REPRESENTATION THEOREM FOR ENRICHED CATEGORIES

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I. MOTIVATION

- Universal properties are useful to make many constructions in mathematics

  Ex: (co)limits, adjunctions, Kan extensions, ...

- Encoded by the representability of a certain presheaf

  Ex: $G: I \to \mathcal{C}$ has a limit $L \in \mathcal{C} \iff \mathcal{C}(-, L) \cong \mathcal{C}^I(\Delta(-), G)$.

- Not always convenient to work with.

  Ex: in ∞-category theory, homotopy type theory.

- Looking for an internal characterization.

  Ex: Limit of a functor G $\leadsto$ terminal object in the category of cones $\Delta \downarrow G$.

II. 1-CATEGORICAL STORY

$\mathcal{C}$ - category

A presheaf $F : \mathcal{C}^{op} \to \mathrm{Set}$ is representable if there is:

- an object $L \in \mathcal{C}$, and
- a natural isomorphism $\mathcal{C}(-, L) \cong F \quad (\in \mathrm{Set}^{\mathcal{C}^{op}})$

Examples:

- **Limit:** $G : I \to \mathcal{C}$ - functor

  set of cones

  $F := \mathcal{C}^I(\Delta(-), G) : \mathcal{C}^{op} \to \mathrm{Set}, \quad C \mapsto \mathcal{C}^I(\Delta C, G)$

  G has a limit $\iff$ F is representable.

- **Adjunction:** $L : \mathcal{C} \to \mathcal{D}$ - functor

  For $D \in \mathcal{D}$, $F_D := \mathcal{D}(L(-), D) : \mathcal{C}^{op} \to \mathrm{Set}, \quad C \mapsto \mathcal{D}(LC, D)$

  L has a right adjoint $\iff$ F_D is representable $\forall D \in \mathcal{D}$.

Grothendieck construction: $\int_{\mathcal{C}} : \mathrm{Set}^{\mathcal{C}^{op}} \longrightarrow \mathrm{Cat}/\mathcal{C}$

$F : \mathcal{C}^{op} \longrightarrow \mathrm{Set} \quad \longmapsto \quad \int_{\mathcal{C}} F \longrightarrow \mathcal{C}$

where $\int_{\mathcal{C}} F$ is the category of elements of F

- obj: pairs (C,x) of $C \in \mathcal{C}$, $x \in FC$
- mor $(C,x) \to (D,y)$: $C \xrightarrow{f} D \in \mathcal{C}$ st $Ff(y) = x$ $\quad (FD \xrightarrow{Ff} FC)$

Remark: It corresponds to the category:

$$\begin{tikzcd} \coprod_{C \in \mathcal{C}} FC \arrow[r, leftarrow, shift left=2, "s"] \arrow[r, "i" description] \arrow[r, leftarrow, shift right=2, "t"'] & \coprod_{C,D \in \mathcal{C}} \mathcal{C}(C,D) \times FD & \cdots \arrow[l, "c"'] \end{tikzcd} \in \mathrm{Set}$$

Example: Limit: $G : I \longrightarrow \mathcal{C}$; $F = \mathcal{C}^I(\Delta(-), G)$

$\int_{\mathcal{C}} F \cong \Delta \downarrow G$ is the category of cones over G

- obj: pairs (C, κ) of $C \in \mathcal{C}$, $\kappa : \Delta C \Rightarrow G$
- mor $(C,\kappa) \longrightarrow (D,\mu)$: $C \xrightarrow{f} D \in \mathcal{C}$ st

$$\begin{tikzcd} \Delta C \arrow[rr, Rightarrow, "\Delta f"] \arrow[dr, Rightarrow, "\kappa"'] & & \Delta D \arrow[dl, Rightarrow, "\mu"] \\ & G & \end{tikzcd}$$

Theorem: $\int_{\mathcal{C}} : \mathrm{Set}^{\mathcal{C}^{op}} \longrightarrow \mathrm{Cat}/_{\mathcal{C}}$ is fully faithful.

A functor $P : \mathcal{A} \to \mathcal{C}$ is a discrete fibration if for all $A \in \mathcal{A}$ and $C \xrightarrow{g} PA \in \mathcal{C}$, there is a unique $B \xrightarrow{f} A \in \mathcal{A}$ st $Pf = g$

Remark: This is equivalent to saying that

$$\begin{tikzcd} \mathrm{mor}\mathcal{A} \arrow[r, "P"] \arrow[d, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{mor}\mathcal{C} \arrow[d, "t"] \\ \mathrm{ob}\mathcal{A} \arrow[r, "P"'] & \mathrm{ob}\mathcal{C} \end{tikzcd}$$

is a pullback in Set

Example: Grothendieck construction

$F : \mathcal{C}^{op} \to \mathrm{Set}$, $\int_{\mathcal{C}} F \to \mathcal{C}$ is a discrete fibration

$$\begin{tikzcd} \coprod_{C \in \mathcal{C}} \mathcal{C}(C,D) \times FD \arrow[r] \arrow[d, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{mor}\mathcal{C} = \coprod_{C,D \in \mathcal{C}} \mathcal{C}(C,D) \arrow[d, "t"] \\ \coprod_{C \in \mathcal{C}} FC \arrow[r] & \mathrm{ob}\mathcal{C} = \coprod_{C \in \mathcal{C}} * \end{tikzcd}$$

<u>Notation</u>: $\mathrm{Fib}_{/\mathcal{C}} \subseteq \mathrm{Cat}_{/\mathcal{C}}$ full subcategory spanned by the discrete fibrations

<u>Theorem</u>: $\int_{\mathcal{C}} : \mathrm{Set}^{\mathcal{C}^{op}} \to \mathrm{Fib}_{/\mathcal{C}}$ is an equivalence of categories.

<u>Representation theorem</u>: A presheaf $F : \mathcal{C}^{op} \to \mathrm{Set}$ is representable $\Leftrightarrow \int_{\mathcal{C}} F$ has a terminal object.

<u>Example</u>: Limit: $G : I \to \mathcal{C}$, $F := \mathcal{C}^{I}(\Delta(-), G)$

G has a limit $\Longleftrightarrow$ F is representable
$\Longleftrightarrow \int_{\mathcal{C}} F = \Delta \downarrow G$ has a terminal object

<u>Remark</u>: An object T in a category $\mathcal{C}$ is terminal $\Longleftrightarrow$ the projection $\mathcal{C}_{/T} \to \mathcal{C}$ is an isomorphism of categories.

$$\begin{tikzcd} \mathcal{C}_{/T} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{2} \arrow[d] \\ \mathcal{C} \arrow[r, "(id, \{T\})"'] & \mathcal{C} \times \mathcal{C} \end{tikzcd} \in \mathrm{Cat}$$

III PROBLEMS IN HIGHER DIMENSIONS

A **2-category** $\mathcal{C}$ is a category enriched in Cat, i.e., it consists of:

$$C \to D \ + \ \begin{tikzcd} C \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & D \end{tikzcd}$$

- a set of objects $\mathrm{ob}\,\mathcal{C}$
- a **category** of morphisms $\mathcal{C}(C,D)$, $\forall C,D \in \mathcal{C}$

\+ associative and unital compositions

<u>Example</u>: **Cat** - 2-category of categories, functors, and natural transformations

$\mathcal{C}$ - 2-category

A **2-presheaf** on $\mathcal{C}$ is a 2-functor $F : \mathcal{C}^{op} \to \mathrm{Cat}$.

<u>Example</u>: **Representable** 2-presheaf: for $L \in \mathcal{C}$,

$$\mathcal{C}(-,L) : \mathcal{C}^{op} \to \mathrm{Cat}, \quad C \mapsto \mathcal{C}(C,L)$$

A 2-presheaf $F:\mathcal{C}^{op}\to \mathrm{Cat}$ is **representable** if there is:

- an object $L\in\mathcal{C}$
- a 2-natural isomorphism $\mathcal{C}(-,L)\cong F$ $(\in 2\mathrm{Cat}(\mathcal{C}^{op},\mathrm{Cat}))$

<u>Example</u>: **2-limit**: $G: I\to\mathcal{C}$ - 2-functor

$F := \mathcal{C}^I(\Delta(-),G):\mathcal{C}^{op}\longrightarrow\mathrm{Cat}$, $C\mapsto \mathcal{C}^I(\Delta C, G)$ $\leftarrow$ category of 2-cones & modifications

G has a 2-limit $\iff$ F is representable

$L\in\mathcal{C}$ - 2-limit of G; then $\forall C\in\mathcal{C}$

$$\mathcal{C}(C,L)\xrightarrow{\cong}\mathcal{C}^I(\Delta C,G) \quad \in \mathrm{Cat}$$

$$C\to L \;\leftrightsquigarrow\; \Delta C\Rightarrow G \quad (=\Delta C\Rightarrow \Delta L\Rightarrow G)$$

$$\begin{tikzcd} C \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & L \end{tikzcd} \;\leftrightsquigarrow\; \begin{tikzcd} \Delta C \arrow[r, Rightarrow, bend left] \arrow[r, Rightarrow, bend right] \arrow[r, phantom, "\Rrightarrow"] & G \end{tikzcd} \quad \left(= \begin{tikzcd} \Delta C \arrow[r, Rightarrow, bend left] \arrow[r, Rightarrow, bend right] \arrow[r, phantom, "\Rrightarrow"] & \Delta L \arrow[r, Rightarrow] & G \end{tikzcd}\right)$$

An object T in a 2-category $\mathcal{C}$ is **2-terminal** if $\mathcal{C}(C,T)\cong\mathbb{1}$ $\forall C\in\mathcal{C}$

<u>Try 1</u>: $\int_{\mathcal{C}} : 2\mathrm{Cat}(\mathcal{C}^{op}, \mathrm{Cat}) \to 2\mathrm{Cat}/_{\mathcal{C}}$, $F : \mathcal{C}^{op} \to \mathrm{Cat} \mapsto \int_{\mathcal{C}} F \to \mathcal{C}$

where $\int_{\mathcal{C}} F$ is the (strict) 2-category of elements of F

- obj: pairs (C, x) of $C \in \mathcal{C}$, $x \in FC$
- mor $(C,x) \to (D,y)$: $C \xrightarrow{f} D \in \mathcal{C}$ st $Ff(y) = x$
- 2-mor $\begin{tikzcd}(C,x) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & (D,y)\end{tikzcd}$: $\begin{tikzcd}C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & D\end{tikzcd} \in \mathcal{C}$ st $(F\alpha)_y = \mathrm{id}_x$ $\quad (Ff \overset{F\alpha}{\Longrightarrow} Fg)$

⚠ Does not see the morphisms of FC

E.g. 2-limit of G $\cancel{\Longleftarrow}$ $\Longrightarrow$ 2-terminal object in the (strict) 2-category of cones $\Delta \downarrow G = \int_{\mathcal{C}} \mathcal{C}^{I}(\Delta(-), G)$.

- obj: pairs (C, κ) of $C \in \mathcal{C}$, $\kappa : \Delta C \Rightarrow G$
- mor $(C,\kappa) \to (D,\mu)$: $C \xrightarrow{f} D \in \mathcal{C}$ st $\begin{tikzcd}\Delta C \arrow[rr, Rightarrow, "\Delta f"] \arrow[dr, Rightarrow, "\kappa"'] & & \Delta D \arrow[dl, Rightarrow, "\mu"] \\ & G & \end{tikzcd}$
- 2-mor $\begin{tikzcd}(C,\kappa) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & (D,\mu)\end{tikzcd}$: $\begin{tikzcd}C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & D\end{tikzcd} \in \mathcal{C}$ st $\begin{tikzcd}\Delta C \arrow[rr, Rightarrow, bend left, "\Delta f"] \arrow[rr, Rightarrow, bend right, "\Delta g"'] \arrow[rr, phantom, "\Downarrow\Delta\alpha"] \arrow[dr, Rightarrow, "\kappa"'] & & \Delta D \arrow[dl, Rightarrow, "\mu"] \\ & G & \end{tikzcd}$

[Clingman-M.]: counterexamples.

<u>Try 2:</u> $\int_{\mathcal{C}}^{\mathrm{lax}} : 2\mathrm{Cat}(\mathcal{C}^{op},\mathrm{Cat}) \to 2\mathrm{Cat}/_{\mathcal{C}}$, $F\colon \mathcal{C}^{op} \to \mathrm{Cat} \mapsto \int_{\mathcal{C}}^{\mathrm{lax}} F \to \mathcal{C}$

where $\int_{\mathcal{C}}^{\mathrm{lax}} F$ is the lax 2-category of elements of F

- mor $(C,x) \to (D,y)$: pairs (f,φ) of $C \xrightarrow{f} D \in \mathcal{C}$, $Ff(y) \xrightarrow{\varphi} x \in FC$

<u>Theorem:</u> [Buckley] $\int_{\mathcal{C}}^{\mathrm{lax}} : 2\mathrm{Cat}(\mathcal{C}^{op},\mathrm{Cat}) \to 2\mathrm{Cat}/_{\mathcal{C}}$ is fully faithful.

But: 2-limit of $G \not\Leftarrow\!\not\Rightarrow$ 2-terminal object in the lax 2-category of cones $\Delta 1 \downarrow^{\mathrm{lax}} G = \int_{\mathcal{C}}^{\mathrm{lax}} \mathcal{C}^{\mathcal{I}}(\Delta(-),G)$

- mor $(C,\kappa) \to (D,\mu)$: pairs (f,φ) of $C \xrightarrow{f} D$,

$$\begin{tikzcd} \Delta C \arrow[rr, Rightarrow, "\Delta f"] \arrow[dr, Rightarrow, "\kappa"'] & \arrow[d, phantom, "\Leftarrow\varphi"] & \Delta D \arrow[dl, Rightarrow, "\mu"] \\ & G & \end{tikzcd}$$

[Clingman-M.]: counterexamples

<u>Other approach:</u> [Gagna-Harpaz-Lanari]

2-limit of $G \overset{\sim}{\Leftrightarrow}$ "2-final" object in $\Delta 1 \downarrow^{\mathrm{lax}} G$ with respect to cartesian morphisms (φ invertible)

IV SOLUTION: DOUBLE CATEGORIES

A double category $\mathbb{A}$ is an internal category to categories

$$\mathbb{A}_0 \overset{s}{\underset{t}{\leftleftarrows}} \xrightarrow{i} \mathbb{A}_1 \xleftarrow{c} \mathbb{A}_1 \times_{\mathbb{A}_0} \mathbb{A}_1 \in \mathcal{C}at$$

category of objects & vertical morphisms

category of horizontal morphisms & squares

There is a (horizontal) embedding

$$\mathrm{Int}: 2\mathcal{C}at \to \mathrm{Dbl}\mathcal{C}at, \quad \mathcal{C} \mapsto \mathrm{ob}\,\mathcal{C} \overset{s}{\underset{t}{\leftleftarrows}} \xrightarrow{i} \coprod_{C,D \in \mathcal{C}} \mathcal{C}(C,D)$$

Sees $\mathcal{C}$ as a double cat. with trivial vertical morphisms

The functor Int has a right adjoint

$$\mathrm{Enr}: \mathrm{Dbl}\mathcal{C}at \to 2\mathcal{C}at, \quad \mathbb{A} \mapsto \begin{cases} \mathrm{ob}(\mathbb{A}_0) \\ \mathrm{Enr}\,\mathbb{A}(A,B) = \mathbb{A}_1 \times_{\mathbb{A}_0 \times \mathbb{A}_0} \{(A,B)\} \end{cases}$$

Forgets the vertical morphisms of $\mathbb{A}$.

[Paré] $\iint_{\mathcal{C}} : 2\mathrm{Cat}(\mathcal{C}^{op}, \mathrm{Cat}) \to \mathrm{DblCat}/_{\mathrm{Int}\,\mathcal{C}}$

$F : \mathcal{C}^{op} \to \mathrm{Cat} \mapsto \iint_{\mathcal{C}} F \to \mathrm{Int}\,\mathcal{C}$ where $\iint_{\mathcal{C}} F$ has:

where $\iint_{\mathcal{C}} F$ is the double category of elements of F

- obj: pairs (C,x) of $C \in \mathcal{C}$, $x \in FC$
- hor mor $(C,x) \to (D,y)$: $C \xrightarrow{f} D \in \mathcal{C}$ st $Ff(y) = x$
- ver mor $(C,x) \nrightarrow (C,x')$: $x \xrightarrow{\varphi} x' \in FC$
- sq

$$\begin{tikzcd}
(C,x) \arrow[r, "f"] \arrow[d, "\varphi"', "|" marking] \arrow[dr, phantom, "\Downarrow"] & (D,y) \arrow[d, "\psi", "|" marking] \\
(C,x') \arrow[r, "g"'] & (D,y')
\end{tikzcd}
: \quad
\begin{tikzcd}
C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & D
\end{tikzcd} \in \mathcal{C} \text{ st }
\begin{tikzcd}
x = Ff(y) \arrow[r, "Ff(\psi)"] \arrow[d, "(F\alpha)_y"'] \arrow[dr, "\varphi"] & Ff(y') \arrow[d, "(F\alpha)_{y'}"] \\
Fg(y) \arrow[r, "Fg(\psi)"'] & Fg(y') = x'
\end{tikzcd}$$

<u>Remark</u>: It corresponds to the double category:

$$\coprod_{C \in \mathcal{C}} FC \overset{s}{\underset{t}{\leftleftarrows}} \xrightarrow{i} \coprod_{C,D \in \mathcal{C}} \mathcal{C}(C,D) \times FD \xleftarrow{c} \cdots \in \mathrm{Cat}$$

<u>Observation</u>: $\int_{\mathcal{C}} = \mathrm{Enr} \iint_{\mathcal{C}}$

[Paré] An object T in a double category $\mathbb{A}$ is **double terminal** if

- $\forall A \in \mathbb{A}, \exists! A \to T \in \mathbb{A}$
- $\forall \begin{tikzcd} A \arrow[d, "|" marking] \\ B \end{tikzcd} \in \mathbb{A}, \exists! \begin{tikzcd} A \arrow[r] \arrow[d, "|" marking] \arrow[dr, phantom, "\Downarrow"] & T \arrow[d, equals, "|" marking] \\ B \arrow[r] & T \end{tikzcd} \in \mathbb{A}$

<u>Remark</u>: This is equivalent to requiring that the projection $\mathbb{A}/_T \to \mathbb{A}$ is an isomorphism of double categories.

$$\begin{tikzcd} \mathbb{A}/_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{A}^{\mathrm{Int}\,2} \arrow[d] \\ \mathbb{A} \arrow[r, "{(id, \{T\})}"'] & \mathbb{A} \times \mathbb{A} \end{tikzcd} \in \mathrm{DblCat}$$

bilimits, pseudo-functors

<u>Theorem</u>: [Lingman-M.] A 2-presheaf $F: \mathcal{C}^{op} \to \mathrm{Cat}$ is representable $\Leftrightarrow \iint_{\mathcal{C}} F$ has a double terminal object.

<u>Example</u>: 2-limits: $G: \mathcal{I} \to \mathcal{C}$, $F := \mathcal{C}^{\mathcal{I}}(\Delta(-), G)$

G has a 2-limit $\Leftrightarrow$ the double category of cones $\Delta \downarrow\downarrow G = \iint_{\mathcal{C}} F$ has a double terminal object

- ver mor $(C, \kappa) \to (C, \kappa')$: $\begin{tikzcd} \Delta C \arrow[r, bend left, "\kappa"] \arrow[r, bend right, "\kappa'"'] \arrow[r, phantom, "\Downarrow\Delta v"] & G \end{tikzcd}$

Theorem: [Clingman-M.] Let $\mathbb{A}$ be a double category with tabulators. Then an object $T \in \mathbb{A}$ is double terminal $\iff T \in \operatorname{Enr}\mathbb{A}$ is 2-terminal.

Lemma: Let $F : \mathcal{C}^{op} \to \mathcal{C}at$ be a 2-presheaf. Suppose that $\mathcal{C}$ has tensors by $\mathbb{2}$ and F preserves them, then $\iint_{\mathcal{C}} F$ has tabulators.

Cor: Under the hypotheses of the Lemma, F is representable $\iff \int_{\mathcal{C}} F = \operatorname{Enr} \iint_{\mathcal{C}} F$ has a 2-terminal object.

Observation: $G : I \to \mathcal{C}$ - 2-functor.
If $\mathcal{C}$ has tensors by $\mathbb{2}$, then $F = \mathcal{C}^I(\Delta(-), G)$ preserves them.

Cor: Let $G : I \to \mathcal{C}$ be a 2-functor and suppose that $\mathcal{C}$ has tensors by $\mathbb{2}$. Then G has a 2-limit $\iff$ the 2-category of cones $\Delta \downarrow G$ has a 2-terminal object.

V. CASE OF ENRICHED CATEGORIES

$(\mathcal{V}, \times, 1)$ - cartesian closed category with pullbacks.

Need $\mathcal{V}$ to be extensive, i.e., it has all small coproducts that "behave well"

Consequence: $\mathrm{Set} \underset{\perp}{\hookrightarrow} \mathcal{V}$, $S \mapsto \coprod_S 1$

$U = \mathcal{V}(1,-)$

A $\mathcal{V}$-enriched category $\mathcal{C}$ consists of:

- a set of objects
- a hom-object $\mathcal{C}(C,D) \in \mathcal{V}$ $\forall C, D \in \mathcal{C}$

+ associative and unital compositions

Example: $\mathcal{V}$ is $\mathcal{V}$-enriched with:

- object set $\mathrm{ob}\mathcal{V}$
- homs Y^X given by the internal homs $\forall X, Y \in \mathcal{V}$

Remark: $\mathrm{Set} \hookrightarrow \mathcal{V}$ induces an inclusion $\mathrm{Cat} \hookrightarrow \mathcal{V}\text{-}\mathrm{Cat}$

$\mathcal{C}$ – $\mathcal{V}$-enriched category

A $\mathcal{V}$-presheaf on $\mathcal{C}$ is a $\mathcal{V}$-functor $F: \mathcal{C}^{op} \to \mathcal{V}$

Example: Representable $\mathcal{V}$-presheaf: for $L \in \mathcal{C}$,

$$\mathcal{C}(-,L): \mathcal{C}^{op} \to \mathcal{V}, \quad C \mapsto \mathcal{C}(C,L).$$

A $\mathcal{V}$-presheaf $F: \mathcal{C}^{op} \to \mathcal{V}$ is representable if there is:
- an object $L \in \mathcal{C}$
- a $\mathcal{V}$-natural isomorphism $\mathcal{C}(-,L) \cong F \quad (\in \mathcal{V}\text{-}\mathrm{Cat}(\mathcal{C}^{op}, \mathcal{V}))$

Example: $\mathcal{V}$-enriched limits: $G: \mathcal{I} \to \mathcal{C}$ – $\mathcal{V}$-functor

$$F := \mathcal{C}^{\mathcal{I}}(\Delta(-), G): \mathcal{C}^{op} \to \mathcal{V}, \quad C \mapsto \mathcal{C}^{\mathcal{I}}(\Delta C, G)$$

object of $\mathcal{V}$-cones in $\mathcal{V}$

G has a $\mathcal{V}$-enriched limit $\Leftrightarrow$ F is representable.

An internal category to $\mathcal{V}$ $\mathbb{A}$ is a diagram

$$\mathbb{A}_0 \overset{s}{\underset{t}{\leftleftarrows}} \!\!\xrightarrow{i}\! \mathbb{A}_1 \xleftarrow{c} \mathbb{A}_1 \underset{\mathbb{A}_0}{\times} \mathbb{A}_1 \in \mathcal{V} \quad \text{+relations}$$

There is an embedding

$$\mathrm{Int}: \mathcal{V}\text{-}\mathrm{Cat} \to \mathrm{Cat}(\mathcal{V}),\ \mathcal{C} \mapsto \mathrm{ob}\,\mathcal{C} \overset{s}{\underset{t}{\leftleftarrows}} \!\!\xrightarrow{i}\! \coprod_{C,D\in\mathcal{C}} \mathcal{C}(C,D)$$

[Cottrell–Fujii–Power] Int has a right adjoint

$$\mathrm{Enr}: \mathrm{Cat}(\mathcal{V}) \to \mathcal{V}\text{-}\mathrm{Cat},\ \mathbb{A} \mapsto \begin{cases} U\mathbb{A}_0 \\ \mathrm{Enr}\mathbb{A}(A,B) = \mathbb{A}_1 \underset{\mathbb{A}_0\times\mathbb{A}_0}{\times} \{(A,B)\} \end{cases}$$

<u>Grothendieck construction:</u>

$$\textstyle\int_{\mathcal{C}}: \mathcal{V}\text{-}\mathrm{Cat}(\mathcal{C}^{op},\mathcal{V}) \to \mathrm{Cat}(\mathcal{V})/_{\mathrm{Int}\mathcal{C}},\ F:\mathcal{C}^{op}\to\mathcal{V} \mapsto \int_{\mathcal{C}} F \to \mathrm{Int}\,\mathcal{C}$$

where $\int_{\mathcal{C}} F$ is the internal category to $\mathcal{V}$

$$\coprod_{C\in\mathcal{C}} FC \overset{s}{\underset{t}{\leftleftarrows}} \!\!\xrightarrow{i}\! \coprod_{C,D\in\mathcal{C}} \mathcal{C}(C,D)\times FD \xleftarrow{c} \cdots \in \mathcal{V}$$

<u>Remark</u>: [Beardsley–Wong] have a "lax" version

Theorem: [M.-Sarazola-Verdugo]
$\int_{\mathcal{C}} : \mathcal{V}\text{-Cat}(\mathcal{C}^{op}, \mathcal{V}) \longrightarrow \mathrm{Cat}(\mathcal{V})/_{\mathrm{Int}\mathcal{C}}$ is fully faithful

An internal functor $P : \mathbb{A} \to \mathrm{Int}\mathcal{C}$ in $\mathrm{Cat}(\mathcal{V})$ is a discrete fibration if

$$\begin{tikzcd} \mathbb{A}_1 \arrow[r, "P"] \arrow[d, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathrm{Int}\mathcal{C})_1 = \coprod_{C,D \in \mathcal{C}} \mathcal{C}(C,D) \arrow[d, "t"] \\ \mathbb{A}_0 \arrow[r, "P"'] & (\mathrm{Int}\mathcal{C})_0 = \mathrm{ob}\mathcal{C} \end{tikzcd}$$

is a pullback in $\mathcal{V}$

Example: Grothendieck construction

$F : \mathcal{C}^{op} \to \mathcal{V}$, $\int_{\mathcal{C}} F \to \mathrm{Int}\mathcal{C}$ is a discrete fibration

Notation: $\mathrm{Fib}/_{\mathcal{C}} \subseteq \mathrm{Cat}(\mathcal{V})/_{\mathrm{Int}\mathcal{C}}$ full subcategory spanned by the discrete fibrations over $\mathrm{Int}\mathcal{C}$

Theorem: [M.-Sarazola-Verdugo]
$\int_{\mathcal{C}} : \mathcal{V}\text{-Cat}(\mathcal{C}^{op}, \mathcal{V}) \longrightarrow \mathrm{Fib}/_{\mathcal{C}}$ is an equivalence of categories.

An object T in an internal category $\mathbb{A}$ to $\mathcal{V}$ is **terminal** if $\mathbb{A}_{/T} \longrightarrow \mathbb{A}$ is an isomorphism in $\mathrm{Cat}(\mathcal{V})$.

$$\begin{tikzcd}
\mathbb{A}_{/T} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{A}^{\mathrm{Int}\,2} \arrow[d] \\
\mathbb{A} \arrow[r, "{(\mathrm{id},\{T\})}"'] & \mathbb{A} \times \mathbb{A}
\end{tikzcd} \quad \in \mathrm{Cat}(\mathcal{V})$$

Theorem: [M.-Sarazola-Verdugo]
A $\mathcal{V}$-presheaf $F : \mathcal{C}^{op} \to \mathcal{V}$ is representable $\Longleftrightarrow$ $\int_e F$ has a terminal object.

Example: $\mathcal{V}$-limits: $G : I \to \mathcal{C}$, $F := \mathcal{C}^I(\Delta(-), G)$

G has a $\mathcal{V}$-limit $\Longleftrightarrow$ the internal category of $\mathcal{V}$-cones $\Delta \downarrow G := \int_e F$ has a terminal object.

Conj: If $\mathcal{V}$ is "generated" by $\{X_i\} \subseteq \mathcal{V}$, $\mathcal{C}$ has tensors by the X_i's and $F : \mathcal{C}^{op} \to \mathcal{V}$ preserves them, then F is representable $\Longleftrightarrow$ $\mathrm{Enr} \int_e \mathcal{V}$ has a $\mathcal{V}$-terminal object T

homs to T are isomorphic to 1

VI. APPLICATIONS

- $\mathcal{V} = \mathrm{Set}$: Set-Cat = Cat = Cat(Set)
  $\rightsquigarrow$ retrieve 1-categorical case
- $\mathcal{V} = \mathrm{Cat}$: Cat-Cat = 2Cat, Cat(Cat) = DblCat
  $\rightsquigarrow$ another proof of the representation theorem
- $\mathcal{V} = (n-1)\mathrm{Cat}$: $(n-1)$Cat-Cat = n-Cat, Cat($(n-1)$Cat)
  $\rightsquigarrow$ the Grothendieck construction takes values in internal categories to $(n-1)$-categories ($\neq$ n-fold categories)
  $\rightsquigarrow$ n-limits $\leftrightarrow$ terminal objects in the internal category to $(n-1)$Cat of n-cones
  $\rightsquigarrow$ with Rasekh & Rovelli, we construct limits in (∞,n)-categories using this characterization.
  $\rightsquigarrow$ n-limits in an n-category with tensors by $\Sigma^{n-1}\mathbb{1}$
  $\leftrightarrow$ n-terminal objects in the n-category of n-cones.