A double categorical look at different flavours of factorisation system

Virtual double cats workshop

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Plan

1. From factorisation systems to double cats of maps.

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2. Orthogonality & factorisation for double cats of maps

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1. From factorisation systems to double cats of maps
2. Orthogonality & factorisation for double cats of maps
3. Algebraic weak fact. systems

① Factorisation systems

- $\mathscr{C}$ a cat, $\mathcal{E}, M \subseteq \mathrm{Mor}(\mathscr{C})$

Eg. (Surjections, injections) in Set or algebraic cat.

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A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \\
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$\mathcal{E} \ni e \qquad m \in \mathcal{M}$

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What is relationship between $\mathcal{E}$'s & $\mathcal{M}$'s though?

They are orthogonal.

Orthogonality

Say $\mathcal{E} \perp \mathcal{M}$ if

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "e \in \mathcal{E}"'] & C \arrow[d, "m \in \mathcal{M}"] \\ B \arrow[r, "s"'] \arrow[ur, "\exists!"] & D \end{tikzcd}$$

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- $\mathcal{E}^{\perp} = \{ g \in \mathcal{C} : \mathcal{E} \perp g \}$
- ${}^{\perp}\mathcal{M} = \{ f \in \mathcal{C} : f \perp \mathcal{M} \}$

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Axiom of orthogonality

- $\mathcal{E} = {}^{\perp}\mathcal{M}$ & $\mathcal{M} = \mathcal{E}^{\perp}$

ie. $f \in \mathcal{E}$ iff it has unique left lifting prop. wrt morphs of $\mathcal{M}$, & $f \in \mathcal{M}$ iff ...

- Factorisation system $(\mathcal{E}, \mathcal{M})$ satisfies axioms of

1) Factorisation & 2) orthogonality.

• <u>Factorisation system</u> $(\mathcal{E},\mathcal{M})$ satisfies axioms of 1) <u>factorisation</u> & 2) <u>orthogonality</u>.

• Changing to <u>weak</u> orthogonality

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "e"'] & C \arrow[d, "m"] \\ B \arrow[r, "s"'] \arrow[ur, "\exists"] & D \end{tikzcd} \quad \text{in } 2$$

we get <u>weak factorisation systems</u>.

• Call these OFS / WFS.

- In OFS / WFS $(\mathcal{E}, \mathcal{M})$ both classes closed under composition & contain ids

$\Rightarrow$ wide subcat.

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of arrow category.

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of arrow category
- Both cats assoc. to $\mathcal{E}$ (or $\mathcal{M}$) are useful. Can we put them in 1 structure?

Yes - into a double cat

- $D(\mathcal{E})$: same obs & horizontal arrows as $\mathcal{C}$, vertical arrows as in $\mathcal{E}$, squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "="] & C \arrow[d, "e'"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

comm. squares

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comm. squares

- $Sq(\mathcal{C}) = D(\mathcal{C})$ - horizontal, vert. arrows as in $\mathcal{C}$, all squares

Forgetful double Fun

$$U : \mathbb{D}(\mathcal{E}) \longrightarrow Sq(\mathcal{E})$$

$$\forall \; \begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle e}\downarrow & = & \downarrow{\scriptstyle e'} \\ B & \xrightarrow[g]{} & D \end{array} \in \mathcal{E} \longmapsto \begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle e}\downarrow & = & \downarrow{\scriptstyle e'} \\ B & \xrightarrow[g]{} & D \end{array}$$

forgets v. maps in $\mathcal{E}$.

Forgetful double fun

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forgets v. maps in $\mathcal{E}$.

- Encodes $\mathcal{C}_{\mathcal{E}} \hookrightarrow \mathcal{C}$ via cats of vert. morphs
- Encodes $\mathcal{E} \hookrightarrow Mor(\mathcal{C})$ via cats of vert arrows & squares.

Double cats of maps

- Think of $U : \mathbb{A} \to Sq(\mathcal{C})$ as double cat of maps over $\mathcal{C}$

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- In practice, U usually id. on horiz. cats, faithful on squares – concrete double cat of maps.

- Eg. $U : \mathbb{D}(\mathcal{E}) \to \mathrm{Sq}(\mathcal{C})$

Examples

① $\mathbb{D}(\mathrm{Surj}) \longrightarrow \mathrm{Sq}(\mathrm{Set})$

e.g.
$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, two heads, "f"'] \arrow[dr, phantom, "="] & C \arrow[d, two heads, "g"] \\ B \arrow[r, "s"'] & D \end{tikzcd}$$

ie. double cat of surjections

Examples

① $D(\mathrm{Surj}) \longrightarrow \mathrm{Sq}(\mathrm{Set})$

e.g.
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ie. double cat of surjections

Encodes property
(being surjection)
but can encode
structure too.

Examples

② $U : \mathrm{SplEpi} \longrightarrow \mathrm{Sq}(\mathrm{Set})$

U. arrows : $A \xrightarrow{(f,u)} B$ section of f

squares :

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, shift left, "f"] & C \arrow[d, shift left, "g"] \\ B \arrow[r, "s"'] \arrow[u, shift left, "u"] & D \arrow[u, shift left, "v"] \end{tikzcd}$$

- Vert. comp : compose sections

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- Vert. comp : compose sections

Maps w' structure -
U faithful ✓ (full ✗) on squares

Examples

- On Cat, conc. double cats of
  - adjunctions
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  any flavour of map

- In connection with hott - structured fibrations on simplicial/cubical sets.

Useful guiding example

① $\mathrm{SplRef} \longrightarrow \mathrm{Sq}(\mathrm{Cat})$

vert. maps "split reflection":

$$\begin{tikzcd} A \arrow[r, bend right, phantom, ""] \arrow[rr, "u"', shift right] & & B \arrow[ll, bend right, "f"'] \end{tikzcd}$$

$$A \underset{u}{\overset{f}{\underset{\perp}{\leftrightarrows}}} B$$

$+ \, (f \dashv u, \, \eta : 1 \Rightarrow uf, \, \varepsilon = \mathrm{id})$

Useful guiding example

① $\mathrm{SplRef} \longrightarrow \mathrm{Sq}(\mathrm{Cat})$

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vert maps "split fibrations"

So Far :

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- replaced classes of maps $\mathcal{E}, \mathcal{M} \subseteq \mathrm{Mor}(\mathcal{C})$ by double cats of maps

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- replaced classes of maps $\mathcal{E}, \mathcal{M} \subseteq \mathrm{Mor}(\mathcal{C})$ by double cats of maps

$$\mathbb{L} \xrightarrow{u} \mathrm{Sq}(\mathcal{C}) \xleftarrow{v} \mathbb{R}$$

- Can we understand "fact. systems" in this setting approp. to structured maps?

Orthogonality via double cats?

- Given OFS $(\mathcal{E}, \mathcal{M})$ consider

$$\mathbb{D}(\mathcal{E}) \xrightarrow{u} \mathbb{S}\mathrm{q}(\mathcal{C}) \xleftarrow{v} \mathbb{D}(\mathcal{M}).$$

Orthogonality via double cats?

- Given OFS $(\mathcal{E}, \mathcal{M})$ consider

$$D(\mathcal{E}) \xrightarrow{u} Sq(\mathcal{C}) \xleftarrow{v} D(\mathcal{M}).$$

- Given $f \in D(\mathcal{E})$, $g \in D(\mathcal{M})$ vert.

&

$$\begin{tikzcd} uA \arrow[r, "r"] \arrow[d, "uf"'] & vC \arrow[d, "vg"] \\ uB \arrow[r, "s"'] \arrow[ur, "\exists!"] & vD \end{tikzcd}$$

Thoughts

- For structured maps, unique diagonals too strong.

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→ Structured liftings satisfying compatibilities

Lifting operations

Consider $\mathbb{L} \xrightarrow{u} Sq(\mathcal{C}) \xleftarrow{v} \mathbb{R}$

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& each square

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$$\begin{tikzcd} uA \arrow[r, "r"] \arrow[d, "uf"'] & vC \arrow[d, "vg"] \\ uB \arrow[r, "s"'] \arrow[ur, "\varphi_{f,g}(r,s)"] & vD \end{tikzcd}$$

a diagonal filler

such that

the fillers

$$\begin{tikzcd} UA \arrow[r, "r"] \arrow[d, "Uf"'] & VC \arrow[d, "Vg"] \\ UB \arrow[r, "s"'] \arrow[ur, "\varphi_{f,g}(r,s)"] & VD \end{tikzcd}$$

are compatible with double category structure on $\mathbb{L}$ & $\mathbb{R}$:

① φ is natural in squares :

(1l) At $\alpha : F' \to F$ a square

$$
\begin{tikzcd}
Ua' \arrow[r, "(U\alpha)_0"] \arrow[d, "Uf'"'] & Ua \arrow[r, "r"] \arrow[d, "Uf"] & Vc \arrow[d, "Vg"] \\
Ub' \arrow[r, "(U\alpha)_1"'] & Ub \arrow[r, "s"'] \arrow[ur, "\varphi"] & Vd
\end{tikzcd}
=
\begin{tikzcd}
Ua' \arrow[r, "r.U\alpha_0"] \arrow[d, "Uf'"'] & Vc \arrow[d, "Vg"] \\
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(1r) & sim. on the right;

② φ respects vertical comp:

(2l)
$$\begin{tikzcd}
Ua_1 \arrow[r, "r"] \arrow[d, "Uf_1"'] & Vc \arrow[dd, "Vg"] \\
Ua_2 \arrow[d, "Uf_2"'] \arrow[ur, "① \varphi_{f_1,g}"] & \\
Ua_3 \arrow[r, "s"'] \arrow[uur, "② \varphi_{f_2,g}"] & Vd
\end{tikzcd}
=
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Ua_3 \arrow[r, "s"'] \arrow[uur, "\varphi_{f_2f_1,g}"] & Vd
\end{tikzcd}$$

(2r) & on the right.

Example

- $(\mathbb{L}, \mathbb{R}) = ($split reflections, split fibrations$)$

- Consider
$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "{(u, F \dashv u)}"'] & C \arrow[d, "{(q, \varphi)}"] \\ B \arrow[r, "s"'] \arrow[ur, "k"] & D \end{tikzcd}$$

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so $\quad sb \xrightarrow{s\eta_b} sufb = grfb$

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given lifting of split fib.

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given lifting of split fib.

- Object part of lifting operation - exercise!

I call a triple

$\mathbb{L} \xrightarrow{u} Sq(\mathcal{C})$, $\mathbb{R} \xrightarrow{v} Sq(\mathcal{C})$

& $(\mathbb{L}, \mathbb{R})$-lifting op. φ

a lifting structure

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Def) An algebraic weak fact. system is a lifting structure $(\mathbb{L}, \varphi, \mathbb{R})$ sat. axioms of orthogonality & factorisation.

Axiom of orthogonality

$(\mathbb{L}, \mathbb{R})$-liftings φ ~

① $\mathbb{R} \xrightarrow{\varphi_r} \mathrm{RLP}(\mathbb{L})$ where

$\mathrm{RLP}(\mathbb{L})$ has vert. arrows

(g, φ) where
$$\begin{tikzcd} UA \arrow[r, "r"] \arrow[d, "Uf"'] & C \arrow[d, "g"] \\ UB \arrow[r, "s"'] \arrow[ur, "\varphi_f(r,s)"] & D \end{tikzcd} \quad g \in \mathcal{C}$$

compat. w/ squares & v. comp. in $\mathbb{L}$.

② $\mathbb{L} \xrightarrow{\varphi_\ell} \mathrm{LLP}(\mathbb{R})$

where $\mathrm{LLP}(\mathbb{R})$ dually defined.

$(\mathbb{L}, \varphi, \mathbb{R})$ satisfies

axion of orthogonality

if $\varphi_\ell : \mathbb{L} \to LLP(\mathbb{R})$

& $\varphi_r : \mathbb{R} \longrightarrow RLP(\mathbb{L})$

are invertible.

Axiom of Factorisation

Lifting str $(\mathbb{L}, \mathcal{C}, \mathbb{R})$ satisfies axiom if each $A \xrightarrow{f} B \in \mathcal{C}$ admits fact:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "u_g"'] & & B \\ & C \arrow[ur, "v_h"'] & \end{tikzcd}$$

vert arrow of $\mathbb{L}$

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vert arrow of $\mathbb{L}$

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s.t.

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "u_g"'] & A \arrow[d, "f"] \\ C \arrow[r, "v_h"'] & B \end{tikzcd}$$ is u-couniversal in $Mor(\mathcal{C})$

&

$$\begin{tikzcd} A \arrow[r, "u_g"] \arrow[d, "f"'] & C \arrow[d, "v_h"] \\ B \arrow[r, "1"'] & B \end{tikzcd}$$ v-universal in $Mor(\mathcal{C})$

Example

$(\mathrm{SplRef}, \mathcal{U}, \mathrm{SplFib})$ is awfs on Cat –

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & B/f \arrow[ur] & \end{tikzcd}$$

comma cat:

obs $(b \xrightarrow{\alpha} fa, a)$

Awfs classically

- Awfs on $\mathcal{C}$ involves:
  - funct. fact $\mathcal{C}^{\to} \xrightarrow{K} \mathcal{C}^{\to\to}$
  - comonad & monad L, R on $\mathcal{C}^{\to}$ related by a distrib. law.

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- Lots of structure.

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  - <u>comonad</u> & <u>monad</u> L, R on $\mathcal{C}^{\to}$ related by a <u>distrib. law</u>.
- <u>Lots</u> of structure.
- Arose to <u>rectify</u> non-funct of wfs (<u>Grandis-Tholen</u>)
- Making small ob. argument converge (<u>Garner</u>)

Applications

- Homotopy Theory (Riehl ...)

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- Higher cats (Nikolaus, Garner ..)

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- Homotopy Theory (Riehl ...)
- Higher cats (Nikolaus, Garner ...)
- Models of hott (Coquand, Gambino, van den Berg ...)

Value of double cats

- Equiv. to double cat. def
(JB22)

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$$\text{Awfs} \xrightarrow{\text{Semantics}} \text{Double cat awfs}$$

$$
\begin{tikzcd}
(L,R) \arrow[r, mapsto] & L\text{-Coalg} \arrow[dr] & & R\text{-Alg} \arrow[dl] \\
& & Sq(\mathcal{C}) &
\end{tikzcd}
$$

Value of double cats

- Equiv. to double cat. defn (JB22)

$$\mathrm{Awfs} \xrightarrow{\text{Semantics}} \text{Double cat awfs}$$

$$(L,R) \longmapsto \quad L\text{-Coalg} \to \mathrm{Sq}(\mathcal{C}) \leftarrow R\text{-Alg}$$

Awfs in fact determined by <u>left</u> or <u>right</u> double cat of maps

(Recognition thm BG16)

- Constructions (eg. limits) of awfs often hard - lots of structure.

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- Much easier to do construction on semantic (double cat) side & apply recognition theorem

See BG16.

MORAL

Whenever you are struggling with a problem about awfs, think double-categorically.

Some refs

Grandis-Tholen: Natural weak Fact. systems

Garner: Understanding small ob. arg.

Riehl - Algebraic model structures

Bourke-Garner: AwfS I - accessible awfs

Bourke: An orthogonal approach to awfs

Is this madness?

$$\mathbb{L} \xrightarrow{u} \cancel{Sq(\mathcal{C})}\ \mathbb{C} \xleftarrow{v} \mathbb{R}$$

$(\mathbb{L}, \mathbb{R})$-lifting op:

$$\begin{tikzcd} UA \arrow[r, "r"] \arrow[d, "Uf"'] \arrow[dr, phantom, "A"] & VC \arrow[d, "Vg"] \\ UB \arrow[r, "s"'] & VD \end{tikzcd} = \begin{tikzcd} UA \arrow[r, "r"] \arrow[d, "Uf"'] \arrow[dr, phantom, "\varphi_1 A"] & VC \arrow[d, "1"] \\ UB \arrow[r, "\varphi A"] \arrow[d] \arrow[dr, phantom, "\varphi_2 A"] & VC \arrow[d, "Vg"] \\ UB \arrow[r, "s"'] & VD \end{tikzcd}$$

"Awfs"-like structures on ~~more gen~~ double cats?