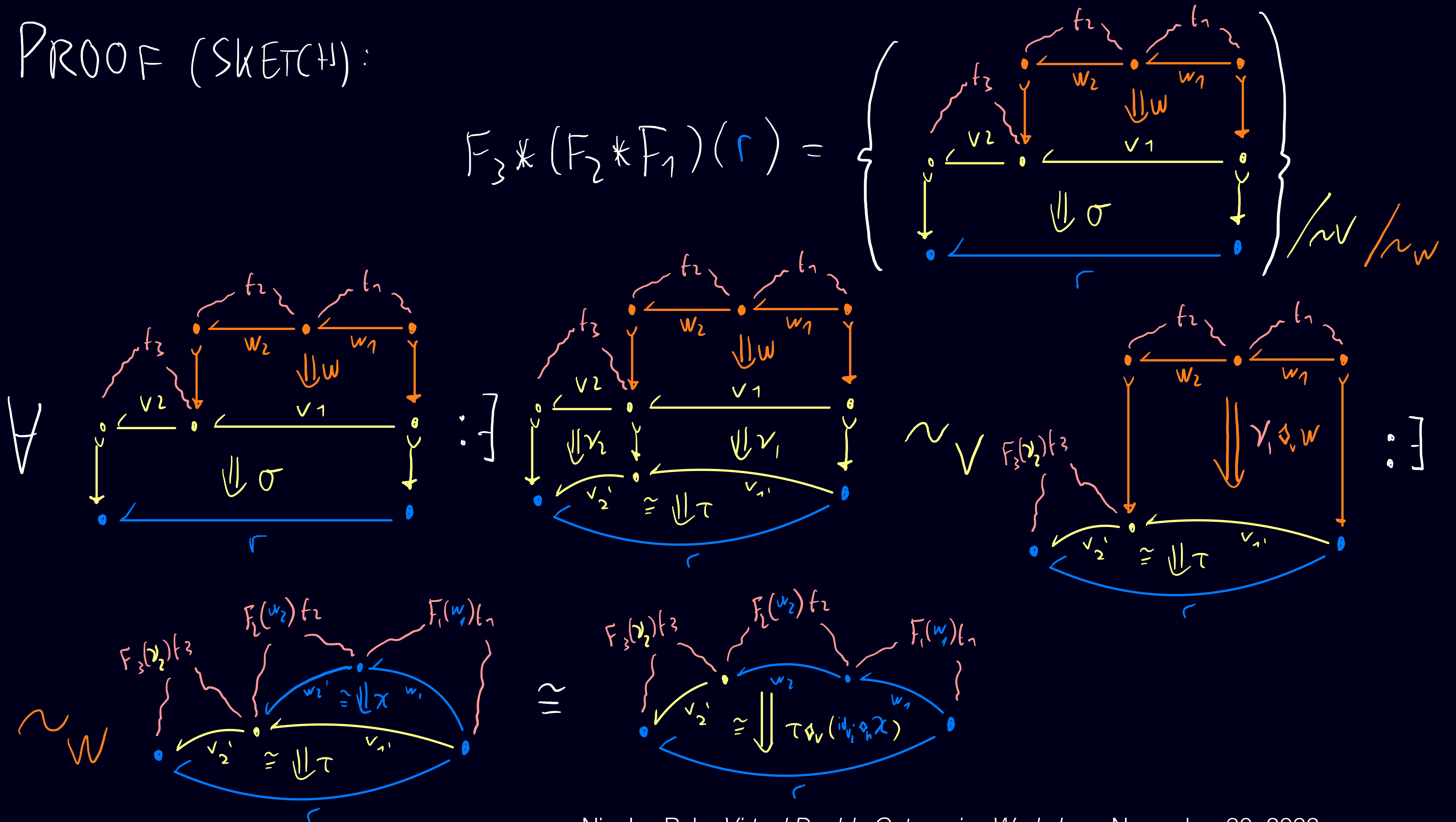
$\sim_W$



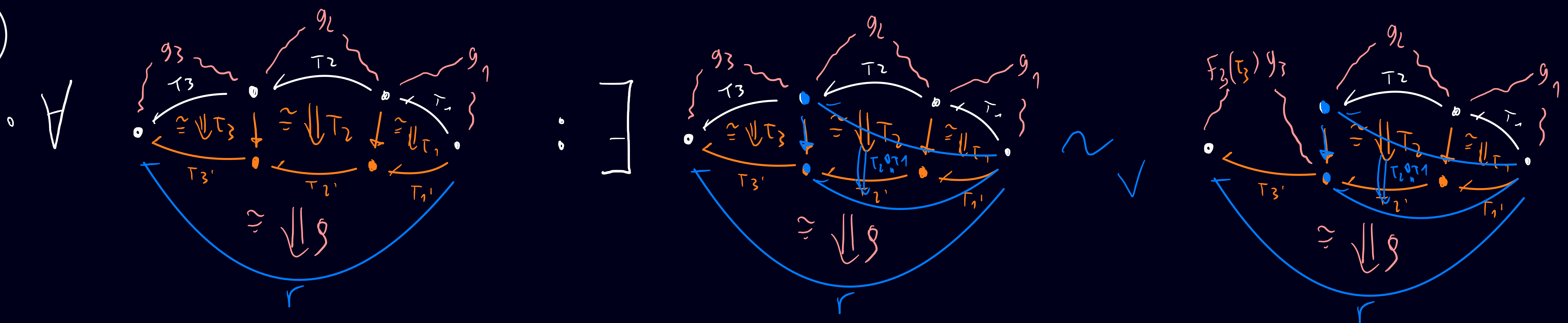
18 KEY RESULT: WEAK ASSOCIATIVITY OF $*$

$$\forall F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \underline{\mathbf{Set}},\ r \in \mathbb{D}_1 :\ F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

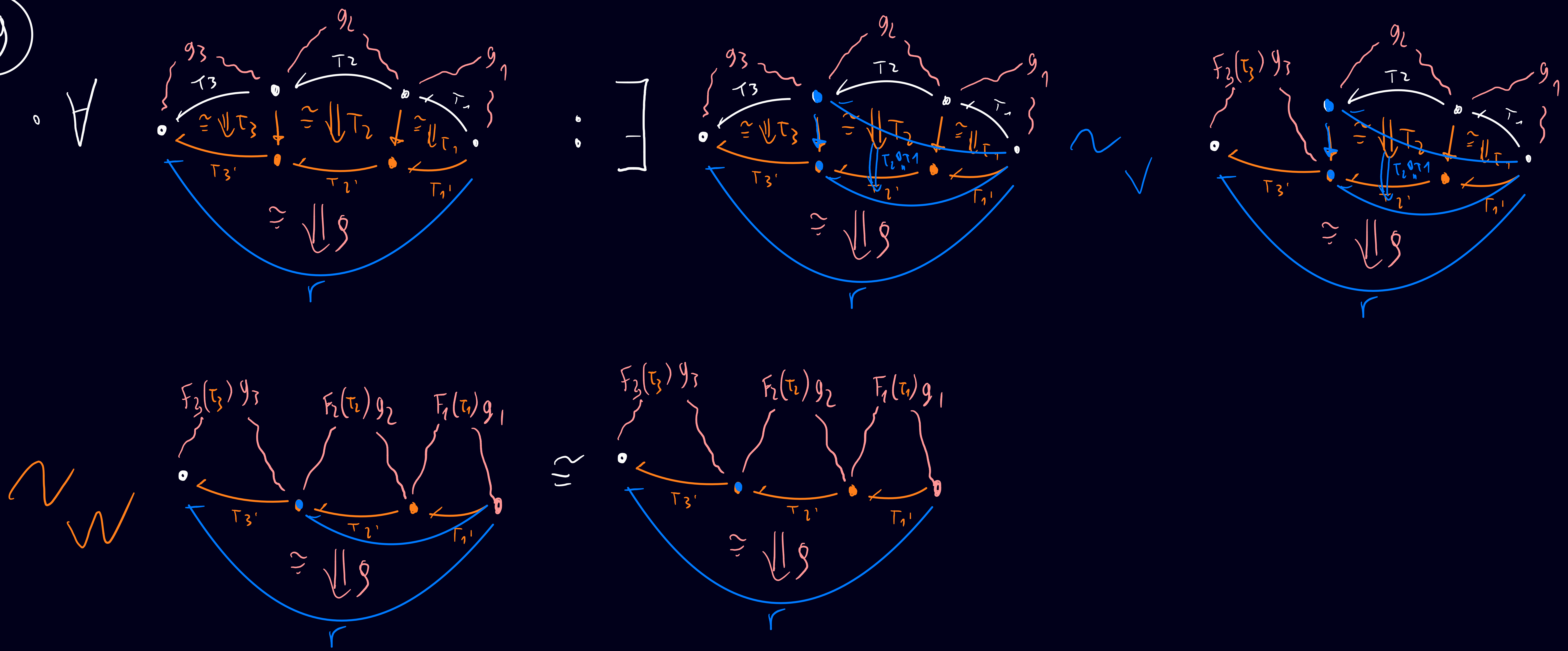
PROOF (SKETCH):



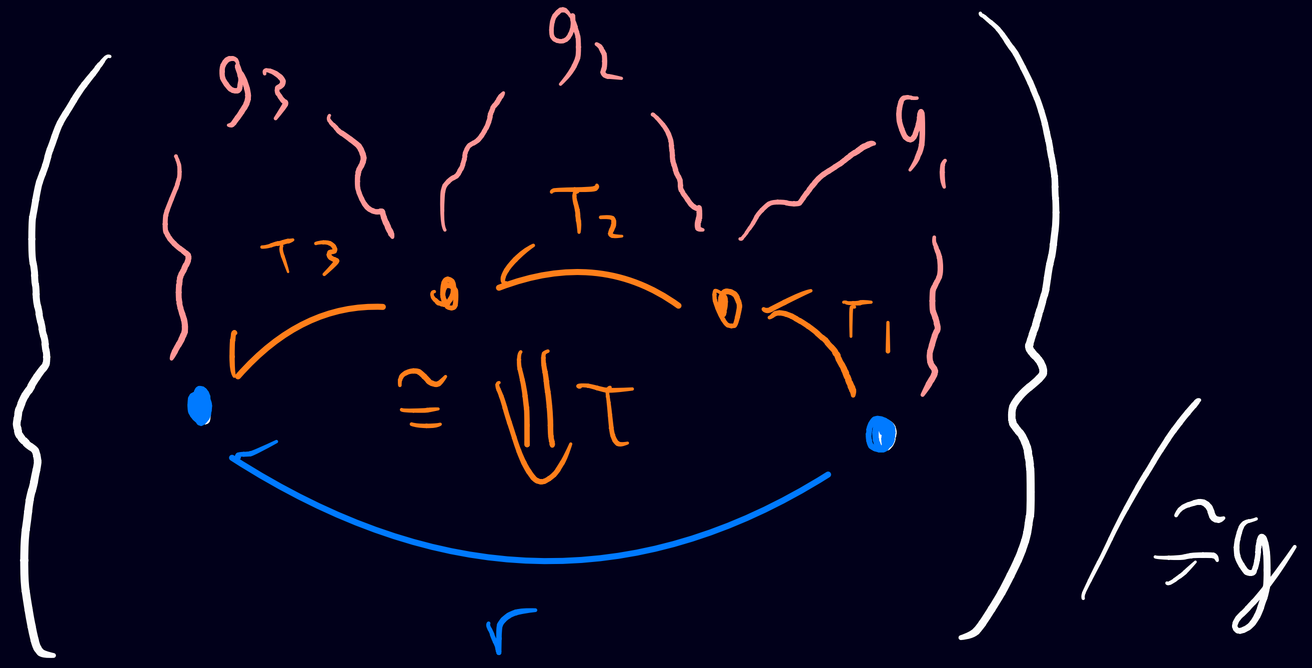
19



19



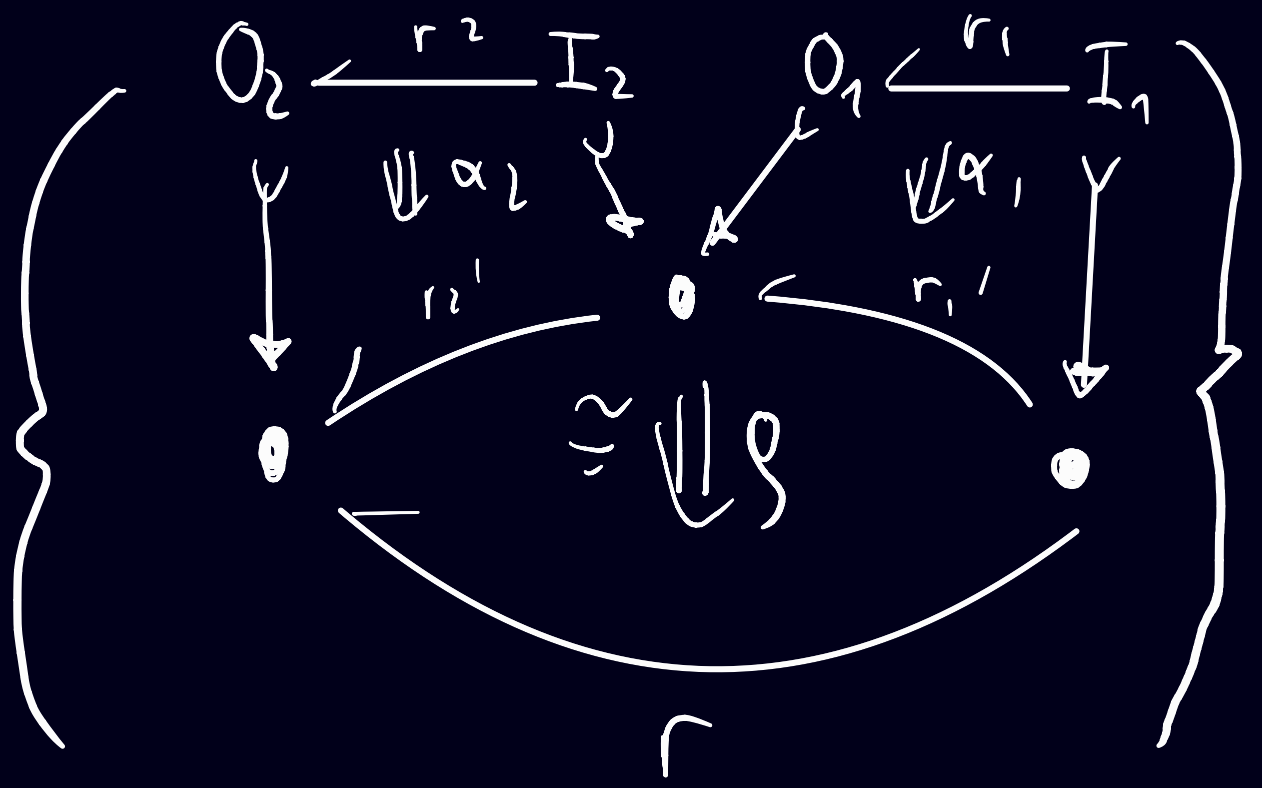
$\hookrightarrow\ F_3 * (F_2 * F_1)(r) \cong$



$= (F_3 * F_2 * F_3)(r)$ □

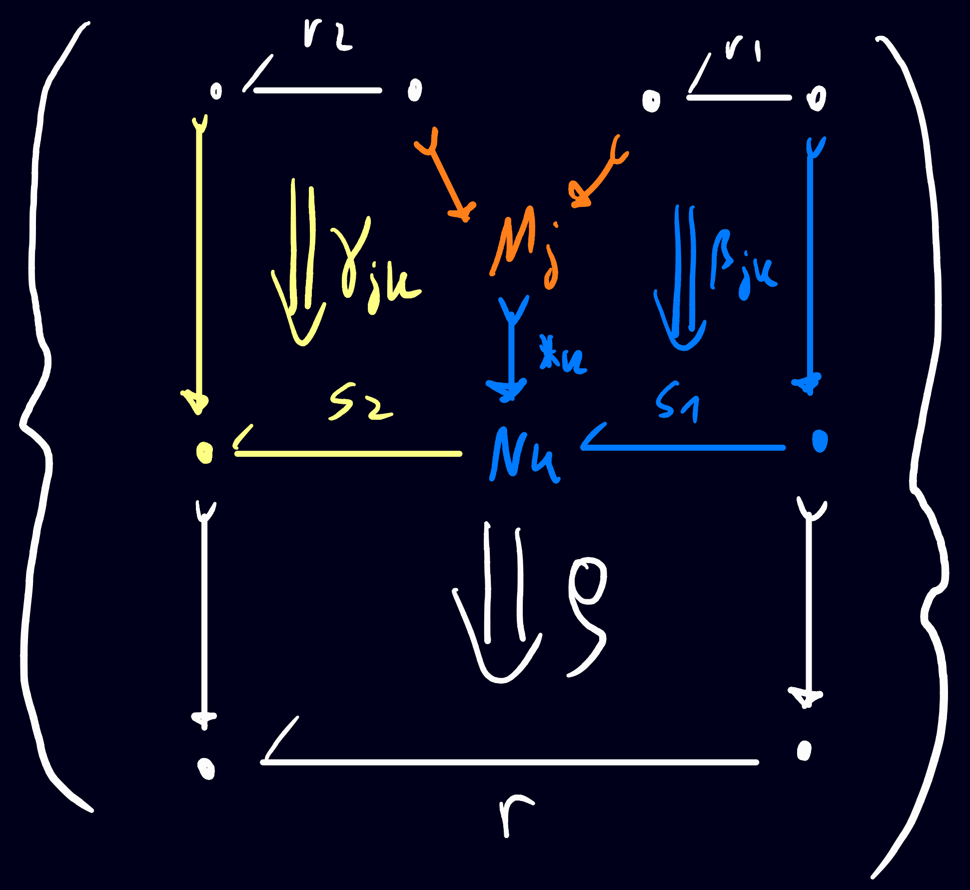
⑳ FINAL INGREDIENT: CATEGORIFICATION OF RULE ALGEBRA

CLAIM: $(\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1})(r) \cong$



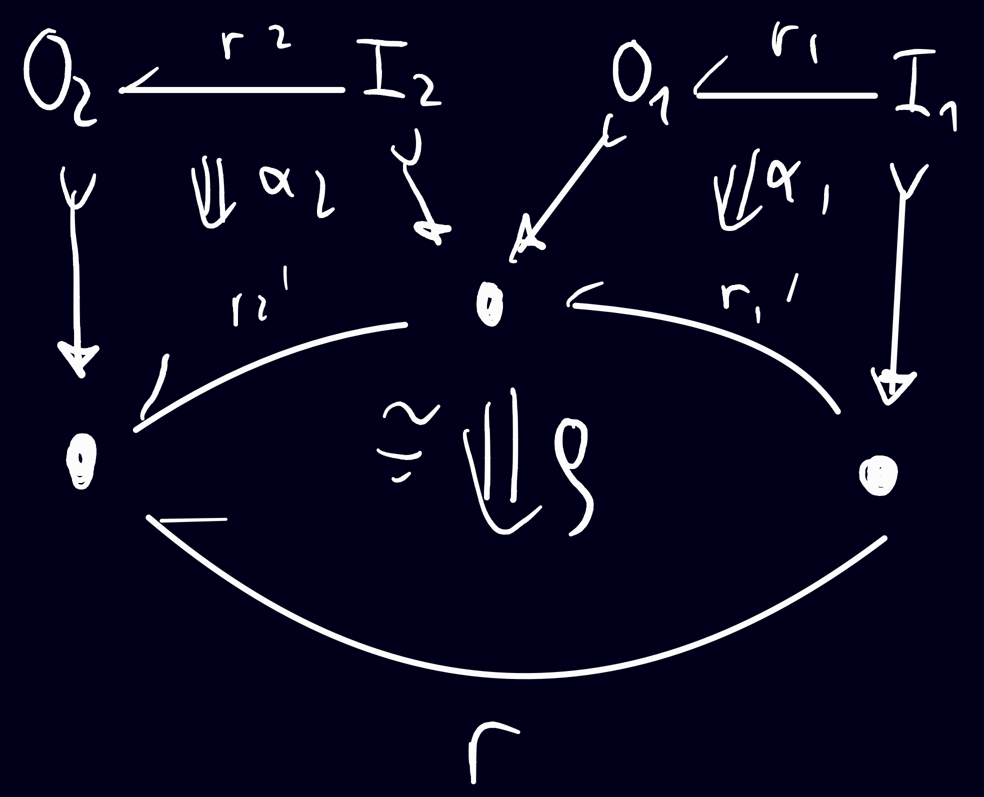
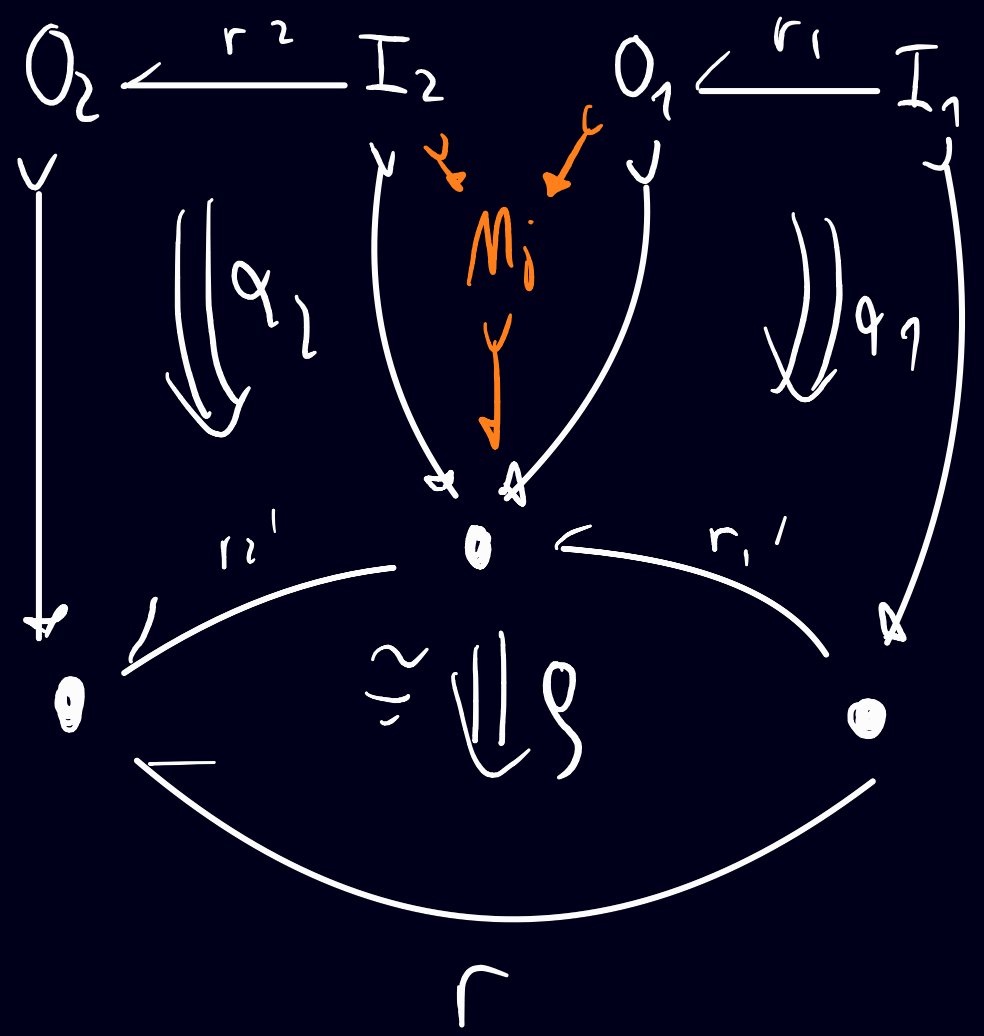
$/\cong_y$

$$\cong \coprod_{\substack{(O_2 \rightarrowtail M_j \leftarrowtail I_1)\\ \in ms(O_2,I_1)}} \;\coprod_{\beta_{jk} \in T^*(O_1 \rightarrowtail M_j)} \;\coprod_{\gamma_{jk} \in S^*(I_2 \rightarrowtail M_j \xrightarrow{*_n} N_n)}$$

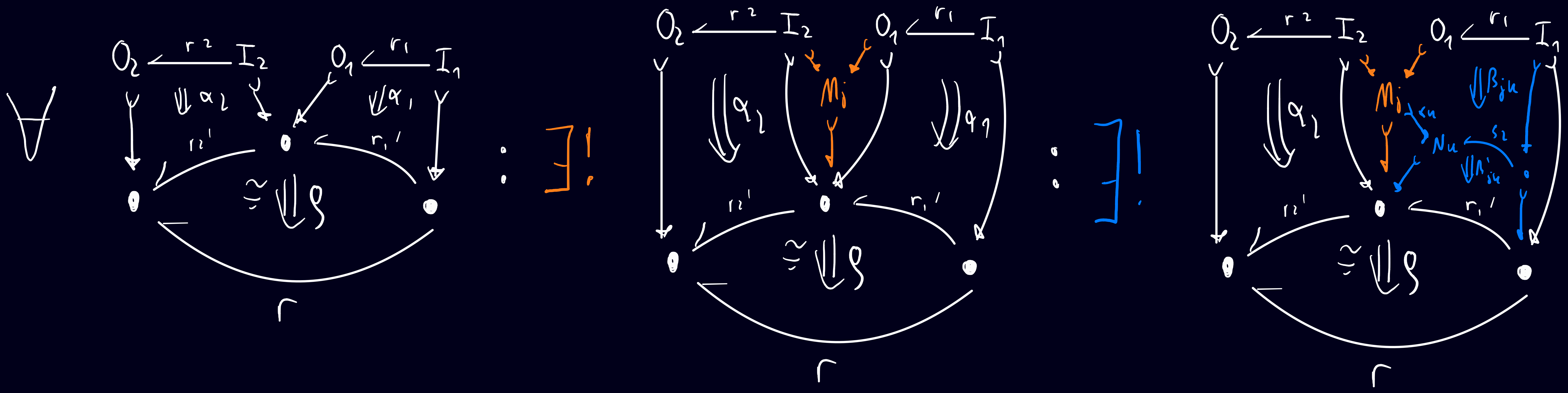


$$=: \hat{\Delta}_{r_2 \circ r_1}(r)$$

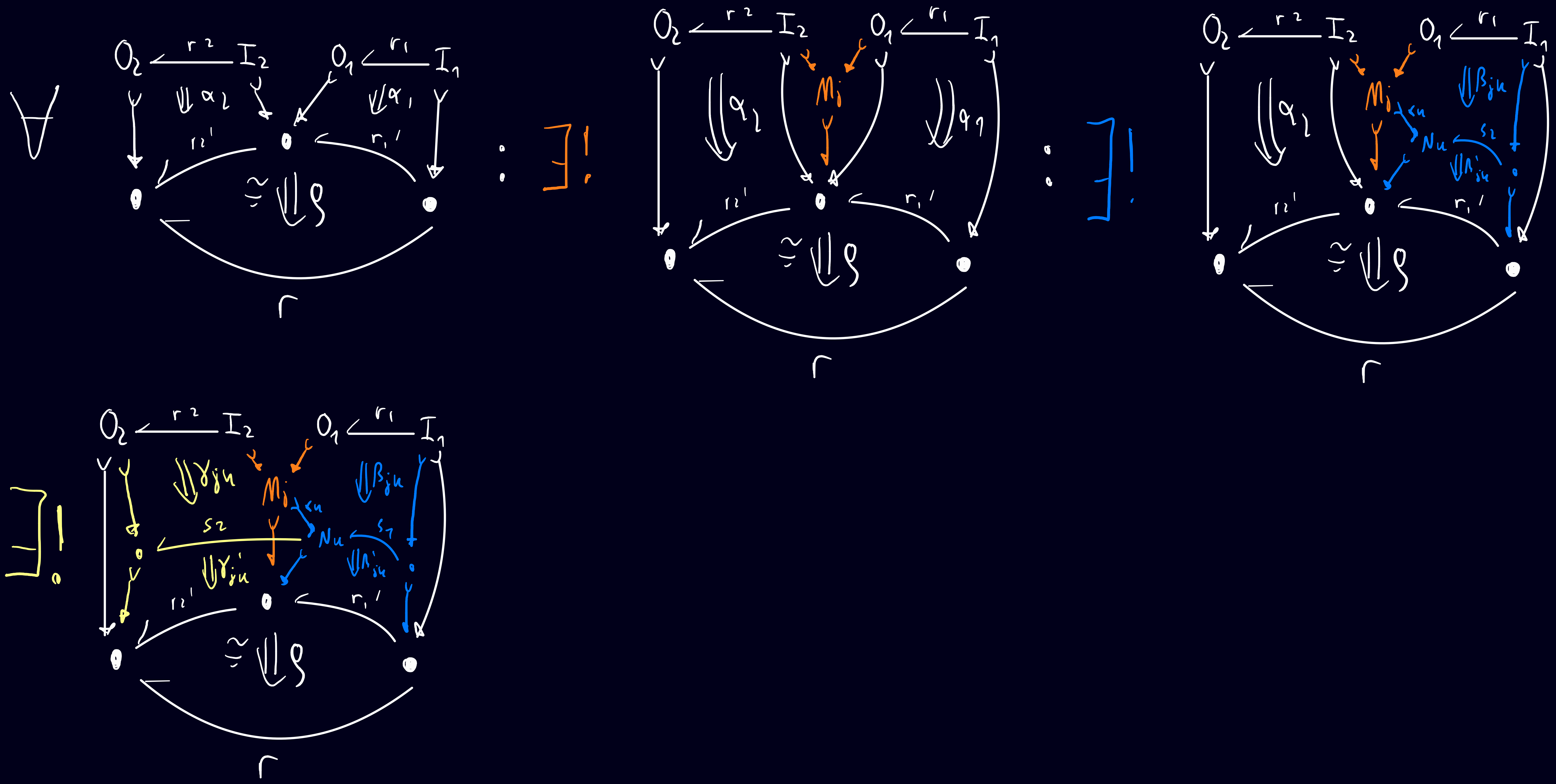
21 PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:

$\forall$  : $\exists!$ 

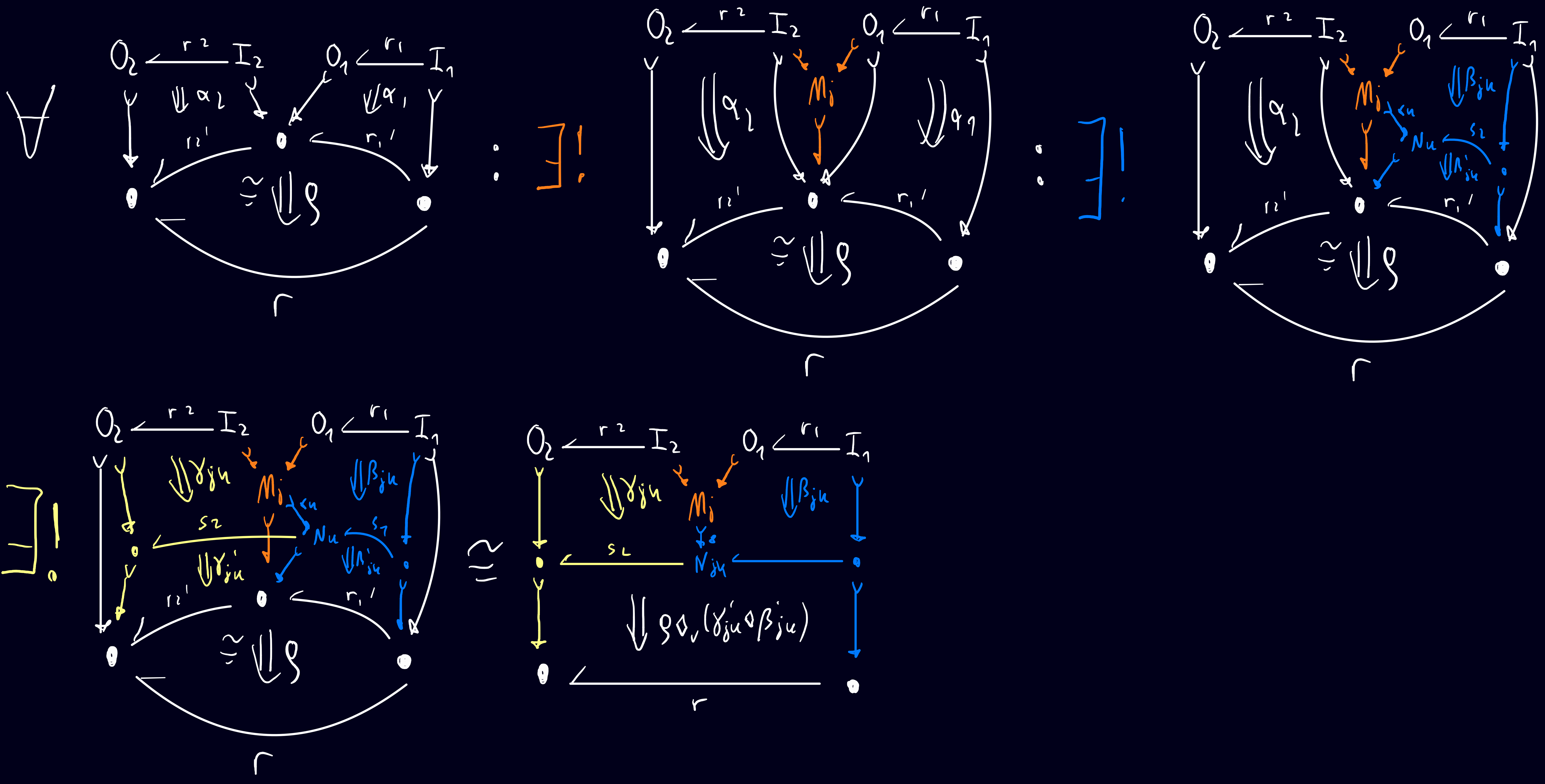
21 PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:



㉑ PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:



(21) PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:

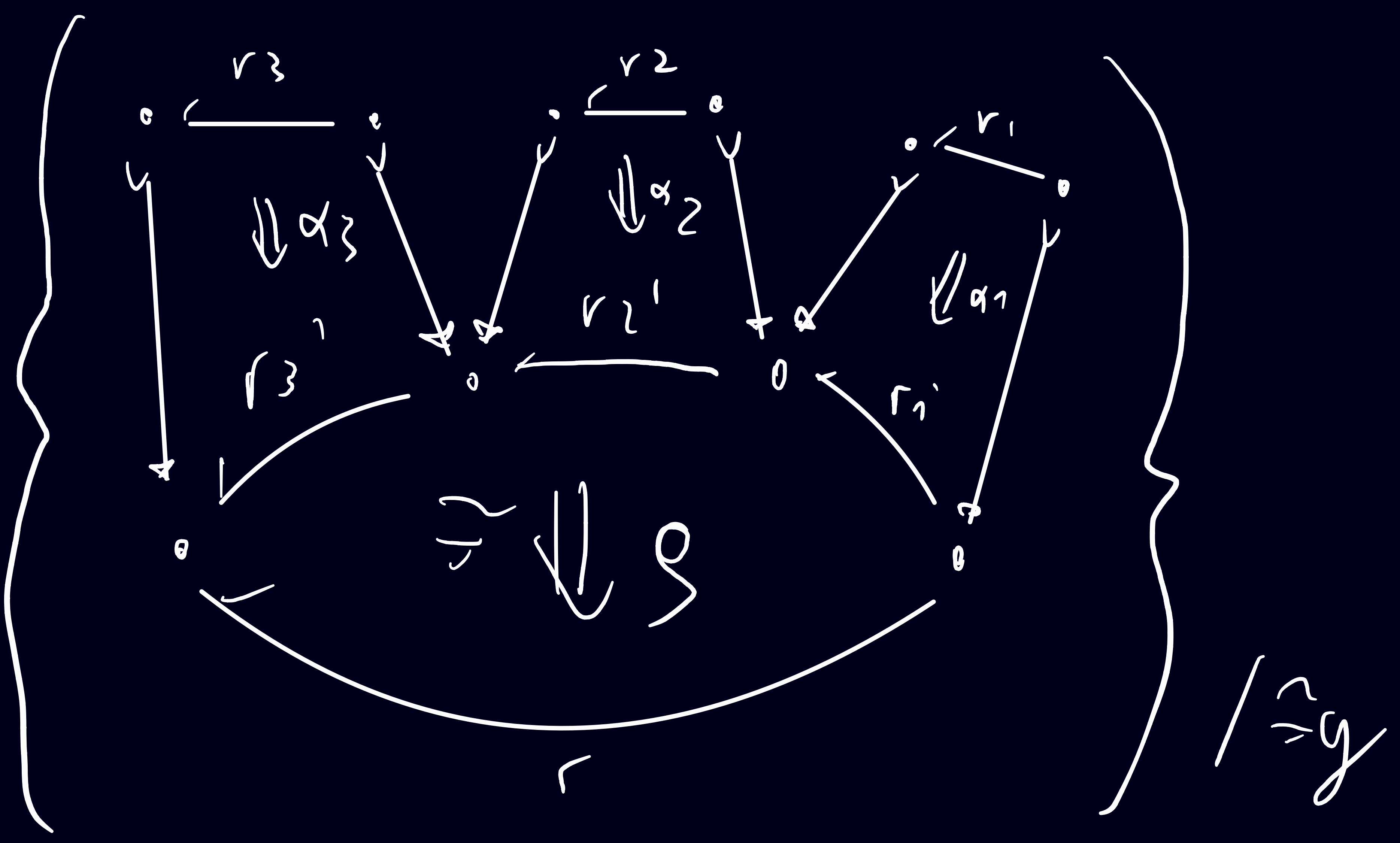


(21) PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:

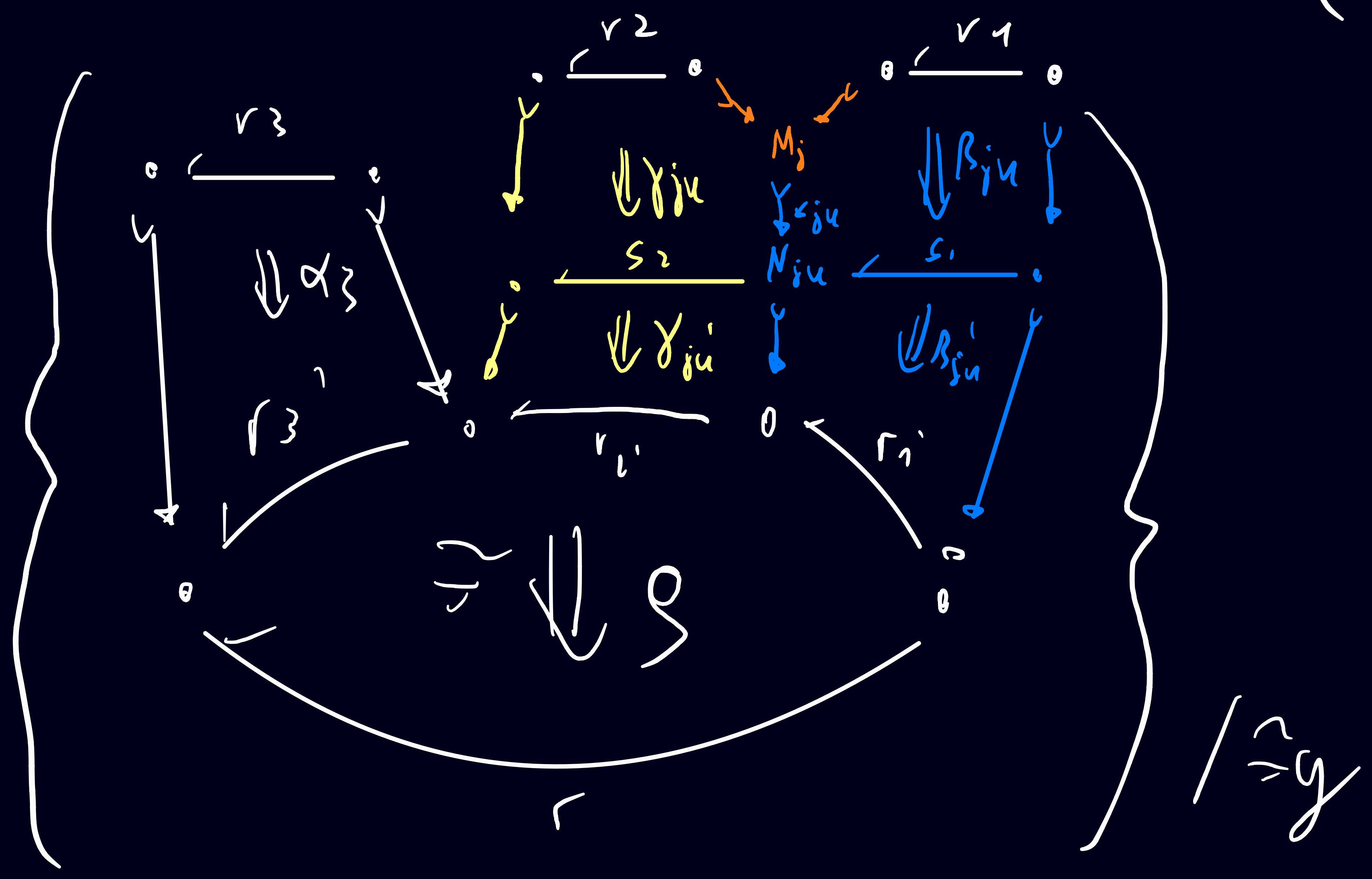
CLAIM: $\hat{\Delta}_{r_3 \odot (r_2 \odot r_1)}(r) \cong \hat{\Delta}_{(r_3 \odot r_2) \odot r_1}$

PROOF (SKETCH):

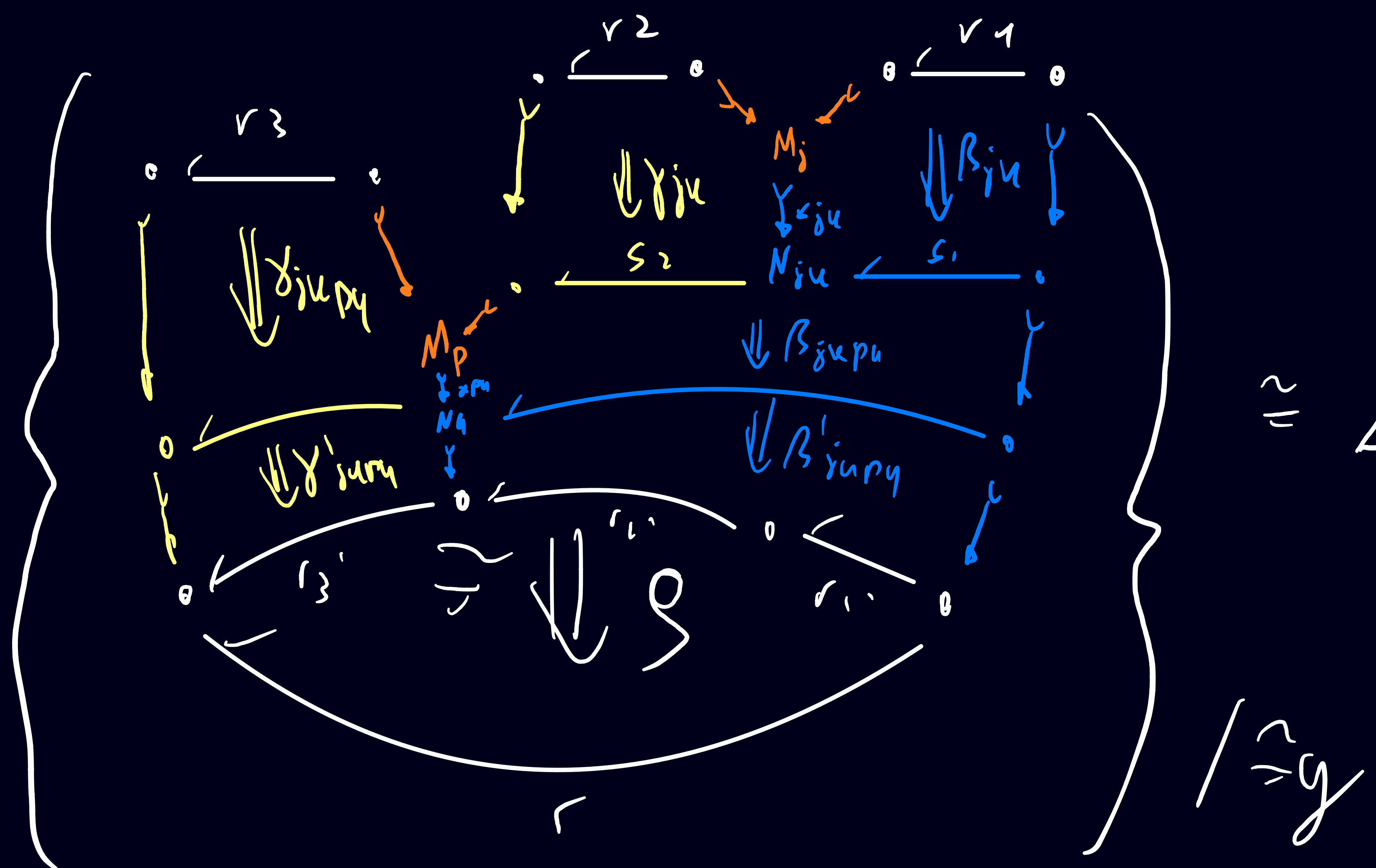
$(\hat{\Delta}_{r_3} * \hat{\Delta}_{r_2} * \hat{\Delta}_{r_1})(r) \cong$



$\cong$



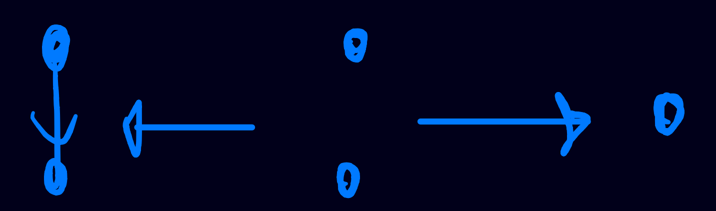
$\cong$

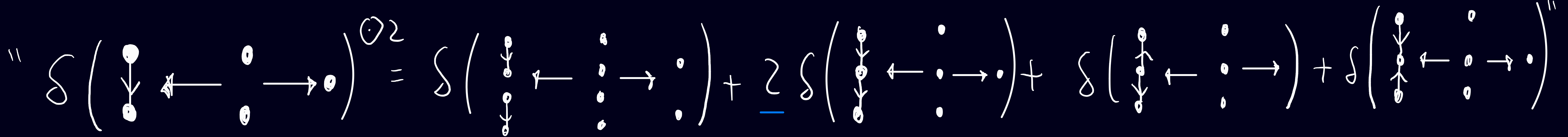


$\cong \hat{\Delta}_{r_3 \odot (r_2 \odot r_1)}(r)$

□

(23) EXAMPLE: SELF-COMPOSITIONS OF THE REWRITING RULE



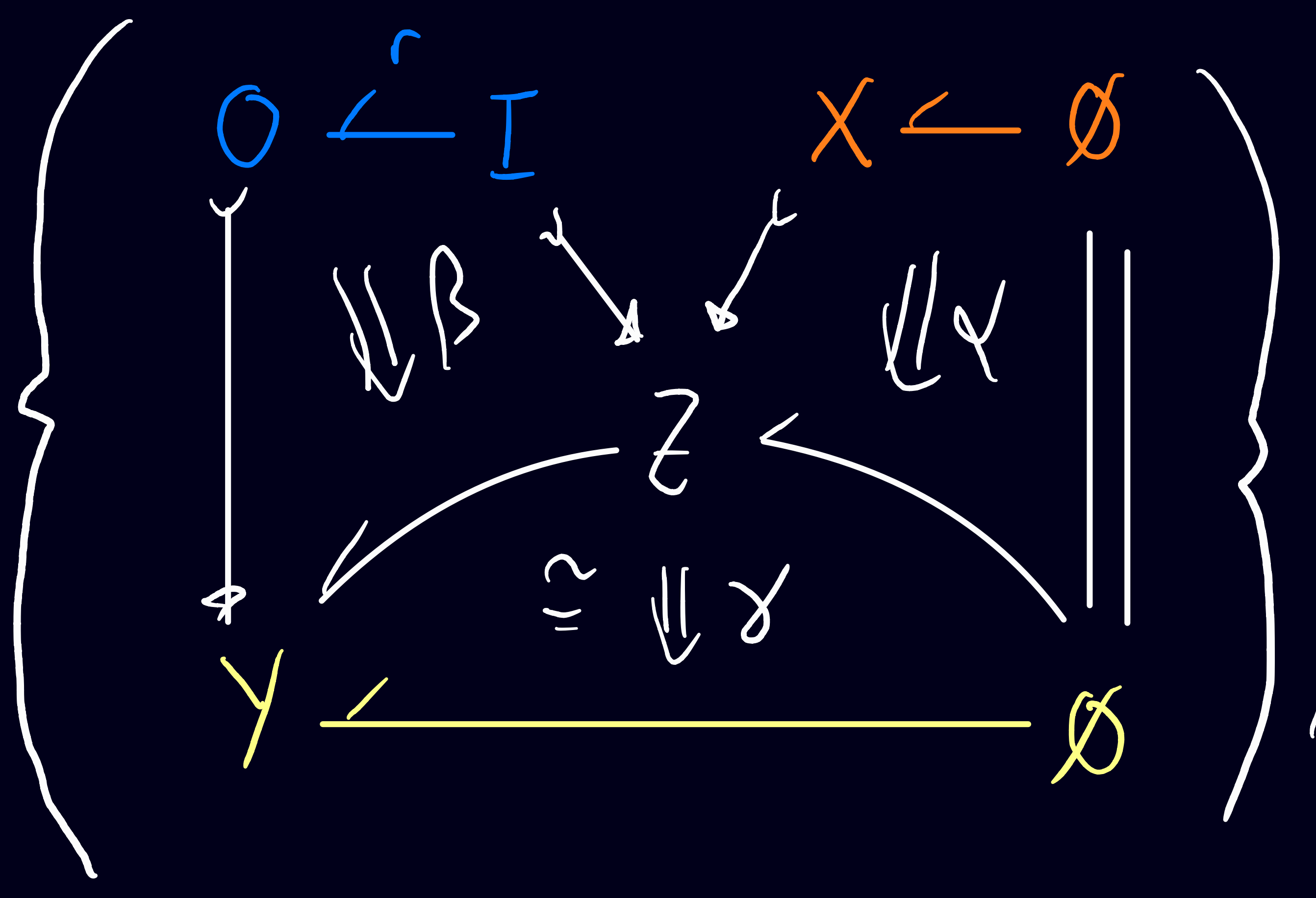


5 CONTRIBUTIONS TO $\hat{A}_{r_2 \circ r_1}$:

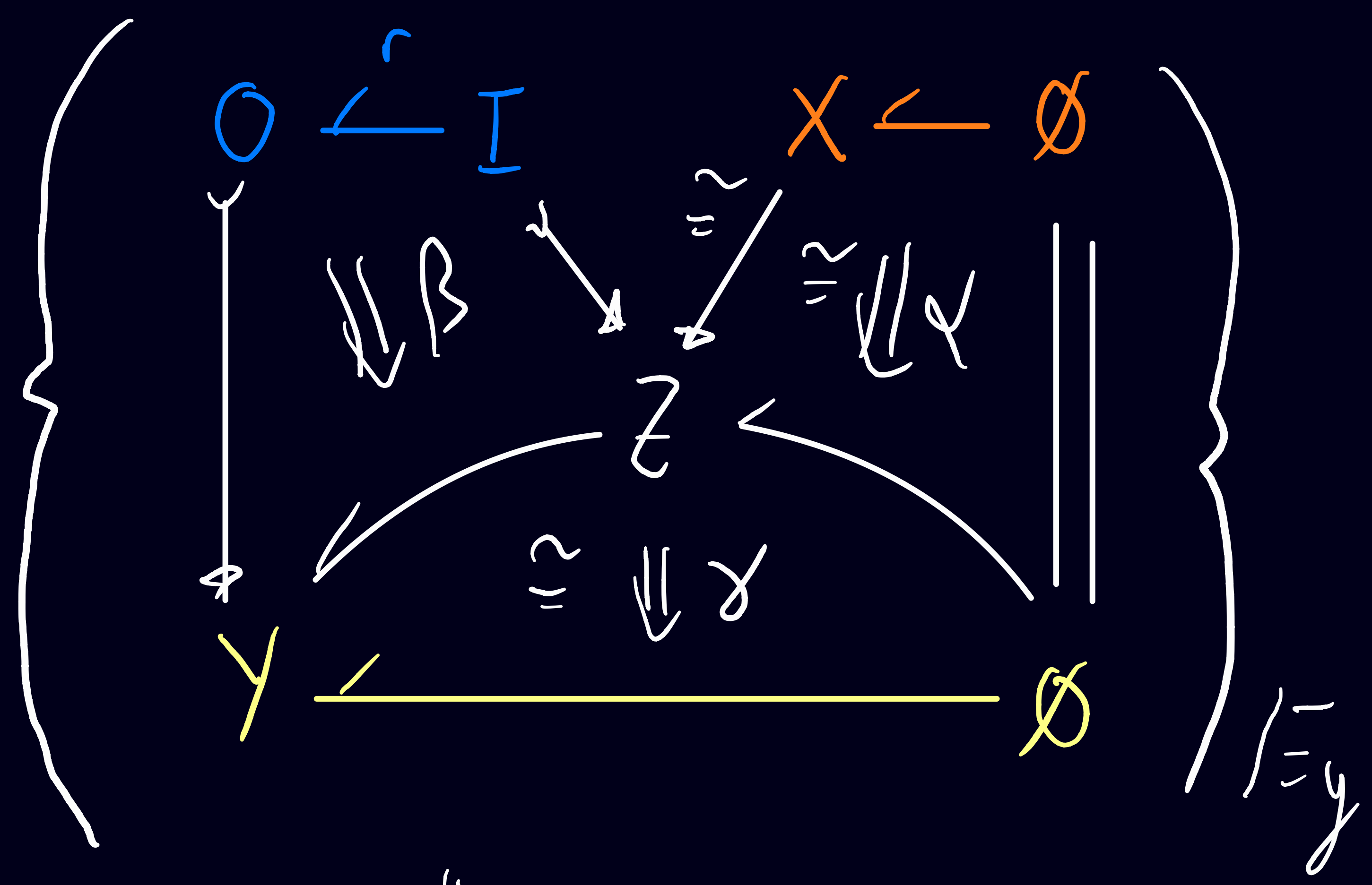
㉔ Counting Rewriting Sequences

- "$\rho(\mathbb{S}(r))|X\rangle = \rho(\mathbb{S}(r))\,\rho(\mathbb{S}(X \leftarrow \emptyset))|\emptyset\rangle = \sum_{\alpha} \rho(\mathbb{S}(r_\alpha(X) \leftarrow \emptyset))|\emptyset\rangle$"

$(\hat{\Delta}_r * \hat{\Delta}_{X\leftarrow\emptyset})(Y \leftarrow \emptyset) \cong$

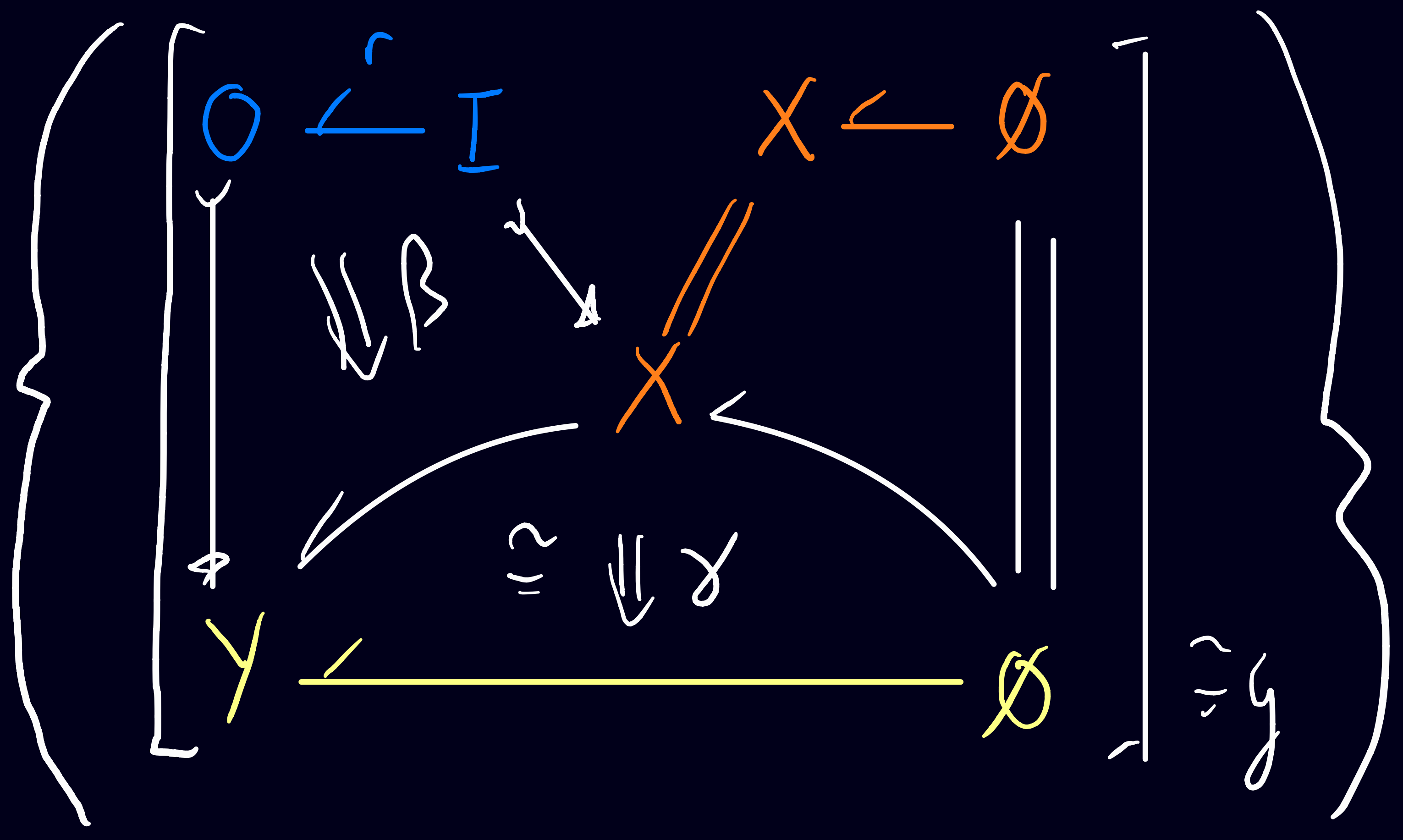


$\cong$

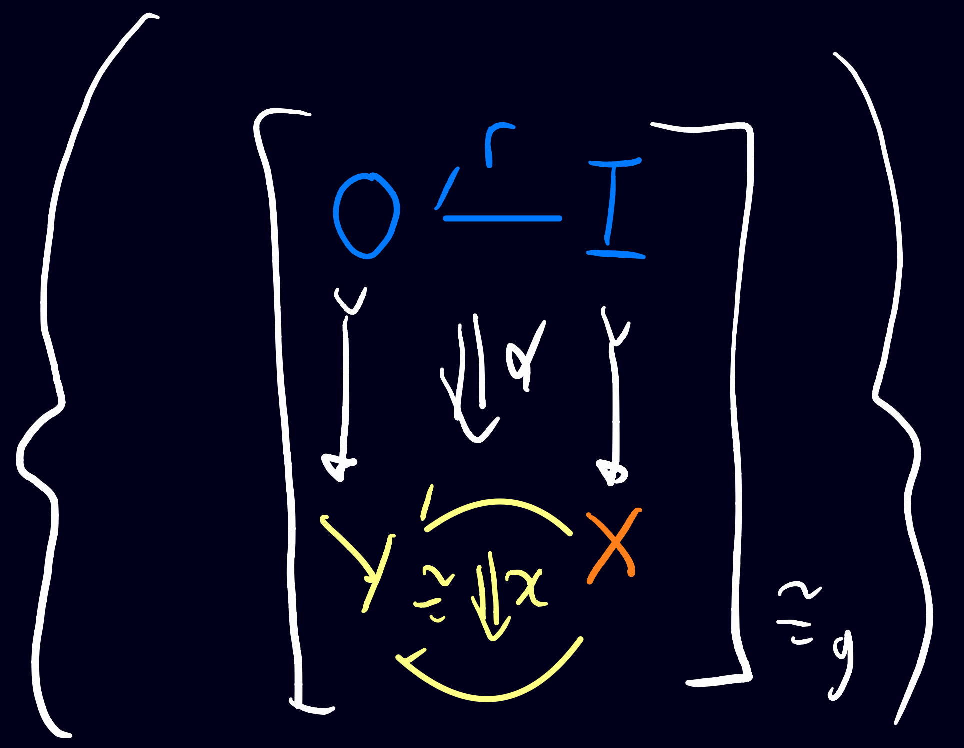


$\mathbb{S}$ is a MoPF ⇒ "lifts" isos!

$\cong$



$\cong$



✓

(25) OUTLOOK

- "# OF WAYS TO REWRITE X VIA APPLYING RULE r":

$$\int^{Y\in\mathbb{D}_0} \left(\hat{\Delta}_r * \hat{\Delta}_{X\leftarrow\emptyset}\right)(Y\leftarrow\emptyset) \cong \coprod_{\alpha\in\mathbb{D}_1} \left\{ \begin{array}{ccc} O & \xleftarrow{r} & I \\ \downarrow & \Downarrow\alpha & \downarrow \\ Y & \xleftarrow[s]{} & X \end{array} \right\} \Big/ \sim_Y$$

WHERE: $\forall$ $\begin{array}{ccc} Y & \xleftarrow{s} & \\ \downarrow\cong & \Downarrow\gamma & X \\ Y' & \xleftarrow[s']{} & \end{array}$: $\begin{array}{ccc} O & \xleftarrow{r} & I \\ \downarrow & \Downarrow\alpha & \downarrow \\ Y & \xleftarrow[s]{} & X \end{array} \sim_Y \begin{array}{ccc} O & \xleftarrow{r} & I \\ \downarrow & \Downarrow\gamma\circ_v\alpha & \downarrow \\ Y' & \xleftarrow[s']{} & X \end{array}$

25 OUTLOOK

- "# OF WAYS TO REWRITE X VIA APPLYING RULE r":

$$\int^{Y\in\mathbb{D}_0} \left(\hat{\Delta}_r * \hat{\Delta}_{X\leftarrow\emptyset}\right)(Y\leftarrow\emptyset) \cong \coprod_{\alpha\in\mathbb{D}_1} \left\{ \begin{array}{ccc} O & \xleftarrow{r} & I \\ \downarrow & \Downarrow\alpha & \downarrow \\ Y & \xleftarrow[s]{} & X \end{array} \right\} \Big/ \sim_Y$$

WHERE: $\forall$ $\begin{array}{ccc} Y & \xleftarrow{s} & \\ \downarrow \cong & \Downarrow\gamma & X \\ Y' & \xleftarrow[s']{} & \end{array}$: $\begin{array}{ccc} O & \xleftarrow{r} & I \\ \downarrow & \Downarrow\alpha & \downarrow \\ Y & \xleftarrow[s]{} & X \end{array} \sim_Y \begin{array}{ccc} O & \xleftarrow{r} & I \\ \downarrow & \Downarrow\gamma\circ_v\alpha & \downarrow \\ Y' & \xleftarrow[s']{} & X \end{array}$

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25 OUTLOOK

- "# OF WAYS TO REWRITE X VIA APPLYING RULE r":

$$\int^{Y\in\mathbb{D}_0}\left(\hat{\Delta}_r * \hat{\Delta}_{X\leftarrow\emptyset}\right)(Y\leftarrow\emptyset) \cong \coprod_{\alpha\in\mathbb{D}_1}\left\{\begin{array}{ccc} O & \xleftarrow{r} & I \\ \downarrow & \Downarrow\alpha & \downarrow \\ Y & \xleftarrow[s]{} & X \end{array}\right\}\Big/\sim_Y$$

WHERE: $\forall \begin{array}{ccc} Y & \xleftarrow{s} & \\ \downarrow\cong & \Downarrow\gamma & X \\ Y' & \xleftarrow[s']{} & \end{array}$: $\begin{array}{ccc} O & \xleftarrow{r} & I \\ \downarrow & \Downarrow\alpha & \downarrow \\ Y & \xleftarrow[s]{} & X \end{array} \sim_Y \begin{array}{ccc} O & \xleftarrow{r} & I \\ \downarrow & \Downarrow\gamma * \alpha & \downarrow \\ Y' & \xleftarrow[s']{} & X \end{array}$

THANK YOU!

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