# Recognising Retromorphisms Retrospectively

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Joint work with Bryce Clarke

Virtual Double Categories Workshop

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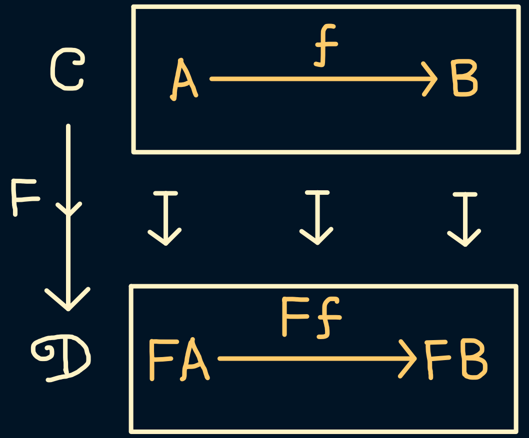
interested in mathematical structures
with two components

morphisms act on both components
in the same direction

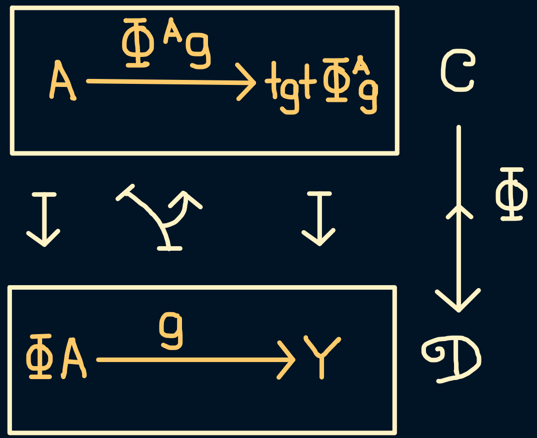
from Paré's retrocells

retromorphisms act on them in
opposite directions

# EXAMPLE: CATEGORIES



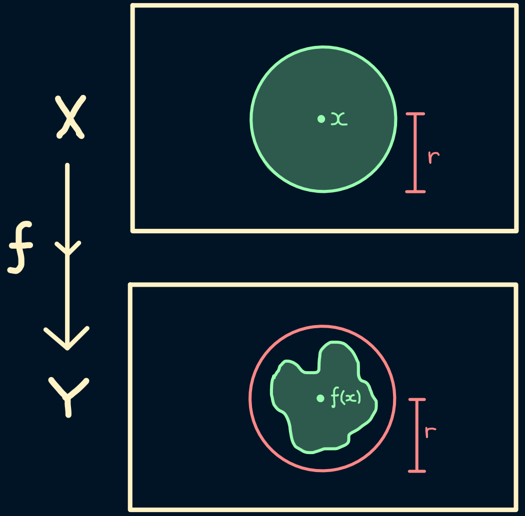

functor




retro
~~co~~functor

functor + retrofunctor = delta lens

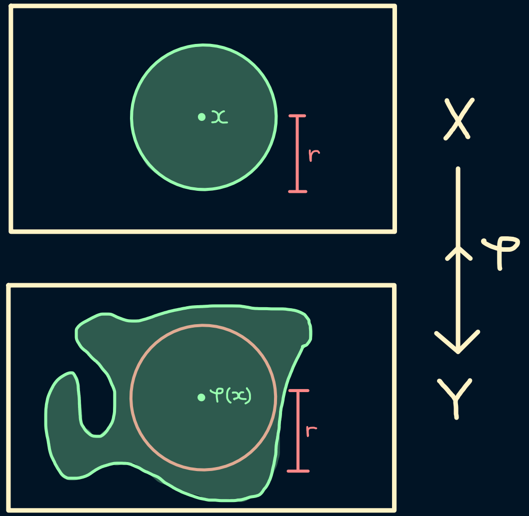
# Example: Metric spaces




$$d(x,a) \geq d(fx,fa)$$

non-expansive map

$$\inf_{a \in \varphi^{-1}\{y\}} d(x,a) \leq d(\varphi x, y)$$

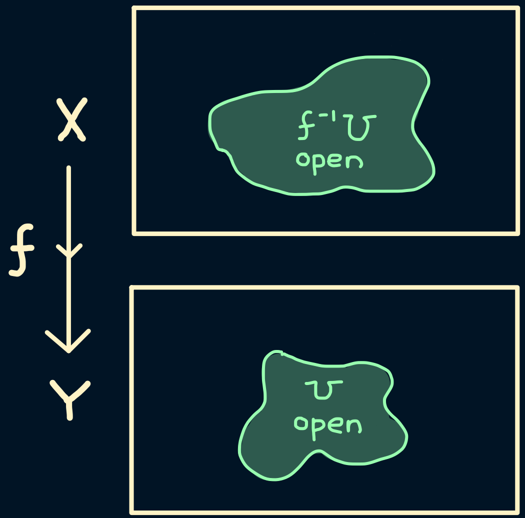



retro-non-expansive map

non-expansive map + retro-non-expansive map = weak submetry

# Example: Topological spaces

filter — filter convergence




$$\mathfrak{X} \longrightarrow x$$

$$\Downarrow$$

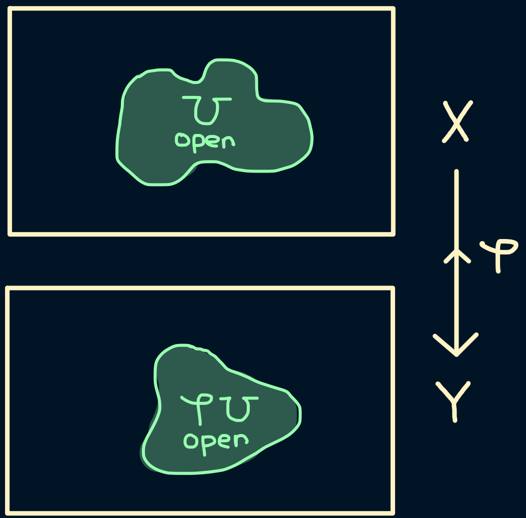
$$f[\mathfrak{X}] \to f(x)$$

continuous map

$$\exists \mathfrak{X}.\ \begin{matrix} \mathfrak{X} \longrightarrow x \\ \mathfrak{Y} = \varphi[\mathfrak{X}] \end{matrix}$$

$$\Uparrow$$

$$\mathfrak{Y} \to \varphi(x)$$




open map

# EXAMPLE SUMMARY



| | Object | Morphism | Retromorphism | Lens |
|---|---|---|---|---|
| enriched | category | functor | retrofunctor | delta lens |
| enriched | metric space | non-expanding map | retro-non-expanding map | weak submetry |
| | topological space | continuous map | open map | continuous open map |

# Double categories

distributive monoidal category

filter monad

| | notation | $\mathcal{V}$-Mat | $\mathbb{K}\mathrm{l}_{lax}(F)$ |
|---|---|---|---|
| objects | $A$ | sets $A$ | sets $A$ |
| arrows | $A \xrightarrow{f} B$ | functions $f: A \to B$ | functions $f: A \to B$ |
| proarrows | $A \overset{M}{\nrightarrow} B$ | matrices $M(a,b) \in \mathcal{V}_{ob}$ | functions $M: A \to FB$ |
| cells | $\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle M}\big\downarrow & \overset{\alpha}{\Rightarrow} & \big\downarrow{\scriptstyle N} \\ B & \xrightarrow[g]{} & D \end{array}$ | matrices $\alpha_{a,b}: M(a,b) \to N(fa, gb)$ | exists if $g[M_a] \supseteq N_{fa}$ for all $a \in A$ |

# Monads

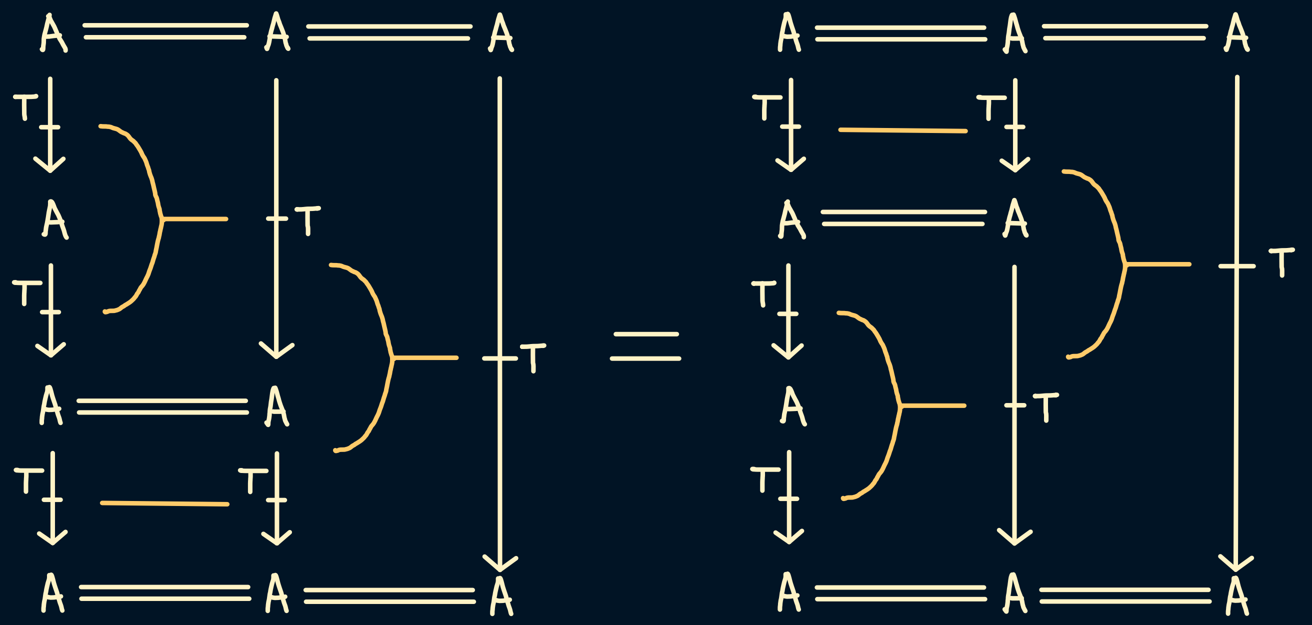
A **monad** is an endoproarrow $T : A \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} A$ equipped with cells

$$\text{unit} \quad \begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[d, "T"] \\ A \arrow[r, equal] & A \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "T"'] & A \arrow[dd, "T"] \\ A \arrow[d, "T"'] & \\ A \arrow[r, equal] & A \end{tikzcd} \quad \text{multiplication}$$

that satisfy the usual unitality and associativity axioms.

# Monad Unitality

# MONAD ASSOCIATIVITY



# Monads in $\mathcal{V}$-Mat are $\mathcal{V}$-categories

$$C_{ob} \xrightarrow{\;C\;} C_{ob}$$

set $C_{ob}$ of objects and hom objects $C(a,b)$

$$\begin{tikzcd} C_{ob} \arrow[r, equal] \arrow[d, equal] & C_{ob} \arrow[d, "C"] \\ C_{ob} \arrow[r, equal] & C_{ob} \end{tikzcd}$$

identity elements

$$j_a : I \longrightarrow C(a,a)$$

$$\begin{tikzcd} C_{ob} \arrow[r, equal] \arrow[d, "C"'] & C_{ob} \arrow[dd, "C"] \\ C_{ob} \arrow[d, "C"'] & \\ C_{ob} \arrow[r, equal] & C_{ob} \end{tikzcd}$$

composition maps

$$m_{a,b,c} : C(a,b) \otimes C(b,c) \longrightarrow C(a,c)$$

# Monads in $\mathbb{Kl}_{lax}(\mathcal{F})$ are topological spaces

$$X \overset{N}{\longmapsto} X$$

set $X$ of points and
neighbourhood filters $N_x \in \mathcal{F}X$

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\bullet"] & X \arrow[d, "N", "\shortmid"{marking}] \\ X \arrow[r, equal] & X \end{tikzcd} \iff \{A \subseteq X : x \in A\} \supseteq N_x \iff$$

every neighbourhood of $x$ contains $x$

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "N"', "\shortmid"{marking}] & X \arrow[dd, "N", "\shortmid"{marking}] \\ X \arrow[d, "N"', "\shortmid"{marking}] & \\ X \arrow[r, equal] & X \end{tikzcd} \iff \bigcup_{M \in N_x} \bigcap_{y \in M} N_y \supseteq N_x \iff$$

every subset of $X$ has open interior

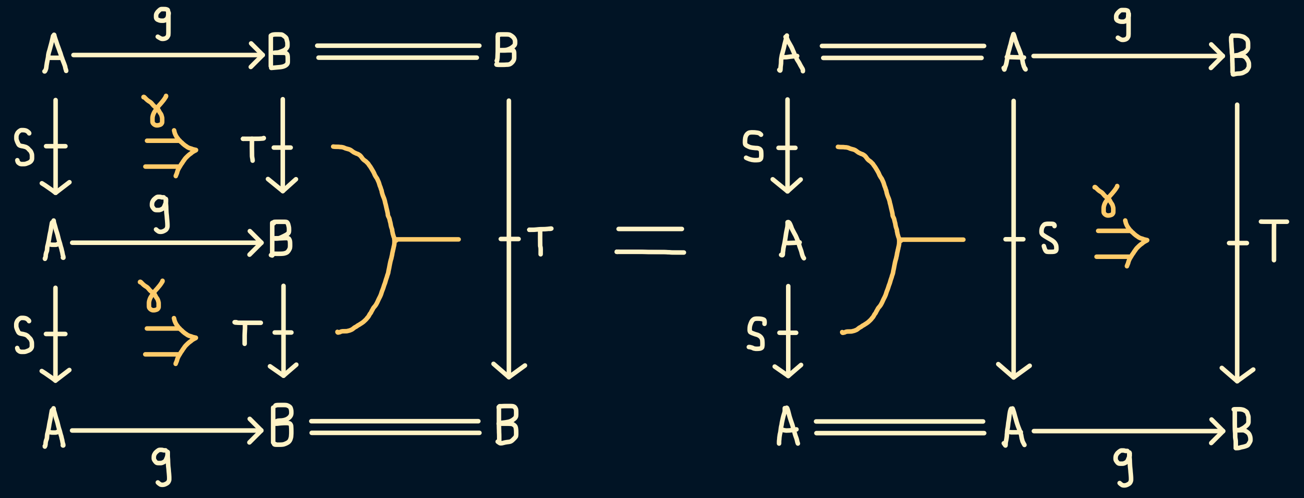
# Monad morphisms

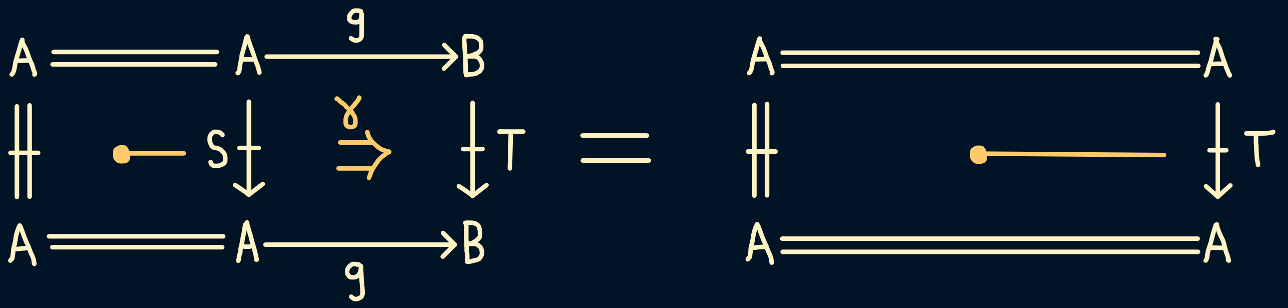
A **monad morphism** from $S : A \rightarrow\mkern-14mu\shortmid\mkern6mu A$ to $T : B \rightarrow\mkern-14mu\shortmid\mkern6mu B$ is a cell

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "S"', "\shortmid" marking] \arrow[dr, phantom, "\overset{\gamma}{\Rightarrow}"] & B \arrow[d, "T", "\shortmid" marking] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

compatible with the units and multiplications of $S$ and $T$.

# Compatibility conditions



# Monad morphisms in $\mathcal{V}$-Mat are $\mathcal{V}$-functors

object map

$$\begin{tikzcd} C_{ob} \arrow[r, "F_{ob}"] \arrow[d, "C"', "|" marking] \arrow[dr, phantom, "\overset{F}{\Rightarrow}"] & D_{ob} \arrow[d, "D", "|" marking] \\ C_{ob} \arrow[r, "F_{ob}"'] & D_{ob} \end{tikzcd}$$

hom maps

$$F_{a,b} : C(a,b) \to D(Fa, Fb)$$

# Monad morphisms in $\mathbb{Kl}_{lax}(F)$ are continuous maps

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{N}"'] \arrow[dr, phantom, "\supseteq"] & Y \arrow[d, "\mathcal{N}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \iff f[\mathcal{N}_x] \supseteq \mathcal{N}_{fx} \iff \begin{array}{c} N \in \mathcal{N}_{fx} \\ \text{implies} \\ f^{-1}N \in \mathcal{N}_x \end{array}$$

$f[M] = \{B \subseteq Y : f^{-1}B \in M\}$

# Companions

A **companion** of an arrow $f : A \to B$ is a proarrow $f_\bullet : A \nrightarrow B$

equipped with cells
$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d, "f_\bullet", "\shortmid"{marking}] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$
and
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f_\bullet"', "\shortmid"{marking}] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, equal] \\ B \arrow[r, equal] & B \end{tikzcd}$$
such that

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner"] & A \arrow[r, "f"] \arrow[d, "f_\bullet", "\shortmid"{marking}] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, equal] \\ A \arrow[r, "f"'] & B \arrow[r, equal] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\mid"] & A \arrow[d, equal] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d, "f_\bullet", "\shortmid"{marking}] \\ A \arrow[r, "f"'] \arrow[d, "f_\bullet"', "\shortmid"{marking}] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, equal] \\ B \arrow[r, equal] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, equal] \arrow[d, "f_\bullet"', "\shortmid"{marking}] \arrow[dr, phantom, "-"] & A \arrow[d, "f_\bullet", "\shortmid"{marking}] \\ A \arrow[r, equal] & B \end{tikzcd}$$

# Examples of companions

$f : A \longrightarrow B$ has companion $f_\bullet : A \nrightarrow B$ given by

## $\mathcal{V}$-Mat

$$[f_\bullet]_{a,b} = \delta_{fa,b} = \begin{cases} I & \text{if } fa = b \\ 0 & \text{otherwise} \end{cases}$$

## $\mathbb{Kl}_{lax}(F)$

$$f_\bullet = \left(A \xrightarrow{f} B \xrightarrow{\eta_B} FB\right)$$

$$f_\bullet(a) = \{S \subseteq B : f(a) \in S\}$$

# Retrocells

A **retrocell**
$$\begin{tikzcd} C \arrow[d, "N"', "|" marking] \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] & A \arrow[l, "f"'] \arrow[d, "M", "|" marking] \\ D & B \arrow[l, "g"] \end{tikzcd}$$
is a cell
$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f_{\bullet}"', "|" marking] & A \arrow[d, "M", "|" marking] \\ C \arrow[d, "N"', "|" marking] \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "g_{\bullet}", "|" marking] \\ D \arrow[r, equal] & D \end{tikzcd}$$
.

introduced by Paré

# Monad retromorphisms

A **monad retromorphism** from $S : A \nrightarrow A$ to $T : B \nrightarrow B$ is a retrocell

$$\begin{tikzcd} B & A \arrow[l, "g"'] \arrow[d, "S"] \\ B \arrow[u, phantom] & A \arrow[l, "g"] \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[d, "T"'] & A \arrow[l, "g"'] \arrow[d, "S"] \arrow[dl, phantom, "\overset{\varphi}{\Rightarrow}"] \\ B & A \arrow[l, "g"] \end{tikzcd}$$

compatible with the units and multiplications of $S$ and $T$.

# Compatibility conditions

$$
\begin{tikzcd}
B \arrow[d, "T"'] & A \arrow[l, "g"'] \arrow[r, equal] \arrow[d, "S"'] \arrow[dl, phantom, "\overset{\varphi}{\Rightarrow}"] & A \arrow[dd, "S"] \\
B \arrow[d, "T"'] & A \arrow[l, "g"'] \arrow[d, "S"'] \arrow[dl, phantom, "\overset{\varphi}{\Rightarrow}"] & \\
B & A \arrow[l, "g"] \arrow[r, equal] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "T"'] & B \arrow[dd, "T"] \arrow[ddr, phantom, "\overset{\varphi}{\Rightarrow}"] & A \arrow[l, "g"'] \arrow[dd, "S"] \\
B \arrow[d, "T"'] & & \\
B \arrow[r, equal] & B & A \arrow[l, "g"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, equal] & B \arrow[d, "T"] & A \arrow[l, "g"'] \arrow[d, "S"] \arrow[dl, phantom, "\overset{\varphi}{\Rightarrow}"] \\
B \arrow[r, equal] & B & A \arrow[l, "g"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[d, "S"] \\
A \arrow[r, equal] & A
\end{tikzcd}
$$

# Monad retromorphisms in $\mathcal{V}$-Mat are $\mathcal{V}$-retrofunctors

$$\begin{tikzcd}
C_{ob} \arrow[r, equal] \arrow[d, "(\Phi_{ob})_\bullet"', "\shortmid" marking] & C_{ob} \arrow[d, "C", "\shortmid"' marking] \\
D_{ob} \arrow[r, phantom, "\overset{\Phi}{\Rightarrow}"] \arrow[d, "D"', "\shortmid" marking] & C_{ob} \arrow[d, "(\Phi_{ob})_\bullet", "\shortmid"' marking] \\
D_{ob} \arrow[r, equal] & D_{ob}
\end{tikzcd}$$

$$\begin{tikzcd}
D(\Phi c, d') \arrow[d, "\wr"'] \arrow[rr, dashed] & & \displaystyle\sum_{c' \in \Phi^{-1}\{d'\}} C(c,c') \\
\displaystyle\sum_{d \in D_{ob}} \delta_{\Phi c, d} \otimes D(d,d') \arrow[r, "\Phi_{c,d'}"'] & \displaystyle\sum_{c' \in C_{ob}} C(c,c') \otimes \delta_{\Phi c', d'} \arrow[ur, "\wr"'] &
\end{tikzcd}$$

# Monad retromorphisms in $\mathbb{K}\mathrm{l}_{lax}(\mathcal{F})$ are open maps

Simplify using monad axioms

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "\varphi_\bullet"'] & X \arrow[d, "\mathcal{N}"] \\
Y \arrow[r, phantom, "\supseteq"] \arrow[d, "\mathcal{N}"'] & X \arrow[d, "\varphi_\bullet"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}
\iff
\mathcal{N}_{\varphi x} \supseteq \varphi[\mathcal{N}_x]
\iff
$$

$N \in \mathcal{N}_x$
implies
$\varphi N \in \mathcal{N}_{\varphi x}$

$\varphi[\mathcal{M}] = \{B \subseteq Y : B \supseteq \varphi A \text{ for some } A \in \mathcal{M}\}$

# Monad transformations

A monad transformation

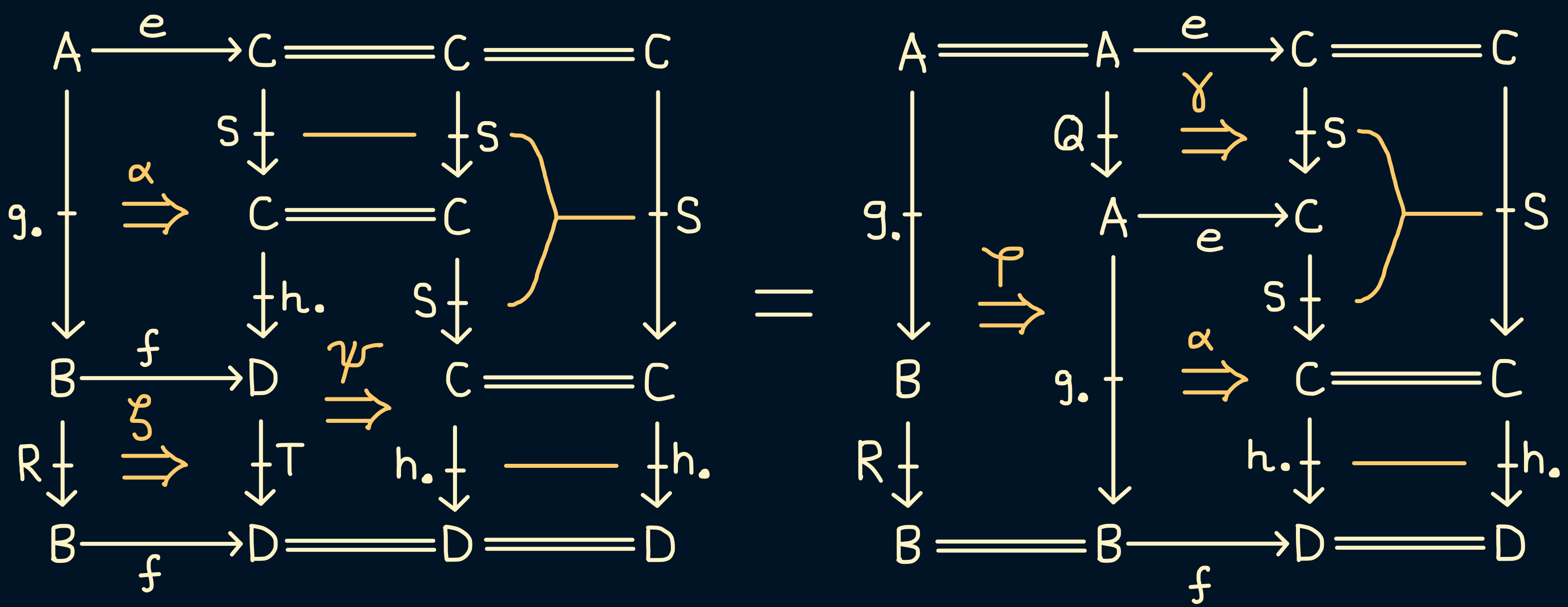
$$\begin{tikzcd}
(A,Q) \arrow[r, "{(e,\gamma)}"] \arrow[d, "{(g,\varphi)}"'] \arrow[dr, phantom, "\searrow\alpha"] & (C,S) \arrow[d, "{(h,\psi)}"] \\
(B,R) \arrow[r, "{(f,\xi)}"'] & (D,T)
\end{tikzcd}$$

is a cell

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[dd, "g"'] & C \arrow[d, "s"] \\
\arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] & C \arrow[d, "h"] \\
B \arrow[r, "f"'] & D
\end{tikzcd}$$

that satisfies the following condition.

# Compatibility condition



# Double category $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$



Let $\mathbb{D}$ be a double category with chosen companions.
Then $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ has

objects:
monads in $\mathbb{D}$

proarrows:
monad retromorphisms

arrows:
monad morphisms

cells:
monad transformations

$\mathcal{V}\text{-}\mathbb{C}\mathrm{at}_{\mathrm{ret}} = \mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at})$

$\mathbb{T}\mathrm{op} = \mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{K}\mathrm{l}_{\mathrm{lax}}(F))$

# Summary

Double categories with companions are a good setting for understanding general notions of lifting.

# Omitted

- Compatible squares
- Lenses
- Internal categories example

# Future work

- relationship between monad retromorphisms and bimodules.
- retromorphisms as spans of morphisms; and tabulators
- proxy pullback of lenses
- more examples

# FURTHER READING

An introduction to enriched cofunctors
Bryce Clarke and Matthew Di Meglio

Retrocells Redux
Robert Paré

Monoidal topology
Dirk Hofmann, Gavin J. Seal, Walter Tholen