From categorical systems theory to categorical cybernetics

Matteo Capucci

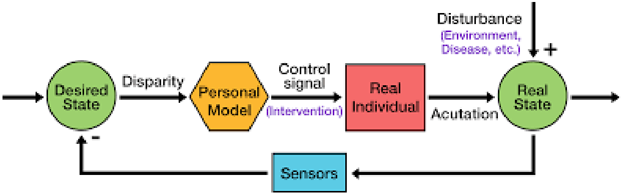
MSP group, University of Strathclyde

Virtual Double Categories Workshop

December 2nd, 2022

Cybernetic systems are

systems with control mechanisms and interactive feedback



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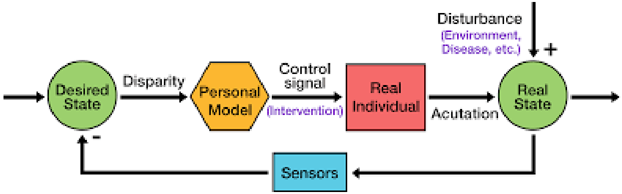

control theory

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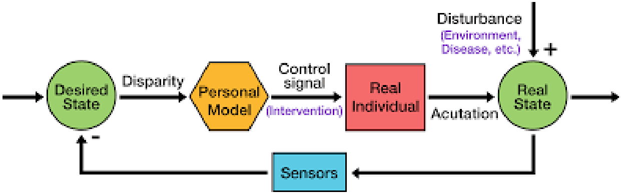

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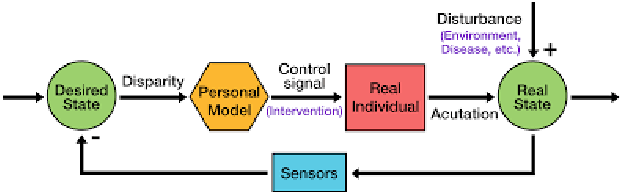

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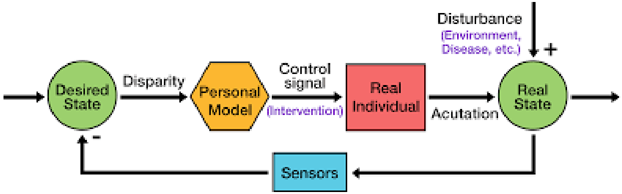
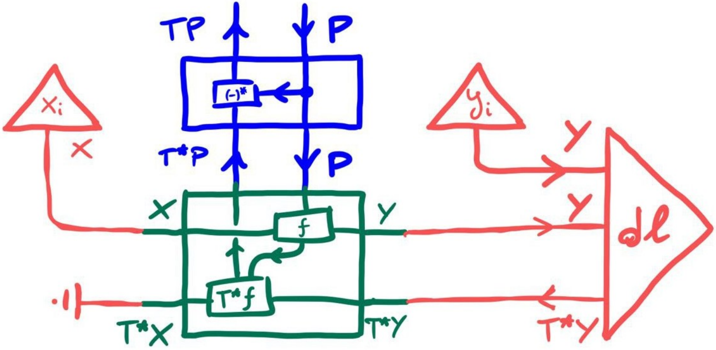
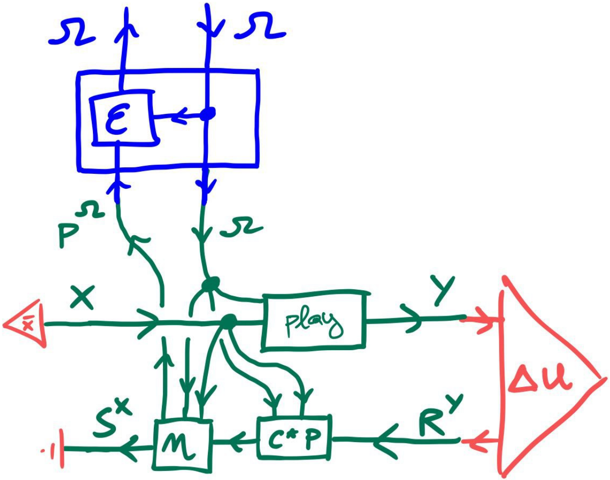


control theory, game theory, learning theory, etc...



A model trained by gradient-descent
(Fong, D. Spivak, and Tuyéras 2019; Cruttwell, Gavranović, Ghani, Wilson, and Zanasi 2021; Capucci 2022)



A strategic game

(Ghani, Hedges, Winschel, and Zahn 2018; Capucci, Gavranović, Hedges, and Rischel 2021; Capucci 2022)

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Ontology

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What is it made of?

How does it interact with other cybernetic systems?

What is its interface? How do we describe it?

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Phenomenology

What does a cybernetic system 'do'?

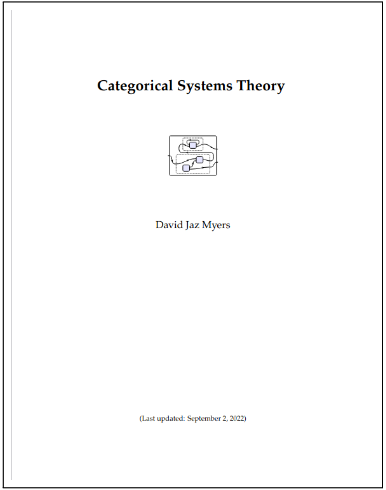
What can it be observed about it?

How do observations of different parts relate to observations on the whole?

When are two cybernetic systems observationally interchangeable?

Categorical Systems Theory

A principled mathematical framework for the ontology and phenomenology of systems is given by **categorical systems theory** (CST)


Categorical Systems Theory

David Jaz Myers

(Last updated: September 2, 2022)

Myers 2021

Categorical Systems Theory

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2. **Bicategories of transition systems** (Katis, Sabadini, and Walters 1997a; Katis, Sabadini, and Walters 1997b; Katis, Sabadini, and Walters 2002; Gianola, Kasangian, and Sabadini 2017; Di Lavore, Gianola, Román, Sabadini, and Sobociński 2021)

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4. **Symmetric monoidal categories** (Abramsky and Coecke 2004; Coecke and Paquette 2010; Coecke and Kissinger 2018)

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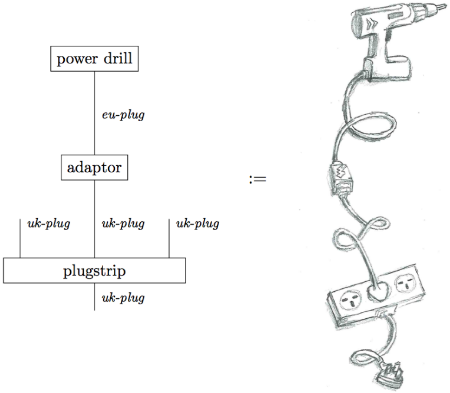

From (Coecke and Kissinger 2018)

Categorical Systems Theory

In fact, CST is inspired by characteristics shared across them all:

1. **Processes compose like morphisms of a category**
2. **Behaviours are relations/spans/matrices**, and are (lax) functorially determined:

If $(U, \alpha) : X \to Y$ is a circuit, an element of U is called a *state* of (U, α), and an element of X or Y is referred to as an *input* or an *output* for (U, α) respectively.

Definition 3. The following data define the homomorphism

$$\text{equilibrium} : \mathbf{Circ} \to \mathbf{Span}.$$

(a) If X is an object of **Circ**, $\text{equilibrium}(X) = X$.
(b) If $(U, \alpha) : X \to Y$ is a circuit, let

$$W = \{(x, u) \in XU \mid p_U \cdot \alpha(x, u) = u\},$$

$$p : W \to X : (x, u) \mapsto x,$$

and

$$q : W \to Y : (x, u) \mapsto p_Y \cdot \alpha(x, u).$$

From (Katis, Sabadini, and Walters 1997a)

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In Theorem 23, we show that the map sending P to $\blacksquare P$ extends to a lax double functor

$$\blacksquare \colon \mathbb{O}\mathbf{pen}(\mathsf{Petri}) \to \mathbb{R}\mathbf{el}.$$

From (Baez and Master 2020)

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Definition 2.28. Let $F = (S, f^{\mathrm{rdt}}, f^{\mathrm{upd}})$ be an (A, B)-discrete system. For $a \in A$ and $b \in B$, recall the set of (a, b)-steady states from Definition 2.4 and its count

$$\mathrm{Stst}(F)_{a,b} = \#\{s \in S \mid f^{\mathrm{rdt}}(s) = b \text{ and } f^{\mathrm{upd}}(a, s) = s\}$$

We can consider this as a matrix $\mathrm{Stst}(F) \in \mathsf{Mat}(A, B)$, which we call the *steady state matrix of* F.

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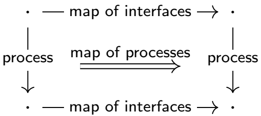
1. **Processes organize in double categories**.



This explains why behaviours land in $\mathbf{Set}\text{-}\mathbf{Mat}/\mathbb{S}\mathbf{pan}(\mathbf{Set})$: behaviours are often horizontal corepresentables!

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2. **There is a distinction between processes and systems**.

   Processes mediate interaction between systems, by acting on their interfaces:

$$\mathbf{Sys} : \mathbb{P}^\top \xrightarrow{\text{unitary lax}} \mathbb{C}\mathbf{at}$$

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Example

Dynamical systems (Rosen 1978) are endomorphisms $T : S \to S$ on some objects of states, hopefully equipped with an observable $X : S \to O$. The notion of process between these systems is functions $O \to O'$ that remap observations. Very different!

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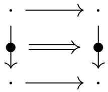
    All in double dimension

2. A glimpse of **categorical cybernetic systems theory**,
   and why we need the triple dimension

Notation & conventions

1. Double categories are **weak** by default, (double) functors are **lax** by default
2. For the rest I mostly follow Grandis (2019)
3. ‘Loose’ arrows are vertical (denoted: $\mathrel{-\!\!\bullet\!\!\to}$), ‘tight’ arrows are horizontal (denoted: $\to$)



4. $\mathbb{S}\mathbf{et}$ denotes the **double category of spans** in $\mathbf{Set}$,
5. $\mathbb{C}\mathbf{at}$ denotes the **double category of** categories, functors, **profunctors** and natural transformations thereof.

I diverged from Myers (2021) on some notational and terminological choices.

Categorical systems theory

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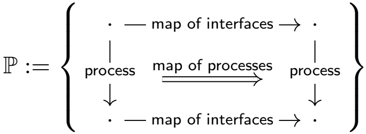

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$$\mathbb{P} := \left\{ \begin{array}{ccc} \cdot & \xrightarrow{\text{map of interfaces}} & \cdot \\ \text{process}\big\downarrow & \overset{\text{map of processes}}{\Longrightarrow} & \big\downarrow\text{process} \\ \cdot & \xrightarrow{\text{map of interfaces}} & \cdot \end{array} \right\}$$

2. Then **processes are used to index systems**, giving rise to doubly indexed categories of systems:

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3. Finally, **behaviour is studied by describing maps into the span-ish ‘observational theory’**:

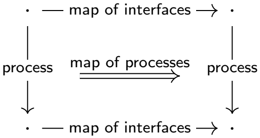
$$\begin{array}{ccc} \mathbb{P}^{\top} & \xrightarrow{B^{\top}} & \mathbb{S}\mathbf{et}^{\top} \\ \Big\downarrow{\scriptstyle \mathbf{Sys}} & \overset{B^{\flat}}{\Longrightarrow} & \Big\downarrow{\scriptstyle \mathbf{Obs}} \\ \mathbb{C}\mathbf{at} & = & \mathbb{C}\mathbf{at} \end{array}$$

Process theories

Definition

A **process theory** is a double category with attitude where:

1. objects are **boundaries** or **interfaces**,
2. vertical morphisms are **processes**,
3. horizontal morphisms are **maps of boundaries**,
4. squares are **maps of processes**



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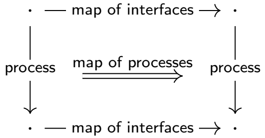

Adding structure to the double category refines the kind of processes we are talking about,

e.g.: **concurrent process theories** have a monoidal product with the attitude of spatial juxtaposition of processes,

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3. maps of boundaries are charts:

$$\begin{pmatrix} A^- \\ A^+ \end{pmatrix} \underset{h}{\overset{h^\flat}{\rightrightarrows}} \begin{pmatrix} C^- \\ C^+ \end{pmatrix}$$

   where

$$h : A^+ \to C^+, \qquad h^\flat : A^+ \times A^- \to C^-$$

Example: lenses

3. maps of processes are 'commutative squares':

$$\begin{array}{ccc} \begin{pmatrix} A^- \\ A^+ \end{pmatrix} & \overset{h^\flat}{\underset{h}{\rightleftarrows}} & \begin{pmatrix} C^- \\ C^+ \end{pmatrix} \\ {\scriptstyle p^\sharp}\uparrow\downarrow{\scriptstyle p} & & {\scriptstyle q^\sharp}\uparrow\downarrow{\scriptstyle q} \\ \begin{pmatrix} B^- \\ B^+ \end{pmatrix} & \overset{k^\flat}{\underset{k}{\rightleftarrows}} & \begin{pmatrix} D^- \\ D^+ \end{pmatrix} \end{array}$$

meaning for every $a^+ \in A^+$ and $b^- \in B^-$ we have

$$q(h(a^+)) = k(p(a^+)),$$
$$h^\flat(a^+, p^\sharp(a^+, b^-)) = q^\sharp(h(a^+), k^\flat(p(a^+), b^-)).$$

Example: observational theories

If $\mathbf{C}$ has pullbacks, $\mathbb{S}\mathbf{pan}(\mathbf{C})$ is the **observational theory of $\mathbf{C}$-processes** where

1. boundaries are objects in $\mathbf{C}$,
2. processes are spans in $\mathbf{C}$,
3. maps of boundaries are maps in $\mathbf{C}$,
4. maps of processes are squares in $\mathbb{S}\mathbf{pan}(\mathbf{C})$:

$$\begin{array}{ccc} A & \xrightarrow{\;h\;} & C \\ {\scriptstyle p_\ell}\uparrow & & \uparrow{\scriptstyle q_\ell} \\ S & \xrightarrow{\;\sigma\;} & R \\ {\scriptstyle p_r}\downarrow & & \downarrow{\scriptstyle q_r} \\ B & \xrightarrow[\;k\;]{} & D \end{array}$$

This is an *observational theory*: processes are described by what we observe about their internal states.

Systems vs processes

We have setup a way to talk about processes and maps thereof, but in practice we often care about specific processes, namely **stateful**/**one-sided** ones.

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Among (non-deterministic) lenses, stateful ones are known as **(non-deterministic) Moore machines**:

$$\begin{pmatrix} \text{update} \\ \text{observe} \end{pmatrix} : \begin{pmatrix} S \\ S \end{pmatrix} \leftrightarrows \begin{pmatrix} I \\ O \end{pmatrix}$$

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Among all spans, we can consider 'one-sided' ones as state spaces exposing some observables:

$$\text{observe} : S \longrightarrow O$$

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A **systems theory** over the process theory $\mathbb{P}$ is a doubly indexed category with attitude:

$$\mathbf{Sys} : \mathbb{P}^\top \xrightarrow{\text{unitary lax}} \mathbb{C}\mathbf{at}^*$$

*More precisely, this is a right module of $\mathbb{P}$, considered as a (pseudo)monad in $\mathbb{S}\mathbf{pan}(\mathbf{Cat})$, since we can't really exchange loose and tight arrows (thanks to DJM for clarifying this).

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Again, one can consider more structure, e.g. **concurrent systems theories**, which are systems theories over concurrent process theories (monoidal dbl cats) given by lax monoidal indexings.

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3. maps of boundaries $h : A \to C$ act profunctorially (and laxly!):

$$\mathbf{Sys}(h) : \mathbf{Sys}(A) \mathrel{\longrightarrow\!\!\!\!\!\!\!+\;\;} \mathbf{Sys}(C)$$

   and we think of $\mathbf{Sys}(h)(\mathsf{S}, \mathsf{R})$ as the **possible simulations of** $\mathsf{S} : \mathbf{Sys}(A)$ **in** $\mathsf{R} : \mathbf{Sys}(C)$ **mediated by the map** h on their boundaries.

Systems theories

4. maps of processes

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle p}\big\downarrow\bullet & \overset{\sigma}{\Longrightarrow} & \bullet\big\downarrow{\scriptstyle q} \\ B & \xrightarrow[k]{} & D \end{array}$$

yield squares

$$\begin{array}{ccc} \mathbf{Sys}(A) & \xrightarrow{\mathbf{Sys}(p)} & \mathbf{Sys}(B) \\ {\scriptstyle \mathbf{Sys}(h)}\big\downarrow & \overset{\mathbf{Sys}(\sigma)}{\Longrightarrow} & \big\downarrow{\scriptstyle \mathbf{Sys}(k)} \\ \mathbf{Sys}(C) & \xrightarrow[\mathbf{Sys}(q)]{} & \mathbf{Sys}(D) \end{array}$$

which **extend** a given simulation of systems along the given map of systems σ.

Example: observable dynamical systems

Traditionally, a **dynamical system** (DS) (Giunti and Mazzola 2012) is an endomorphism

$$\mathrm{update} : S \to S$$

in some category of spaces $\mathbf{S}$ (e.g. $\mathbf{Smooth}$, $\mathbf{Meas}$, ...)

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We can arrange ODS in a system theory, over a slightly trivial process theory, $\mathbf{S}^{\to}$:

$$\begin{array}{ccc} O & \xrightarrow{h} & O' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Q & \xrightarrow[k]{} & Q' \end{array}$$

Depending on the context, one might make vertical and horizontal maps different, e.g. vertical maps might be effectful while horizontal ones aren't.

Example: observable dynamical systems

Then

$$\mathbf{DynSys} : \mathbf{S}^{\rightarrow} \xrightarrow{\text{unitary lax}} \mathbb{C}\mathbf{at}$$

$$\mathbf{DynSys}(O) = \left\{ \begin{array}{ccc} S & \xrightarrow{g} & R \\ {\scriptstyle \mathrm{update}_{\mathsf{S}}}\downarrow & & \downarrow{\scriptstyle \mathrm{update}_{\mathsf{R}}} \\ S & \xrightarrow{g} & R \\ {\scriptstyle \mathrm{observe}_{\mathsf{S}}}\downarrow & & \downarrow{\scriptstyle \mathrm{update}_{\mathsf{R}}} \\ O & = & O \end{array} \right\}$$

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Given a process $p : O \to Q$, we define

$$\mathbf{DynSys}(p) : \mathbf{DynSys}(O) \longrightarrow \mathbf{DynSys}(Q)$$

$$(S, \mathrm{update}_{\mathsf{S}}, \mathrm{observes}_{\mathsf{S}}) \longmapsto (S, \mathrm{update}_{\mathsf{S}}, \mathrm{observes}_{\mathsf{S}} \fatsemi p)$$

Example: observable dynamical systems

Given a map of boundaries $h : O \to O'$, we define a profunctor

$$\mathbf{DynSys}(h) : \mathbf{DynSys}(O)^{\mathsf{op}} \times \mathbf{DynSys}(O') \longrightarrow \mathbf{Set}$$

$$(\mathsf{S}, \mathsf{R}) \longmapsto \left\{ g : S \to R \;\middle|\; \begin{array}{ccc} S & \overset{g}{\dashrightarrow} & R \\ {\scriptstyle \mathrm{update}_{\mathsf{S}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{update}_{\mathsf{R}}} \\ S & \overset{g}{\dashrightarrow} & R \\ {\scriptstyle \mathrm{observe}_{\mathsf{S}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{update}_{\mathsf{R}}} \\ O & \underset{h}{\longrightarrow} & O' \end{array} \right\}$$

Example: observable dynamical systems

Finally, given a square $\begin{array}{ccc} O & \xrightarrow{h} & O' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Q & \xrightarrow[k]{} & Q' \end{array}$ we define a square

$$\begin{array}{ccc} \mathbf{DynSys}(O) & \xrightarrow{\mathbf{DynSys}(p)} & \mathbf{DynSys}(Q) \\ {\scriptstyle \mathbf{DynSys}(h)}\big\downarrow\!\!\!+ & \xRightarrow{\mathbf{DynSys}(\square)} & +\!\!\!\big\downarrow{\scriptstyle \mathbf{DynSys}(k)} \\ \mathbf{DynSys}(O') & \xrightarrow[\mathbf{DynSys}(q)]{} & \mathbf{DynSys}(Q') \end{array}$$

'sending $g : S \to R$ to itself', with the proof of commutativity got by stacking squares vertically:

$$\begin{array}{ccc} S & \overset{g}{\dashrightarrow} & R \\ {\scriptstyle \mathrm{update}_\mathsf{S}}\downarrow & & \downarrow{\scriptstyle \mathrm{update}_\mathsf{R}} \\ S & \overset{g}{\dashrightarrow} & R \\ {\scriptstyle \mathrm{observe}_\mathsf{S}}\downarrow & & \downarrow{\scriptstyle \mathrm{update}_\mathsf{R}} \\ O & \xrightarrow[h]{} & O' \end{array} \quad\longmapsto\quad \begin{array}{ccc} S & \overset{g}{\dashrightarrow} & R \\ {\scriptstyle \mathrm{update}_\mathsf{S}}\downarrow & & \downarrow{\scriptstyle \mathrm{update}_\mathsf{R}} \\ S & \overset{g}{\dashrightarrow} & R \\ {\scriptstyle \mathrm{observe}_\mathsf{S}}\downarrow & & \downarrow{\scriptstyle \mathrm{update}_\mathsf{R}} \\ O & \xrightarrow[h]{} & O' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Q & \xrightarrow[k]{} & Q' \end{array}$$

Example: Moore machines

Moore machines are (one possible notion of) system associated to the process theory $\mathbb{L}\mathbf{ens}$:

$$\mathbf{Moore} : \mathbb{L}\mathbf{ens}^\top \xrightarrow{\text{unitary lax}} \mathbb{C}\mathbf{at}$$

A Moore machine $\mathsf{S} : \mathbf{Moore}\binom{I}{O}$ is an **open dynamical systems**:

$$\mathrm{update}_{\mathsf{S}} : S \times I \longrightarrow S, \mathrm{observe}_{\mathsf{S}} : S \longrightarrow O$$

It's not just open **towards** the outside (*observable*), it's also open **from** the outside.

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It's not just open **towards** the outside (*observable*), it's also open **from** the outside.

Morphisms are maps that commute with the dynamics

$$\mathbf{Moore}\binom{I}{O} = \left\{ \begin{array}{ccc} \binom{S}{S} & \overset{\pi_2 ⨾ g}{\underset{g}{\rightrightarrows}} & \binom{R}{R} \\ \text{update}_{\mathsf{S}}\uparrow\downarrow\text{observe}_{\mathsf{S}} & & \text{update}_{\mathsf{R}}\uparrow\downarrow\text{observe}_{\mathsf{R}} \\ \binom{I}{O} & = & \binom{I}{O} \end{array} \right\}$$

Example: Moore machines

Given a lens $\begin{pmatrix} p^\sharp \\ p \end{pmatrix} : \begin{pmatrix} I \\ O \end{pmatrix} \rightrightarrows \begin{pmatrix} I' \\ O' \end{pmatrix}$ we get a functor:

$$\mathbf{Sys}\begin{pmatrix} p^\sharp \\ p \end{pmatrix} : \mathbf{Sys}\begin{pmatrix} I \\ O \end{pmatrix} \longrightarrow \mathbf{Sys}\begin{pmatrix} I' \\ O' \end{pmatrix}$$

$$\begin{array}{ccc}
\begin{pmatrix} S \\ S \end{pmatrix} & & \begin{pmatrix} S \\ S \end{pmatrix} \\
 & & \text{update}_S \uparrow\downarrow \text{observe}_S \\
\text{update}_S \uparrow\downarrow \text{observe}_S & \longmapsto & \begin{pmatrix} I \\ O \end{pmatrix} \\
 & & p \downarrow\uparrow p^\sharp \\
\begin{pmatrix} I \\ O \end{pmatrix} & & \begin{pmatrix} I' \\ O' \end{pmatrix}
\end{array}$$

Example: Moore machines

Given a chart $\begin{pmatrix} h^\flat \\ h \end{pmatrix} : \begin{pmatrix} I \\ O \end{pmatrix} \rightrightarrows \begin{pmatrix} J \\ Q \end{pmatrix}$ we get a profunctor:

$$\mathbf{Moore}\begin{pmatrix} I \\ O \end{pmatrix}^{\mathsf{op}} \times \mathbf{Moore}\begin{pmatrix} J \\ Q \end{pmatrix} \xrightarrow{\mathbf{Moore}\begin{pmatrix} h^\flat \\ h \end{pmatrix}} \mathbf{Set}$$

$$\left(\begin{array}{c} \begin{pmatrix} S \\ S \end{pmatrix} \\ \mathrm{update}_{\mathsf{S}} \upharpoonleft\downharpoonright \mathrm{observe}_{\mathsf{S}} \\ \begin{pmatrix} I \\ O \end{pmatrix} \end{array} , \begin{array}{c} \begin{pmatrix} R \\ R \end{pmatrix} \\ \mathrm{update}_{\mathsf{R}} \upharpoonleft\downharpoonright \mathrm{observe}_{\mathsf{R}} \\ \begin{pmatrix} J \\ Q \end{pmatrix} \end{array} \right) \longmapsto \left\{ \begin{array}{ccc} \begin{pmatrix} S \\ S \end{pmatrix} & \overset{\pi_2 ⨾ g}{\underset{g}{\dashrightarrow}} & \begin{pmatrix} R \\ R \end{pmatrix} \\ \mathrm{update}_{\mathsf{S}} \upharpoonleft\downharpoonright \mathrm{observe}_{\mathsf{S}} & & \mathrm{update}_{\mathsf{R}} \upharpoonleft\downharpoonright \mathrm{observe}_{\mathsf{R}} \\ \begin{pmatrix} I \\ O \end{pmatrix} & \overset{h^\flat}{\underset{h}{\rightrightarrows}} & \begin{pmatrix} J \\ Q \end{pmatrix} \end{array} \right\}$$

Example: Moore machines

Finally, given a square

$$
\begin{array}{ccc}
\begin{pmatrix} I \\ O \end{pmatrix} & \overset{h^\flat}{\underset{h}{\rightrightarrows}} & \begin{pmatrix} J \\ Q \end{pmatrix} \\
{\scriptstyle p^\sharp}\uparrow\downarrow{\scriptstyle p} & & {\scriptstyle q^\sharp}\uparrow\downarrow{\scriptstyle q} \\
\begin{pmatrix} I' \\ O' \end{pmatrix} & \overset{k^\flat}{\underset{k}{\rightrightarrows}} & \begin{pmatrix} J' \\ Q' \end{pmatrix}
\end{array}
$$

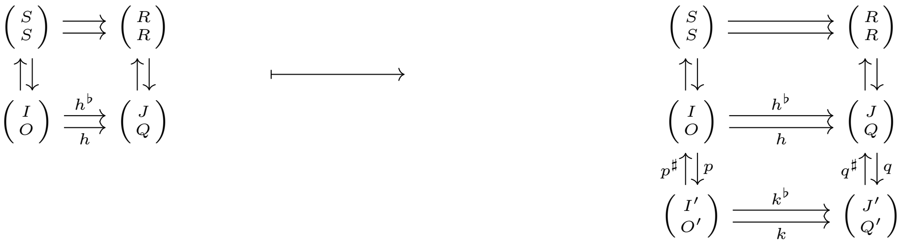
we get a square in $\mathbb{C}\mathbf{at}$...

$$
\begin{array}{ccc}
\mathbf{Moore}\begin{pmatrix} I \\ O \end{pmatrix} & \xrightarrow{\mathbf{Moore}\begin{pmatrix} p^\sharp \\ p \end{pmatrix}} & \mathbf{Moore}\begin{pmatrix} I' \\ O' \end{pmatrix} \\
{\scriptstyle \mathbf{Moore}\begin{pmatrix} h^\flat \\ h \end{pmatrix}}\big\downarrow & \overset{\mathbf{Moore}(\square)}{\Longrightarrow} & \big\downarrow{\scriptstyle \mathbf{Moore}\begin{pmatrix} k^\flat \\ k \end{pmatrix}} \\
\mathbf{Moore}\begin{pmatrix} J \\ Q \end{pmatrix} & \xrightarrow[\mathbf{Moore}\begin{pmatrix} q^\sharp \\ q \end{pmatrix}]{} & \mathbf{Moore}\begin{pmatrix} J' \\ Q' \end{pmatrix}
\end{array}
$$

Example: Moore machines

…given by stacking squares:

$$\mathbf{Moore}\begin{pmatrix} h^\flat \\ h \end{pmatrix}(\mathsf{S}, \mathsf{R}) \xrightarrow{\mathbf{Moore}(\square)_{\mathsf{S},\mathsf{R}}} \mathbf{Moore}\begin{pmatrix} k^\flat \\ k \end{pmatrix}\left(\mathbf{Moore}\begin{pmatrix} p^\sharp \\ p \end{pmatrix}(\mathsf{S}),\ \mathbf{Moore}\begin{pmatrix} q^\sharp \\ q \end{pmatrix}(\mathsf{R})\right)$$



Example: observational theory

Any observational theory of processes $\mathbb{S}\mathbf{pan}(\mathbf{C})$ supports a theory of observational systems

$$\mathbf{Obs}_{\mathbf{C}} : \mathbb{S}\mathbf{pan}(\mathbf{C})^{\top} \xrightarrow{\text{unitary lax}} \mathbb{C}\mathbf{at}$$

$$\mathbf{Obs}_{\mathbf{C}}(I) = \left\{ \begin{array}{ccc} S & \xrightarrow{h} & R \\ {\scriptstyle \mathrm{observe}_{\mathsf{S}}}\downarrow & & \downarrow{\scriptstyle \mathrm{observe}_{\mathsf{R}}} \\ I & = & I \end{array} \right\}$$

In this theory, a system S over the interface $I : \mathbf{C}$ is simply a state space $S : \mathbf{C}$ together with an observation $\mathrm{observe}_{\mathsf{S}} : S \to I$.

One can see maps $S \to I$ as spans $S = S \to I$, thereby fitting this example into a more general pattern of 'systems are processes with a special left boundary'

Example: observational theory

Given a span $I \xleftarrow{p_\ell} X \xrightarrow{p_r} I'$, we can reindex by pull-push (span composition):

$$\mathbf{Obs}(I \xleftarrow{p_\ell} X \xrightarrow{p_r} I') : \mathbf{Obs}(I) \longrightarrow \mathbf{Obs}(I')$$

$$\begin{array}{ccc} S & & p_\ell^* S \\ & & \downarrow{\scriptstyle p_\ell^* \mathrm{observes}_S} \\ {\scriptstyle \mathrm{observes}_S}\downarrow & \longmapsto & X \\ & & \downarrow{\scriptstyle p_r} \\ I & & I' \end{array}$$

Example: observational theory

Given a map $I \xrightarrow{h} J$, we define a profunctor:

$$\mathbf{Obs}_{\mathbf{C}}(I)^{\mathsf{op}} \times \mathbf{Obs}_{\mathbf{C}}(J) \xrightarrow{\mathbf{Obs}_{\mathbf{C}}(h)} \mathbf{Set}$$

$$\begin{array}{c} S \\ \text{observes}_{\mathsf{S}} \downarrow \\ I \end{array} \qquad \begin{array}{c} R \\ \text{observer}_{\mathsf{R}} \downarrow \\ J \end{array} \longmapsto \left\{ \begin{array}{ccc} S & \longrightarrow & R \\ \text{observes}_{\mathsf{S}} \downarrow & & \downarrow \text{observer}_{\mathsf{R}} \\ I & \xrightarrow{h} & J \end{array} \right\}$$

Example: observational theory

Finally, we have a map on squares that sends

$$
\begin{array}{ccc}
I & \xrightarrow{h} & J \\
{\scriptstyle p_\ell}\uparrow & & \uparrow{\scriptstyle q_\ell} \\
X & \overset{\sigma}{\longrightarrow} & Y \\
{\scriptstyle p_r}\downarrow & & \downarrow{\scriptstyle q_r} \\
I' & \xrightarrow[k]{} & J'
\end{array}
\quad \longmapsto \quad
\begin{array}{ccc}
\mathbf{Obs}_{\mathbf{C}}(I) & \xrightarrow{\mathbf{Obs}_{\mathbf{C}}(p)} & \mathbf{Obs}_{\mathbf{C}}(J) \\
{\scriptstyle \mathbf{Obs}_{\mathbf{C}}(h)}\downarrow & \overset{\mathbf{Obs}_{\mathbf{C}}(\sigma)}{\Longrightarrow} & \downarrow{\scriptstyle \mathbf{Obs}_{\mathbf{C}}(k)} \\
\mathbf{Obs}_{\mathbf{C}}(I') & \xrightarrow[\mathbf{Obs}_{\mathbf{C}}(q)]{} & \mathbf{Obs}_{\mathbf{C}}(J')
\end{array}
$$

again by stacking:

$$
\mathbf{Obs}_{\mathbf{C}}(h)(\mathsf{S},\mathsf{R}) \xrightarrow{\mathbf{Obs}_{\mathbf{C}}(\sigma)_{\mathsf{S},\mathsf{R}}} \mathbf{Obs}_{\mathbf{C}}(k)(\mathbf{Obs}_{\mathbf{C}}(f)(\mathsf{S}),\mathbf{Obs}_{\mathbf{C}}(g)(\mathsf{R}))
$$

$$
\begin{array}{ccc}
S & \longrightarrow & R \\
{\scriptstyle \mathrm{observe}_{\mathsf{S}}}\downarrow & {\scriptstyle \mathrm{observe}_{\mathsf{R}}} & \downarrow \\
I & \xrightarrow[h]{} & J
\end{array}
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$$

Example: observational theory

Thus we have a systems theory:

$$\mathbf{Obs}_{\mathbf{C}} : \mathbb{S}\mathbf{pan}(\mathbf{C})^{\top} \xrightarrow{\text{unitary lax}} \mathbb{C}\mathbf{at}$$

In particular, we have

$$\mathbf{Obs}_{\mathbf{Set}} : \mathbb{S}\mathbf{et}^{\top} \xrightarrow{\text{unitary lax}} \mathbb{C}\mathbf{at}$$

which on objects is defined as $I \mapsto \mathbf{Set}/I \cong \mathbf{Set}^{I}$.

When we write $\mathbf{Obs}$, this is what we mean.

Morphisms of system theories

In category theory, we are interested in the way things map into each other...

Definition

A **morphism of system theories** $\Phi : \mathbf{Sys} \to \mathbf{Sys}'$ is a vertical lax natural transformation:

$$\begin{array}{ccc} \mathbb{P}^{\top} & \xrightarrow{\Phi^{\top}} & \mathbb{P}'^{\top} \\ {\scriptstyle \mathbf{Sys}}\downarrow & \overset{\Phi^{\flat}}{\Longrightarrow} & \downarrow{\scriptstyle \mathbf{Sys}'} \\ \mathbb{C}\mathbf{at} & = & \mathbb{C}\mathbf{at} \end{array}$$

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Categorical system theory studies system theories

$$\mathbf{Sys} : \mathbb{P}^\top \xrightarrow{\text{unitary lax}} \mathbb{C}\mathbf{at}$$

through **behaviours**:

$$B : \mathbf{Sys} \longrightarrow \mathbf{Obs}$$

Corepresentable behaviours

Construction

Given $\mathbf{Sys} : \mathbb{P}^\top \to \mathbb{C}\mathbf{at}$, and a system $\mathsf{S} : \mathbf{Sys}(I)$, there is a morphism:

$$\begin{array}{ccc} \mathbb{P}^\top & \xrightarrow{\mathbb{P}_h(I,-)^\top} & \mathbb{S}\mathbf{et}^\top \\ {\scriptstyle \mathbf{Sys}}\downarrow & \overset{\mathbf{Sys}(\mathsf{S},-)}{\Longrightarrow} & \downarrow{\scriptstyle \mathbf{Obs}} \\ \mathbb{C}\mathbf{at} & = & \mathbb{C}\mathbf{at} \end{array}$$

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where, for a given $J : \mathbb{P}$,

$$\mathbf{Sys}(\mathsf{S}, -)_J : \mathbf{Sys}(J) \longrightarrow \mathbf{Set}^{\mathbb{P}_h(I,J)}$$

is a functor equivalently given as

$$\begin{array}{ccc} \mathbf{Sys}(J) \times \mathbb{P}_h(I,J) & \longrightarrow & \mathbf{Set} \\ (\mathsf{T}, \quad I \xrightarrow{h} J) & \mapsto & \mathbf{Sys}(h)(\mathsf{S},\mathsf{T}) \end{array}$$

This picks all simulations of S in T mediated by the boundary map h.

Example: fixpoints behaviours

Let fix : $\binom{1}{1} \Longrightarrow \binom{1}{1}$ be the trivial Moore machine (only one states, no input output).

Then $\mathbf{Moore}(\text{fix}, -)$ is the **behaviour of fixpoints**: given another Moore machine T : $\binom{I}{O}$, and a chart $\binom{i}{o} : \binom{1}{1} \rightrightarrows \binom{I}{O}$, the set

$$\mathbf{Moore}\binom{i}{o}(\text{fix}, \mathsf{T}) = \left\{ t \in T \;\middle|\; \begin{array}{ccc} \binom{1}{1} & \overset{\pi_2 \fatsemi s}{\underset{s}{\Longrightarrow}} & \binom{T}{T} \\ \Vert & & {\scriptstyle p^\sharp}\uparrow\downarrow{\scriptstyle p} \\ \binom{1}{1} & \overset{i}{\underset{o}{\Longrightarrow}} & \binom{I}{O} \end{array} \right\} = \{t \in T \mid p(s) = o,\ p^\sharp(t, i) = t\}$$

is the set of fixpoints for input i (and giving output o).

Example: fixpoint behaviours

Hence: **studying fixpoints of Moore machines amounts to studying**

$$\mathbf{Moore}(\mathsf{fix}, -) : \mathbf{Moore} \longrightarrow \mathbf{Obs}$$

In particular, we can automatically mint lots of compositional structure. For instance, we know for any lens $\begin{pmatrix} f^\sharp \\ f \end{pmatrix} : \begin{pmatrix} I \\ O \end{pmatrix} \leftrightarrows \begin{pmatrix} J \\ Q \end{pmatrix}$ there is a natural map

$$\begin{array}{ccc}
\mathbf{Moore}\begin{pmatrix} I \\ O \end{pmatrix} & \xrightarrow{\mathbf{Moore}(\mathsf{fix},-)_I} & \mathbf{Set}^{O\times I} \\
{\scriptstyle \mathbf{Moore}\begin{pmatrix} f^\sharp \\ f \end{pmatrix}} \downarrow & \swArrow {\scriptstyle \mathbf{Moore}\left(\mathsf{fix}, \begin{pmatrix} f^\sharp \\ f \end{pmatrix}\right)} & \downarrow {\scriptstyle f_*} \\
\mathbf{Moore}\begin{pmatrix} J \\ Q \end{pmatrix} & \xrightarrow[\mathbf{Moore}(\mathsf{fix},-)_J]{} & \mathbf{Set}^{Q\times J}
\end{array}$$

sending fixpoints of $\mathsf{S} : \mathbf{Moore}\begin{pmatrix} I \\ O \end{pmatrix}$ to fixpoints of its extension $\mathbf{Moore}\begin{pmatrix} f^\sharp \\ f \end{pmatrix}(\mathsf{S})$.

And one can prove in this case the map is iso! This recovers (D. I. Spivak 2015), and generalizes more (see (Myers 2021, Theorem 5.3.3.1)).

Recap: categorical systems theory

1. One starts by defining a **process theory**, i.e. a double category with attitude:

$$\mathbb{P} := \left\{ \begin{array}{ccc} \cdot & \xrightarrow{\text{map of interfaces}} & \cdot \\ {\scriptstyle\text{process}}\big\downarrow & \overset{\text{map of processes}}{\Longrightarrow} & \big\downarrow{\scriptstyle\text{process}} \\ \cdot & \xrightarrow{\text{map of interfaces}} & \cdot \end{array} \right\}$$

2. Then **processes are used to index systems**, giving rise to doubly indexed categories of systems:

$$\mathbf{Sys} : \mathbb{P}^{\top} \xrightarrow{\text{unitary lax}} \mathbb{C}\mathbf{at}$$

3. Finally, **behaviour is studied by describing maps into the 'observational theory'**:

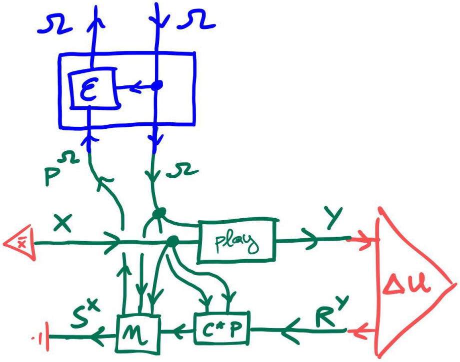
$$\begin{array}{ccc} \mathbb{P}^{\top} & \xrightarrow{B^{\top}} & \mathbb{S}\mathbf{et}^{\top} \\ {\scriptstyle\mathbf{Sys}}\big\downarrow & \overset{B^{\flat}}{\Longrightarrow} & \big\downarrow{\scriptstyle\mathbf{Obs}} \\ \mathbb{C}\mathbf{at} & = & \mathbb{C}\mathbf{at} \end{array}$$

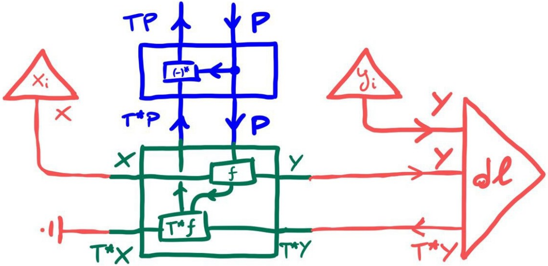
We get many of these just a corepresentable ones, i.e. maps of type $\mathbf{Sys}(\mathsf{S}, -)$.

Categorical cybernetics

Motivation

The systems of categorical system theory are varied and numerous, but they miss some interesting examples. In my work I mostly care about **games** and **learners**:





These (and others) are what I call **cybernetic systems**.

Categorical cybernetics

The conceptual foundation of ‘categorical cybernetics’, as advocated in Capucci, Gavranović, Hedges, and Rischel 2021 and in Smithe 2021 rests on two main pillars:

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1. The fact cybernetic systems are *mereologically* peculiar in having a distinctive boundary between ‘**controller**’ and ‘**controlled**’ subsystems;

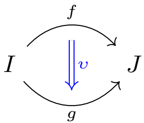
Categorical cybernetics

The conceptual foundation of 'categorical cybernetics', as advocated in Capucci, Gavranović, Hedges, and Rischel 2021 and in Smithe 2021 rests on two main pillars:

1. The fact cybernetic systems are *mereologically* peculiar in having a distinctive boundary between '**controller**' and '**controlled**' subsystems;
2. The fact cybernetic processes tend to organize in **bicategorical structures**, where the first dimension ignores the mereological distinction between controller and controllee, and the second dimension distinguishes the controller system—so that higher dimensions encode deeper control hierarchies;

The compositional structure of cybernetic processes

The basic component of a bicategory of cybernetic processes is a globular 2-cell:



The ‘cybernetic’ interpretation of such a globe is that I and J denote an interface, f and g are controllable processes between the given interfaces, and v ‘reduces’ the controllability of g to that of f, being itself a dynamical process.

The compositional structure of cybernetic processes

Their composition algebra also checks out: **vertical composition of globular cells models sequential composition of controls**, whereas horizontal composition models the 'parallel' composition of controls arising from the sequential composition of the processes they control.

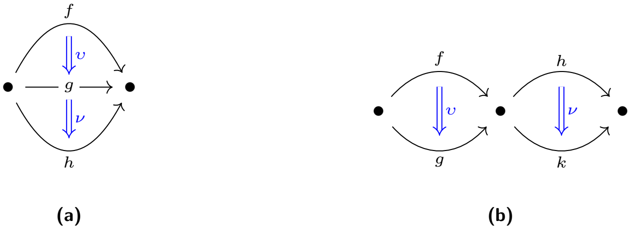
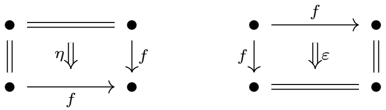


Figure: *(a) sequential composition of controls, (b) parallel composition of controls.*

The compositional structure of cybernetic processes

With little effort, we can concert controllable and simple processes in the same structure. We claim the structure of controllable processes is that a of a **company** a double category where every tight arrow has a companion.



Idea: simple processes are trivially controllable processes, so 'appear both tightly and loosely'.

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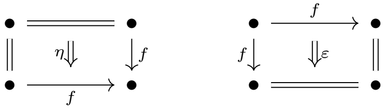
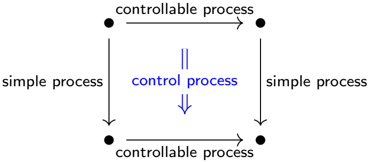


Idea: simple processes are trivially controllable processes, so 'appear both tightly and loosely'.

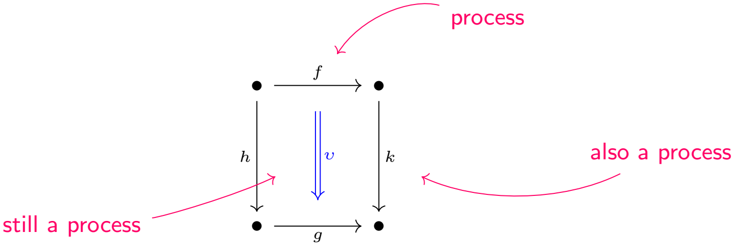
Definition

A **cybernetic company** is a a company with attitude whose objects are **interfaces**, loose arrows are **controllable processes**, tight arrows are **simple processes**, and squares are **control processes**:



Categorical cybernetics

Hence a company could be a good candidate for an objective definition of cybernetic process theory, except we lack maps of interfaces and of processes which are not themselves processes.



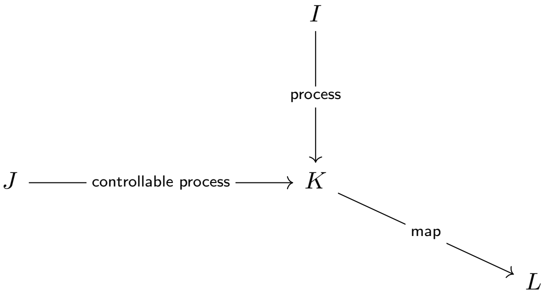
Thus we are forced to go 3D!

Cybernetic process theories

Definition

A **cybernetic process theory** is a triple category with attitude $\mathfrak{P}$ where

1. objects are **boundaries/interfaces**,
2. transversal 2-cells are **maps of boundaries**,
3. vertical 1-cells are **processes**,
4. horizontal 1-cells are **controllable processes**,

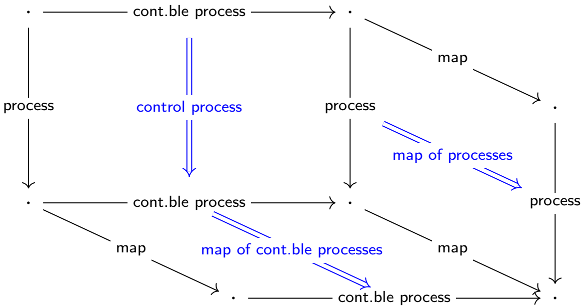


Cybernetic process theories

Definition

A **cybernetic process theory** is a triple category with attitude $\mathfrak{P}$ where

1. frontal 2-cells are **control processes** (and form a *company*),
2. vertical 2-cells are **maps of processes**,
3. horizontal 2-cells are **maps of controllable processes**



Cybernetic process theories

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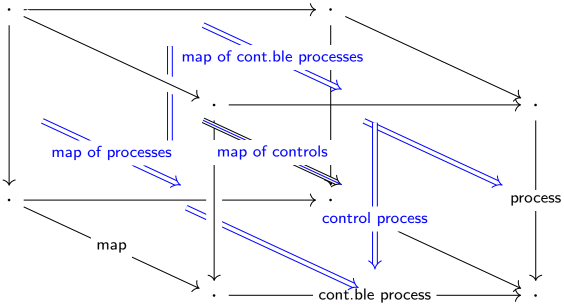
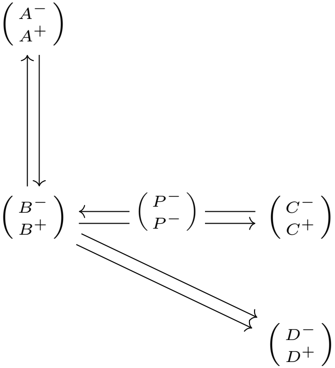
A **cybernetic process theory** is a triple category with attitude $\mathfrak{P}$ where

1. cubes are **maps of control processes**



Example: parametric lenses

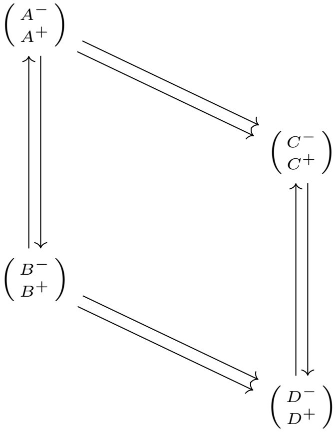
One can promote the process theory of lenses to a cybernetic theory $\mathbb{P}\mathbf{ara}(\mathbb{L}\mathbf{ens})$[1], with parametric lenses in the role of controllable processes:



[1] This notation is actually sound: there's a Para construction that yields this triple category

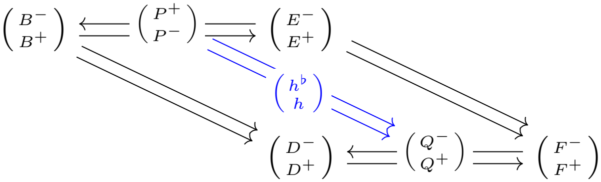
Example: parametric lenses

Vertical 2-cells are the same kind we encountered before ('commutative squares'):



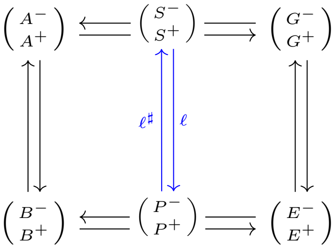
Example: parametric lenses

Horizontal 2-cells are still of the same kind, but there's an extra chart going between the parameters of the parametric lenses:



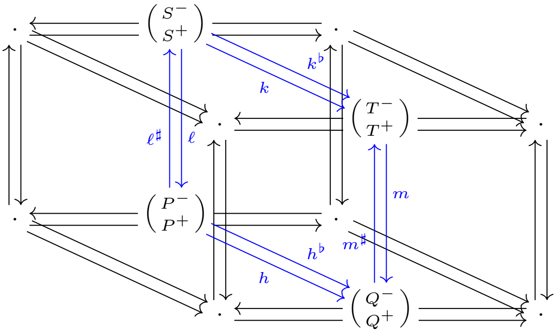
Example: parametric lenses

Basic 2-cells are commutative squares of parametric lenses:



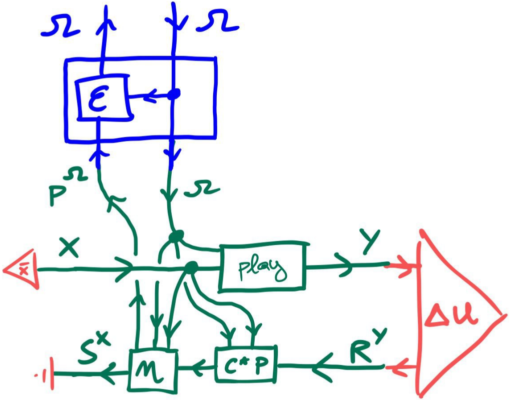
Example: parametric lenses

A cube is an arrangement of faces such that the blue parts form a square in $\mathbb{L}\mathbf{ens}$:



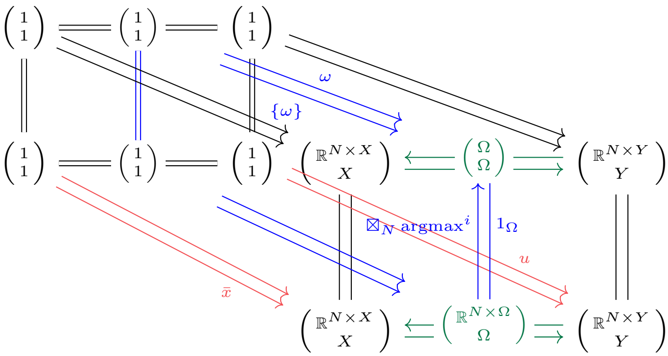
Example: Nash equilibria

Categorical game theory (Capucci 2022; Capucci, Gavranović, Hedges, and Rischel 2021; Ghani, Hedges, Winschel, and Zahn 2018) is nowadays based on the idea games are parametric lenses, controlled by players.



Example: Nash equilibria

Categorical game theory is nowadays based on the idea games are parametric lenses, controlled by players. In the same way fixpoints of Moore machines are maps (squares) from fix into them, one can show **Nash equilibria** of games are maps (cubes) from fix into them:



Here $u : Y \to \mathbb{R}^N$ is a payoff function, $\bar{x}$ an initial state, and $\bar{\omega}$ a strategy profile.

Towards behavioural cybernetics

We would like to recover the compositional formulae for Nash equilibria as functoriality of some corepresentable behaviour "$\mathbf{Plrs}(\mathsf{fix}, -)$", whatever this means.

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It takes time to unpack all of these, so unfortunately I'm only going to be able to tell you about (1)

Cybernetic system theories

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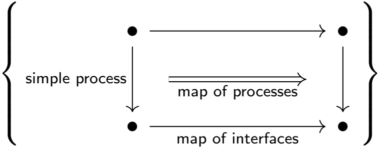
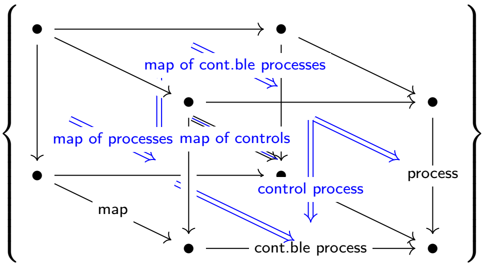


Cybernetic system theories

The crucial idea in the definition of cybernetic system theory is that a cybernetic process theory $\mathfrak{P}$ can be decomposed into two process theories:

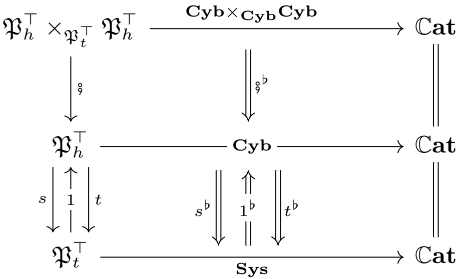
1. A **simple part**, given by the double category of interfaces, maps theoreof, simple processes and maps thereof.
2. A **cybernetic part**, whose objects are cybernetic processes, vertical 1-cells are control processes, horizontal 1-cells are maps of controllable processes, and squares are maps of control processes.

$\mathfrak{P}_h :=$



Cybernetic system theory

This suggests one can define a cybernetic system theory by putting together a system theory $\mathbf{Sys}$ on $\mathfrak{P}_t$ and a system theory $\mathbf{Cyb}$ on $\mathfrak{P}_h$:



Cybernetic system theory

One can show that $\mathbf{Cyb}$ and $\mathbf{Sys}$ arrange to yield a triple functor

$$\mathbf{CybSys} : \mathfrak{P}^\top \xrightarrow{\text{unitary lax-lax}} \mathbb{S}\mathbf{pan}(\mathbb{C}\mathbf{at})$$

where $(-)^\top$ exchanges vertical and transversal 1-cells.

Thus, together with $s^\flat$ and $t^\flat$, $\mathbf{Cyb}$ assigns a **span of functors** to each controllable process:

Idea: the category $\mathbf{Cyb}$ picks out, for a given controllable process $I \xrightarrow{p} J$, is a category control system U together with simple systems S 'closing off' p. This makes sense: **a cybernetic system is a controllable system coupled to a control system**.

$$I \xrightarrow{p} J \quad \longmapsto \quad \mathbf{Sys}(I) \xleftarrow{s^\flat_p} \mathbf{Cyb}(I \xrightarrow{p} J) \xrightarrow{t^\flat_p} \mathbf{Sys}(J)$$

Cybernetic system theory

The triple functor $\mathfrak{P}^\top \xrightarrow{\text{unitary lax}} \mathbb{S}\mathbf{pan}(\mathbb{C}\mathbf{at})$...

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$$s^\flat_{I \xrightarrow{p} J} : \mathbf{Cyb}(I \xrightarrow{p} J) \to \mathbf{Sys}(I),$$
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4. ...features a **compatibility structure with sequential composition of cybernetic processes**, given by

$$\fatsemi^\flat_{\substack{I \xrightarrow{p} J \\ J \xrightarrow{q} K}} : \mathbf{Cyb}(I \xrightarrow{p} J) \times_{\mathbf{Sys}(J)} \mathbf{Cyb}(J \xrightarrow{q} K) \longrightarrow \mathbf{Cyb}(I \xrightarrow{p} J \xrightarrow{q} K)$$

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First in line are applications to game theory, where it can be used to mint compositional structure for equilibrium concepts.

Secondly, how can this framework be useful for other cybernetic systems theories? Can it be used to frutifully map between learning & games? Can we use it to prove theorems across different kinds of systems? (e.g. good regulator theorems/internal model principles/FEP).

Thanks for your attention!

Questions?

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