How long does it take to frame a bicategory?

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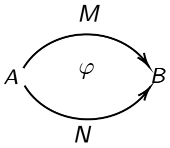
Plan for the talk

1. Mod-like bicategories.
2. Globularly generated double categories.
3. Length
4. Free globularly generated double category.
5. Canonical projection.
6. Framed bicategories.

Mod-like bicategories

The bicategory of algebras

Mod denotes the bicategory whose 2-cells are of the form:

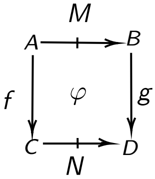


where A and B are unital complex algebras, ${}_A M_B$ and ${}_A N_B$ are bimodules and $\varphi : M \to N$ is a bimodule morphism, i.e. $\varphi(axb) = a\varphi(x)b$.Horizontal identity and horizontal composition in **Mod** are ${}_A A_A$ and $M \otimes_B N$.

A, B algebras. A, B are isomorphic in **Mod** if and only if A and B are Morita equivalent. M is adjoint if and only if M f.g. projective.

The double category of algebras

[**Mod**] denotes the double category whose squares are of the form:



where A, B, C and D are algebras, ${}_A M_B$ and ${}_C N_D$ are bimodules, $f : A \to C$ and $g : B \to D$ are unital algebra morphisms, and $\varphi : M \to N$ is a linear transformation such that the equation:

$$\varphi(a\xi b) = f(a)\varphi(\xi)g(b)$$

holds i.e. the squares of [**Mod**] are equivariant bimodule morphisms.
Horizontal identity and horizontal composition in [**Mod**] are defined by the obvious functorial extensions of $A \mapsto_A A_A$ and $(M_B,_B N) \mapsto M \otimes_B N$.

Relation between **Mod**]*and***Mod**: $H[\mathbf{Mod}] = \mathbf{Mod}$.

Symmetric monoidal structure on **Mod**

What do we get from $H[\mathbf{Mod}] = \mathbf{Mod}$?

Tensor product of algebras, etc. provides **Mod** with the structure of a symmetric monoidal bicategory. Coherence is in the form of invertible bimodules satisfying a bunch of very complicated equations presented in, e.g. [Kapranov, Voevodski 94'], [Schommer-Pries 11'], [Stay 13'] Coherence data for $\otimes$ of algebras is naturally defined in terms of unital morphisms, and satisfies MacLane equations strictly. Need a different language to express this more efficiently.

Tensor product on vertices, edges and squares of $[\mathbf{Mod}]$ provide $[\mathbf{Mod}]$ with the structure of a symmetric monoidal double category. $[\mathbf{Mod}]$ is framed bicategory. Thus coherence isomorphisms of $[\mathbf{Mod}]$ descend to coherence isomorphisms of a symmetric monoidal structure on **Mod** with tensor porduct $H\otimes$. [Shulman10'] $[\mathbf{Mod}]$ is the correct framework to equip algebras with a 2 dim symmetric monoidal structure.

Mod-like bicategories

Observation: There are two types of bicategories, exemplified by **Cat** and **Mod**. **Cat** has objects, 'functions' between objects as 1-morphisms, and morphisms between these 'functions' as 2-morphisms. **Mod** has objects, other types of 'parametrized' objects as 1-morphisms, and 'functions' between 1-dimensional 'objects' as 2-morphisms. There is a correct/natural notion of morphism between objects in **Mod**, not directly included in **Mod**.

Bicategories fitting the above description of **Mod** are called **Mod-like bicategories** in [Shulman 08']. Bicategories whose objects are algebras of some sort, 1-morphisms are bimodules, and 2-morphisms bimodule morphisms are **Mod**-like

Slogan: A **Mod**-like bicategory B should have a category of 'function/correct' morphisms B^*. There should be a clear lift of B to a double category C, such that $C_0 = B^*$ and such that $HC = B$. A natural symmetric monoidal structure on B should better be expressed as a symmetric monoidal structure on C.

An odd example

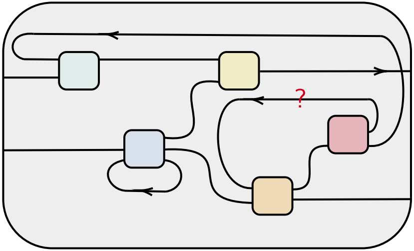
von Neumann algebras are a type of topological algebras used in conformal field theory, low dimensional topology, etc. non-commutative measure spaces. Natural notion of bimodule, bimodule morphism (intertwiners), relative tensor product (CFTP) and relative tensor unit (Haagerup standard form).

von Neumann algebras, bimodules and intertwiners form a bicateogory W^*. Horizontal composition is CFTP, horizontal identity Haagerup standard form. Clearly a **Mod**-like bicategory.

Tensor product of von Neumann algebras, bimodules and intertwiners should make W^* into a symmetric monoidal bicategory. Coherence data in terms of morphisms of von Neumann algebras. Same situation as with **Mod**. **Strategy:** Lift W^* to a double category (framed bicat.) $[W^*]$ of equivariant bimodule morphisms. Problem already considered by [Bartels, Douglas, Hénriques, 14']. **Problem:** No analytic tools to define U nor $\odot$ on all morphisms. **Solution:** define $[W^*]$ where there are tools, i.e. finite Jones index techniques (Factors and finite Jones index). Obtain BDH.

How do we find a double cat of vN algebras?

The theory of von Neumann algebras does not give us direct tools to extend *BDH* to general morphisms. No applicable general lifting technique is avialable. **Strategy:** Solve the problem by understanding any such extension in terms of its 'surrounding' categorical structure, i.e. in terms of other double categories of von Neumann algebras. Pictorially:

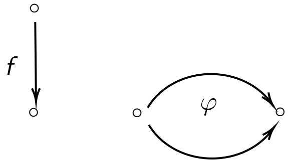


Question: Are there double categories of factors at all? i.e. is the above shaded square $\neq \emptyset$?

Globularly generated double categories

Decorated bicategories

A decorated bicategory is a pair (B^*, B) where B^* is a category and B is a bicategory such that the objects of B^* and B are the same. Represent a decorated bicategory as a bunch of diagrams of the form:



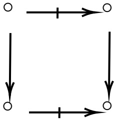
where the sets of vertices of the two types of diagrams are the same.

Example: Let C be a double category. The pair (C_0, HC) is a decorated bicategory. Write H^*C and call it the decorated horizontalization of C.

Internalizations

Problem: Given a decorated bicategory (B^*, B). Find double categories C such that $H^*C = (B^*, B)$. We call any such C an internalization of (B^*, B). Can we understand the internalizations of (B^*, B)?

Problem of existence of internalizations: Is the decorated horizontalization construction generic? Problem as a problem of coherently 'filling' 'hollow' squares of the form:



which we form with the 1-dimensional data provided to us by (B^*, B) in such a way that the 1-dimensional and the globular data we started with is fixed. Problems of filling squares with globular data appear in Ronnie Brown's proof of the 2-dimensional Seifert-van Kampen theorem [Brown, Higgins, Sivera 11'] and in the definition of framed bicategory. [Shulman 08']

The globularly generated piece

Let C be a duble category. Write γC for the minimal sub-double category of C containing every horizontal identity square and every globular square of C.

Lemma (O 18')

Let C be a double category.

1. $H^* C = H^* \gamma C$.
2. *If D is a sub-double category of C satisfying the equation $H^* C = H^* D$ then γC is a sub-double category of D.*

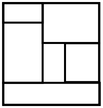
C is a solution to internalization for $H^* C$. (1) says that so is γC. (2) says that γC is the minimal solution in C. γC the globularly generated piece of C.

Globularly generated double categories

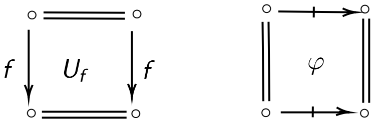
A double category C is globularly generated if any of the following three equivalent conditions is satisfied:

1. $\gamma C = C$.
2. C is generated, as a double category, by its globular squares.
3. C contains no proper sub-double categories D such that $H^*C = H^*D$.

C is globularly generated if every square in C admits a subdivision, say as:



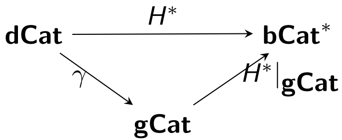
where every smaller square is of one of the two forms:



Globularly generated double categories

$\gamma^2 C = \gamma C$ any C. Thus γC is globularly generated for every C. γC is the maximal globularly generated sub-double category of C.

Categorically: **gCat** sub-2-category of **dCat** generated by globularly generated double categories. **bCat*** category of decorated bicategories, decorated pseudofunctors. The function $C \mapsto \gamma C$ extends to a functor γ :**dCat**→**gCat**. γ is a 2-coreflector and thus a fibration. **dCat** is indexed by globularly generated double categories. The function $C \mapsto H^* C$ extends to a functor H^* :**dCat**→**bCat*** and the following triangle commutes:



H^* is constant along γ-fibers. **Lesson:** If interested in internalizations, first study basis for γ, i.e. study globularly generated double categories.

γ[**Mod**]

Lemma

Let C be a double category. Let $\varphi \in C_1$. If φ is not globular, globularly generated, then $L\varphi = R\varphi$, i.e. every non-globular, globularly generated square in C is a horizontal endomorphism.

We compute γ[**Mod**]. We know objects, horizontal/vertical morphisms, and globular squares of γ[**Mod**]. We need to compute the non-globular, globularly generated horizontal endomorphisms.

Let R, S be algebras. Let $(f, \varphi, f) :_R M_R \to_S N_S$ be a square in [**Mod**]. (f, φ, f) is 2-subcyclic if there exist submodules $L \subseteq K \subseteq N$ such that ${}_R L_R^f$ is R-cyclic, ${}_S K_S$ is cyclic, and $im\varphi \subseteq L$. Example: R, S rings, $f : R \to S$, $U_f = (f, f, f) :_R R_R \to_S S_S$ is 2-subcyclic. Non-example: $(i, i^2, i) :_{\mathbb{Z}} \mathbb{Z}_{\mathbb{Z}} \to_{\mathbb{Q}} \mathbb{Q}_{\mathbb{Q}}$ not 2-subcyclic.

Lemma

*The non-globular squares of γ[**Mod**] are the 2-subcyclic squares.*

Length

What can we say about GG double categories?

Let C be a globularly generated double category. The category of squares C_1 of C is canonically filtered:

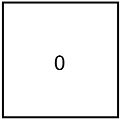
Inductively: Write H_0 for the set of all globular and horizontal identity squares of C. Write V_1 for the category generated by H_0. Suppose V_{n-1} has been defined for some $n > 1$, define H_n as the sub-pseudo-category of $\mathcal{V}C$ generated by V_{n-1} and make V_n to be the subcategory of C_1 generated by H_n. We have:

1. $V_n \subseteq V_{n+1} \subseteq C_1$.
2. $\varinjlim V_n = C_1$.

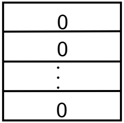
i.e. the chain of V_n's is a filtration for C_1. Call the filtration $... \subseteq V_n \subseteq V_{n+1} \subseteq ...$ of C_1 the vertical filtration of C.

Geometrically

Let C be a globularly generated double category. Think of globular and horizontal identity squares as squares of complexity 0. Draw them as:



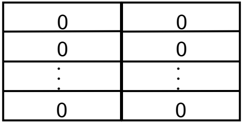
H_0 is the collection of squares of complexity 0. V_1 is the collection of squares that can be subdivided as vertical composition of squares of complexity 0. Geometrically, squares of the form:



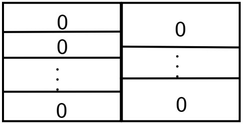
We say that these squares are of complexity ≤ 1 and we draw them as squares marked with 1.

Geometrically

Given two horizontally compatible squares φ, ψ in V_1, it might be the case the we can find compatible complexity ≤ 1 presentations of φ, ψ, i.e. the composition $\varphi \odot \psi$ can be made to look like:



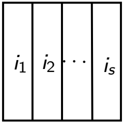
In that case $\varphi \odot \psi$ is of complexity ≤ 1, i.e. is a square in V_1. This doesn't happen in general. $\varphi \odot \psi$ can look like:



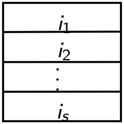
for every complexity ≤ 1 decomposition of ϕ and ψ. In that case we say that $\varphi \odot \psi$ has complexity $\leq 1 + 1/2$. H_1 is the pseudocategory of complexity $\leq 1 + 1/2$ squares.

Geometrically

Let $n > 1$. Assume we have defined squares of complexity $\leq n-1$. Squares of complexity $\leq n - 1/2$ are now squares that can be subdivided as horizontal composition of squares of complexity $\leq n-1$. Geometrically:



with the i_j's are $\leq n-1$. H_{n-1} is the pseudocategory of squares of complexity $\leq n - 1/2$. V_n now is the subcategory of C_1 generated by H_{n-1}. We say that squares in V_n are of complexity $\leq n$. Geometrically look like:

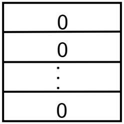


where the $i_j's$ are $\leq n - 1/2$. V_n's form a filtration means that ever square in C has a complexity.

Length

C be a globularly generated. φ square in C. $\ell\varphi$ is $min\{n : \varphi \in V_n\}$, i.e. $\ell\varphi$ is the min n such that φ is of complexity $\leq n$. $\ell\varphi$ the vertical length of φ. ℓC for $Sup\{\ell\varphi : \varphi \in C_1\}$. ℓC the vertical length of C. For general C we define the vertical lenght of C, ℓC, as $\ell\gamma C$.

Intuition: ℓC measures the complexity of mixed compositions of horizontal identity and globular squares in C , e.g. $\ell C = 1$ iff every square in γC can be written as vertical composition of globular and horizontal identity squares, i.e. every globularly generated square can be drawn as:

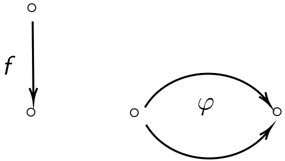


and any two horizontally compatible expressions match. **Examples:** $\ell\mathbb{H}B = 1$, $\ell\mathbf{Q}B = 1$, $\gamma[\mathbf{Mod}] = 1$ and $\ell BDH = 1$. **Question:** Is ℓ trivial?

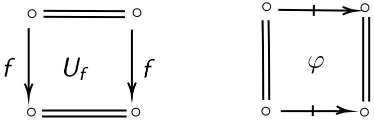
Constructing GG double categories

Let (B^*, B) be a decorated bicategory. We wish to associate to (B^*, B) a globularly generated double category defined only through the data of (B^*, B). **Idea:** Formally reconstruct a vertical filtration with the data of (B^*, B) and then turn that into a globularly generated double category.

Puff up diagrams of the form:



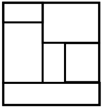
in (B^*, B) into diagrams of the form:



Constructing globularly generated double categories

Constructing GG double categories

Stack the above diagrams vertically, then horizontally, etc. and obtain formal squares admitting formal subdivisions as:



where the smaller squares are of the shape above. Carefully choose an equivalence relation R on the set of such squares, containing both the exchange relation and the composition information of (B^*, B). Write $Q_{(B^*,B)}$ for the quotient.

Theorem (O'19)

Let (B^*, B) be a decorated bicategory. $Q_{(B^*,B)}$ is a globularly generated double category such that the category of objects of $Q_{(B^*,B)}$ is B^* and $B \subseteq H^* Q_{(B^*,B)}$.

$Q_{(B^*,B)}$ the free globularly generated double category associated to (B^*, B). **Warning:** The equality $H^* Q_{(B^*,B)} = (B^*, B)$ does not hold in general.

Saturated decorated bicategories

We say that a decorated bicategory (B^*, B) is saturated if the equation $H^*Q_{(B^*,B)} = (B^*, B)$ holds. We have easy tests to decide if a decorated bicategory is saturated.

Example: If B^* has no sections or retractions, i.e. (B^*, B) is reduced. In particular if B^* is free or ΩM for a reduced monoid M then (B^*, B) is saturated.

Lemma

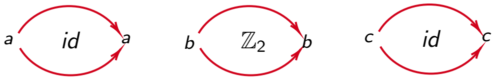
Let (B^*, B) be a decorated bicategory. $Q_{H^*Q_{(B^*,B)}} = Q_{(B^*,B)}$ and thus $H^*Q_{(B^*,B)}$ is saturated.

If (B^*, B) is not saturated we can always enlarge (B^*, B) canically in order to obtain a saturated decorated bicategory.

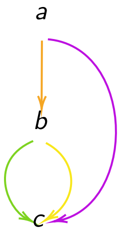
In [O'19] it is proven that the decorated bicategory of factors, is saturated. Also the decorated bicategory of simple algebras by Shur's lemma.

Length is not trivial

We use the free globularly generated double category construction to prove that vertical length is non-trivial. Consider the bicategory B:

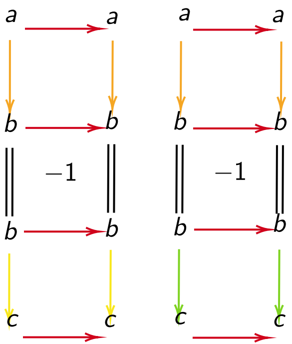


Decorate B by the free category B^* generated by:



Length is not trivial

The horizontal composition of the squares



in $Q_{(B^*,B)}$ is not of complexity 1, i.e. is of length 2 and thus $\ell Q_{(B^*,B)} \geq 2$. The above construction can be adapted to prove the existence of double categories of length $n \forall n$ and of length ∞. Length is not trivial!

Canonical projection

Bounding length

Theorem

Let (B^*, B) be a decorated bicategory. Let C be an internalization of (B^*, B). There exists a unique double functor $\pi^C : Q_{(B^*,B)} \to C$ such that $\pi^C|_{(B^*,B)} = id_{(B^*,B)}$ and π^C is surjective on globular squares of C.

π^C canonical double projection of C. $(B^*, B) \mapsto Q_{(B^, B)}$ extends to a functor Q :$\mathbf{bCat}^* \to \mathbf{gCat}$. $Q \dashv H^*$ with the π^C's as counit. Q is a decorated version of **Q**. $H^*|_{\mathbf{gCat}}$ is faithful. Q is thus free.

Let (B^*, B) be a decorated bicategory. Define the length $\ell(B^*, B)$ of (B^*, B) to be $\ell Q_{(B^*,B)}$. Depends only on (B^*, B). If C is an internalization of (B^*, B) then $\ell C \leq \ell(B^*, B)$.

Every globularly generated internalization can be written uniquely as a double qotient of a free globularly generated double category Construct a new double category of factors, extending BDH 2 non-equivalent double categories of factors. Both of lenght 1!

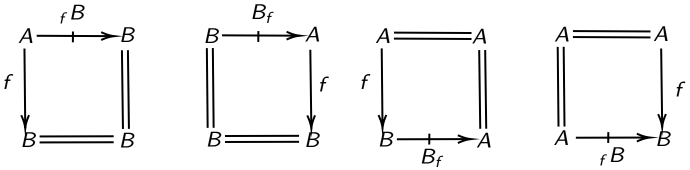
Framed bicategories

Framed bicategories

Definition

A framed bicategory is a double category C such that any of the following three equivalent conditions are satisfied:

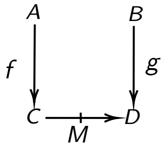
1. $L \times R : C_1 \to C_0 \times C_0$ is a fibration.
2. $L \times R : C_1 \to C_0 \times C_0$ is an opfibration
3. For every vertical morphism $f : A \to B$ in C there exist horizontal arrows ${}_fB : A \to B$ and $B_f : B \to A$ together with squares:



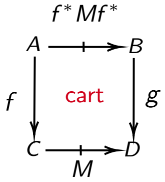
Satisfying certain compatibility conditions. ${}_fB$ and B_f companion and conjoint of f. Examples: [**Mod**], **Bord**, $\rho_2^{\square}(X)$, **Adj**, ${}_L coSpan(C)$, $\mathbb{P}oly$, $coMon_{Poly}$, BDH, etc.

Framed bicategories

C framed bicategory. $L \times R$ is a fibration. This means that for every empty lower frame/shell/boundary:

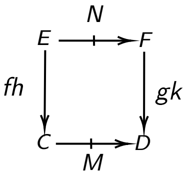


there exists a cartesian filler:

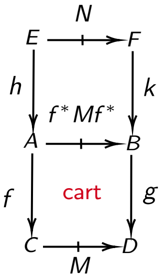


which means that:

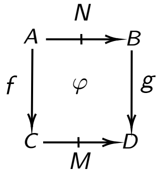
for every square:



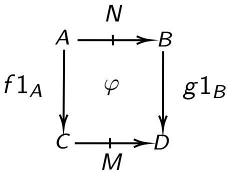
there exists a unique factorization:



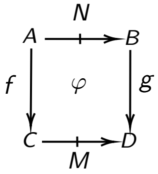
In particular, for every square:



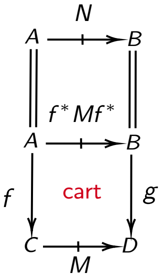
consider the factorization:



The square :



factorizes uniquely as:



i.e. every square in C is determined by a globular square and a cart square.

Framed bicategories organize into a sub 2-category $\mathbf{bCat}^{fr}$ of $\mathbf{dCat}$. A symmetric monoidal framed bicategory is a commutative pseudomonoid in $\mathbf{bCat}^{fr}$. A symmetric monoidal structure on C_0 and C_1 + easy compatibility conditions.

Theorem

Let C be a symmetric monoidal framed bicategory. Every choice of cleavage for $L \times R$ defines a symmetric monoidal structure on HC.

Observation

1. Globularly generated framed bicategories are vertically trivial. No non-trivial companions/conjoints: Every square is either globular or a horizontal endomorphism.2. There exist globularly generated double categories C such that there doesn't exist a framed bicategory D such that $\gamma D = C$.

Question: What can we say about globularly generated double categories having framed bicategories in their γ-fiber?

Question: What is the length of a framed bicategory? Conjecture: 1.

Support: $\ell[\mathbf{Mod}] = 1$, $\ell\mathbf{Bord}_n = 1$, $\ell BDH = 1$.

References

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Thank you!