Double Categories of Relations

Virtual Double Categories Workshop

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Introductory comments/notation:

- Funny arrows $m \colon A \nrightarrow B$ I call "proarrows."
- Display cells horizontally.
- External composition and identity are $m \otimes n$ and y_A.
- Proarrows and cells have external sources and targets; cells have internal proarrow domains and codomains.
- External composition is diagrammatic.
- Monoidal structure on a bicategory is $\boxtimes \colon \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$.

Talk plan: try to give an answer to the question: which double categories are of the form $\mathbb{R}\mathrm{el}(\mathscr{E})$ for a regular category $\mathscr{E}$.

A **relation** in a category is a monomorphism $R \to A \times B$.

$\mathscr{E}$ = regular category

1. $\mathscr{C} \rightsquigarrow \mathrm{Rel}(\mathscr{E})$ ordinary category/locally discrete 2-category
2. $\mathscr{C} \rightsquigarrow \mathfrak{Rel}(\mathscr{E})$ a bicategory
3. $\mathscr{C} \rightsquigarrow \mathbb{R}\mathrm{el}(\mathscr{E})$ a double category.

Bicategorical axiomatization of relations [CW87].

- A **cartesian bicategory** is a bicategory $\mathfrak{B}$ equipped with a tensor pseudo-functor $\boxtimes \colon \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ and identity I where
  - every object is equipped with a commutative comonoid structure;
  - every morphism is a lax comonoid homomorphism;
  - comultiplication and counit have right adjoints;
  - uniqueness condition.
- A **bicategory of relations** is a cartesian bicategory where every object is *discrete*. Every *functionally complete* bicategory of relations is equivalent to one $\mathfrak{Rel}(\mathscr{C})$.
- Our question: which double categories $\mathbb{D}$ are equivalent to those of the form $\mathbb{R}\mathrm{el}(\mathscr{C})$?

- A double category $\mathbb{D}$ is an **equipment** [Shu08] if every f has a **companion** and a **conjoint**: proarrows $f_!$ and f^* and cells

$$
\begin{array}{ccc} \cdot & \xrightarrow[\quad]{y} & \cdot \\ {\scriptstyle 1}\downarrow & \Downarrow & \downarrow{\scriptstyle f} \\ \cdot & \xrightarrow[f_!]{} & \cdot \end{array}
\qquad
\begin{array}{ccc} \cdot & \xrightarrow{f_!} & \cdot \\ {\scriptstyle f}\downarrow & \Downarrow & \downarrow{\scriptstyle 1} \\ \cdot & \xrightarrow[y]{} & \cdot \end{array}
\qquad
\begin{array}{ccc} \cdot & \xrightarrow{y} & \cdot \\ {\scriptstyle f}\downarrow & \Downarrow & \downarrow{\scriptstyle 1} \\ \cdot & \xrightarrow[f^*]{} & \cdot \end{array}
\qquad
\begin{array}{ccc} \cdot & \xrightarrow{f^*} & \cdot \\ {\scriptstyle 1}\downarrow & \Downarrow & \downarrow{\scriptstyle f} \\ \cdot & \xrightarrow[y]{} & \cdot \end{array}
$$

  satisfying some equations [GP04].
- In any equipment $\mathbb{D}$ can form so-called "restrictions" and "extensions" of proarrows along ordinary arrows. In relations, restriction = pullback, extension = image.
- [Ale18] A double category $\mathbb{D}$ is **cartesian** if $\Delta\colon \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ and $\mathbb{D} \to 1$ have right adjoints in Dbl. Each category $\mathbb{D}(A, B)$ has products, written '$\wedge$'.

Definition (Cf. [Sch15])

The **kernel** of a morphism $f\colon A \to B$ is the restriction ρ of the unit on B along f. Dually, the **cokernel** of f is the extension cell ξ

$$
\begin{array}{ccc}
A & \xrightarrow{f_! \otimes f^*} & A \\
{\scriptstyle f}\downarrow & \rho & \downarrow{\scriptstyle f} \\
B & \xrightarrow[y_B]{} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{y_A} & A \\
{\scriptstyle f}\downarrow & \xi & \downarrow{\scriptstyle f} \\
B & \xrightarrow[f^* \otimes f_!]{} & B
\end{array}
$$

A morphism $e\colon A \to E$ in an equipment is a **cover** if the canonical globular cell is an iso $e^* \otimes e_! \cong y_E$. Dually, a morphims $m\colon E \to B$ is an **inclusion** if the canonical globular cell is an iso $m_! \otimes m^* \cong y_E$.

Let's look more closely at relations on a regular category $\mathscr{E}$.

- The double category $\mathbb{R}\mathsf{el}(\mathscr{E})$ is a cartesian equipment.
- $\mathbb{R}\mathsf{el}(\mathscr{E})$ is "unit-pure" i.e. $y \colon \mathbb{D}_0 \to \mathbb{D}_1$ is fully faithful.
- $\mathbb{R}\mathsf{el}(\mathscr{E})$ has tabulators. Tabulators are basic for bicategories of relations.
- Goal: characterize cartesian equipments of the form $\mathbb{R}\mathsf{el}(\mathscr{E})$ in terms of extra conditions on tabulators.

Definition
A **bicategory of relations** is a locally posetal cartesian bicategory where every object is *discrete*: for each object X, the comultiplication $\Delta\colon X \to X \boxtimes X$ satsifies the *Frobenius identity*:

$$\Delta\Delta^* = (\Delta^* \boxtimes 1)(1 \boxtimes \Delta).$$

Discrete in what way? In bicategory of semilattices [JT84] discrete objects are precisely the discrete spaces [CW87, Remark 2.9(iv)]

Definition
A bicategory of relations is **functionally complete** if every arrow $X \to I$ has a *tabulation*: an arrow $TX \to X$ satisfying equations.

Theorem (Theorem 3.5 [CW87])
Given a functionally complete bicategory of relations $\mathfrak{B}$, the category of maps[1] *in $\mathfrak{B}$ is regular and $\mathfrak{B}$ is equivalent to relations on the category of maps.*

So, refine our question again. If there is a concept of *double category of relations*, which double categories of relations are actually of the form $\mathbb{R}\mathrm{el}(\mathscr{E})$ for some regular category $\mathscr{E}$?

[1] An **map** is a morphism having a right adjoint in the bicategory.

Here's the key observation:

Lemma (Proposition 3.1 [Lam22b])
The horizontal bicategory of any locally posetal cartesian equipment is a cartesian bicategory.

Definition
A locally posetal cartesian equipment $\mathbb{D}$ is a '**double category of relations**' if the discreteness axiom

$$\Delta \otimes \Delta^* = (\Delta^* \times 1) \otimes (1 \times \Delta).$$

holds in the horizontal bicategory of $\mathbb{D}$.

So, which double categories of relations are of the form $\mathbb{R}\mathrm{el}(\mathscr{E})$ for $\mathscr{E}$ regular?

For inspiration, review the case of spans.

Theorem (§5 of [Nie12])
Let $\mathbb{D}$ denote a double category with pullbacks. The following are equivalent:

1. *The identity functor* $1\colon \mathbb{D}_0 \to \mathbb{D}_0$ *extends to an oplax/lax adjunction*[2] $F\colon \mathbb{S}\mathsf{pan}(\mathbb{D}_0) \rightleftarrows \mathbb{D}\colon G$.
2. $\mathbb{D}$ *is an equipment with tabulators.*

F takes the proarrow cokernel of a span; and G takes the legs of the tabulator of a proarrow.

[1] An **oplax/lax adjunction** is a conjoint pair in the strict double category of double categories with oplax and lax functors. Basically it's an adjunction between double categories where F is oplax and G is lax.

The oplax/lax adjunction is a strong (both functors are pseudo!) equivalence under some further conditions:

Theorem (§5 of [Ale18])
For a double category $\mathbb{D}$ the following are equivalent:

1. $\mathbb{D}$ *is equivalent to* $\mathbb{S}\mathrm{pan}(\mathscr{C})$ *for finitely-complete* $\mathscr{C}$*.*
2. $\mathbb{D}$ *is a unit-pure cartesian equipment admitting certain Eilenberg-Moore objects.*
3. $\mathbb{D}_0$ *has pullbacks satisfying a Beck-Chevalley condition and the canonical functor* $\mathbb{S}\mathrm{pan}(\mathbb{D}_0) \to \mathbb{D}$ *is an equivalence of double categories.*

The precise statement of these conditions isn't really important for now; they are completeness conditions.

Our goal however was to characterize in terms of tabulators.

What are they?

Definition
A double category $\mathbb{D}$ has **tabulators** if $y\colon \mathbb{D}_0 \to \mathbb{D}_1$ has a right adjoint $\top\colon \mathbb{D}_1 \to \mathbb{D}_0$ in Dbl. The **tabulator** of $m\colon A \nrightarrow B$ is the object $\top m$ together with a counit cell $\top m \Rightarrow m$.

Tabulators in $\mathbb{R}\mathrm{el}(\mathscr{E})$ satisfy:

1. $\langle l, r\rangle\colon \top m \to A \times B$ is monic;
2. and $m = l^* \otimes r_!$ holds (tabulators are strong).

Profunctors: the tabulator of a profunctor is basically the category of elements construction! These are strong but not monic.

- If a bicategory of relations is functionally complete (every arrow $X \to I$ has a tabulation), then ordinary arrows $m\colon X \to Y$ have tabulations too. These satisfy
  - $lr^\circ = m$ (strong condition)
  - uniqueness condition $\approx$ jointly monic
- More explicitly: an allegory[3]is **tabular** if every arrow R has a tabulator: a pair of arrows f and g such that
  1. $gf^\circ = R$ "tabulators are strong"
  2. $f^\circ f \wedge g^\circ g = 1$ "tabulators are monic."

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[1] **Allegories** are another approach to formalizing a calculus of relations [FS90]. In particular they have local products $\wedge$ and a duality involution $(-)^\circ$ behaving somewhat like the conjoint $(-)^*$.

Definition
A cartesian equipment is **functionally complete** if it has tabulators and they are strong and monic in the sense that

1. $m \cong l^* \otimes r_!$
2. $l_! \otimes l^* \wedge r_! \otimes r^* \cong y$

both hold for any proarrow $m \colon A \nrightarrow B$ and its tabulator $\langle l, r \rangle \colon Tm \to A \times B$.

At least the "monic" hypothesis gives us the following:

Theorem (Theorem 6.5 [Lam22b])
For a double category $\mathbb{D}$ *with* $\mathbb{D}_0$ *regular, the identity functor* $\mathbb{D}_0 \to \mathbb{D}_0$ *extends to a normalized oplax/lax adjunction* $F \colon \mathbb{R}\mathrm{el}(\mathbb{D}_0) \rightleftarrows \mathbb{D} \colon G$ *if, and only if,*

1. $\mathbb{D}$ *is a unit-pure equipment;*
2. *has monic tabulators;*
3. *the unit cell* y_e *is an extension for each cover* e.

Strictly speaking, the cartesian hypothesis isn't needed here.

When is G pseudo?

- "Relations tabulate" = every relation in $\mathbb{D}$ tabulates its cokernel.
- If relations tabulate, G is pseudo.
- Relations tabulate implies that $\mathbb{D}$ is unit-pure.
- This is a strong assumption we'll discuss more shortly. Certainly it's true of actual relations.

Theorem (Theorem 7.5 [Lam22b])
The identity functor $1 \colon \mathbb{D}_0 \to \mathbb{D}_0$ *extends to an adjoint equivalence of pseudo-functors*

$$F \colon \mathbb{R}\mathsf{el}(\mathbb{D}_0) \rightleftarrows \mathbb{D} \colon G$$

if, and only if,

1. $\mathbb{D}$ *is a functionally complete;*
2. *relations tabulate;*
3. y_e *is an extension cell for each cover* e.

NB: Roughly, “strong” assumption in “functionally complete” implies a Beck-Chevalley condition which implies that normal oplax F is pseudo.

Wrap up conditions 1. and 2. in a single one: existence of a *subobject comprehension scheme*

Functionally complete implies at least that

$$\top\colon \mathbb{D}(A, B) \to \mathrm{Sub}_{\mathbb{D}_0}(A \times B)$$

is well defined.

Lemma
The morphism $\top$ as above is an equivalence if, and only if, relations tabulate.

Definition
$\mathbb{D}$ admits a **subobject comprehension scheme** if $\top$ as above is a well-defined (strong) equivalence of categories.

Some consequences and reflections:

- If $\mathbb{D}$ admits a subobject comprehension scheme:
    1. $\mathbb{D}$ is functionally complete;
    2. relations tabulate;
    3. $\mathbb{D}$ is unit-pure;
- Why is this reasonable?
    1. Subobject classifier in a topos induces a bijection $\mathscr{E}(X,\Omega) \cong \mathrm{Sub}(X)$;
    2. Elements construction for set-valued functors has a genuine pseudo-inverse (fibers construction).
    3. Size issues are subtle!

Theorem (Theorem 8.3 [Lam22b])
For a double category $\mathbb{D}$ *with* $\mathbb{D}_0$ *regular, the identity functor* $\mathbb{D}_0 \to \mathbb{D}_0$ *extends to an adjoint equivalence of pseudo-functors* $F\colon \mathbb{R}\mathsf{el}(\mathbb{D}_0) \rightleftarrows \mathbb{D}\colon G$ *if, and only if,*

1. $\mathbb{D}$ *admits a subobject comprehension scheme;*
2. y_e *is an extension cell for each cover* e*.*

So, when is $\mathbb{D}_0$ regular? Are these conditions sufficient?

Lemma (Theorem 9.4 [Lam22b])
If $\mathbb{D}$ is a 'double category of relations' with a subobject comprehension scheme, then $\mathbb{D}_0$ is regular.

Discreteness enters here non-trivially in the form of the modular law; also to show that equipment covers are precisely covers in $\mathbb{D}_0$.

Putting everything together:

Theorem (Theorem 10.2 [Lam22b])
If $\mathbb{D}$ is a 'double category of relations' with a subobject comprehenseion scheme, then the identity functor on $\mathbb{D}_0$ extends to a strong adjoint equivalence of double categories $\mathbb{R}\mathsf{el}(\mathbb{D}_0) \simeq \mathbb{D}$.

Question: why?

- ‘bicategories of relations’ used for knowledge representation (KR) in [Pat17]. This is a bicategorical approach to the functional OLOGs of [KS12].
- improvement, but if bicategories $\approx$ categorified logics/theories, they don’t quite have a type theory, so equational reasoning about terms is hard but not impossible
- enter double categories: these supply the missing type theories enabling a cleaner approach to equational reasoning, computer implementation and data migration
- for more: [Lam22a] also some slides at https://michaeljlambert.github.io/main.pdf

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