

A double categorical
look at different
flavours of factorisation
system

Virtual double cats
workshop

John Bourke
Masaryk University

Plan

- ① From factorisation systems to double cats of maps.

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② Orthogonality & factorisation for double cats of maps

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- ① From factorisation systems to double cats of maps.
- ② Orthogonality & factorisation for double cats of maps
- ③ Algebraic weak Fact. systems

① Factorisation systems

- \mathcal{C} a cat, $\mathcal{E}, \mathcal{M} \subseteq \text{Mor}(\mathcal{C})$

Eg. (Surjections, injections)
in Set or algebraic cat.

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They are orthogonal.

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- $\mathcal{E}^\perp = \{g \in \mathcal{C} : \mathcal{E} \perp g\}$
- ${}^\perp \mathcal{M} = \{f \in \mathcal{C} : f \perp \mathcal{M}\}$

Orthogonality

Say $\mathcal{E} \perp M$ if

$$\begin{array}{ccc} A & \xrightarrow{r} & C \\ x \in \mathcal{E} \downarrow & \nearrow \exists! & \downarrow m \in M \\ B & \xrightarrow{s} & D \end{array}$$

- $\mathcal{E}^\perp = \{g \in \mathcal{C} : \mathcal{E} \perp g\}$
- ${}^\perp M = \{f \in \mathcal{C} : f \perp M\}$

Axiom of orthogonality

• $\mathcal{E} = {}^\perp M$ & $M = \mathcal{E}^\perp$

ie. $f \in \mathcal{E}$ iff it has unique left lifting prop. wrt morphs of M , & $f \in M$ iff ...

- Factorisation system $(\mathcal{E}, \mathcal{M})$ satisfies axioms of
1) factorisation & 2) orthogonality.

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- Changing to weak orthogonality



we get weak factorisation systems.

- Call these OFS / WFS.

- In OFS / WFS $(\mathcal{E}, \mathcal{M})$
both classes closed under
composition & contain ids
 \Rightarrow wide subcat.

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$$\mathcal{E} \hookrightarrow \text{Arr}(\mathcal{C})$$

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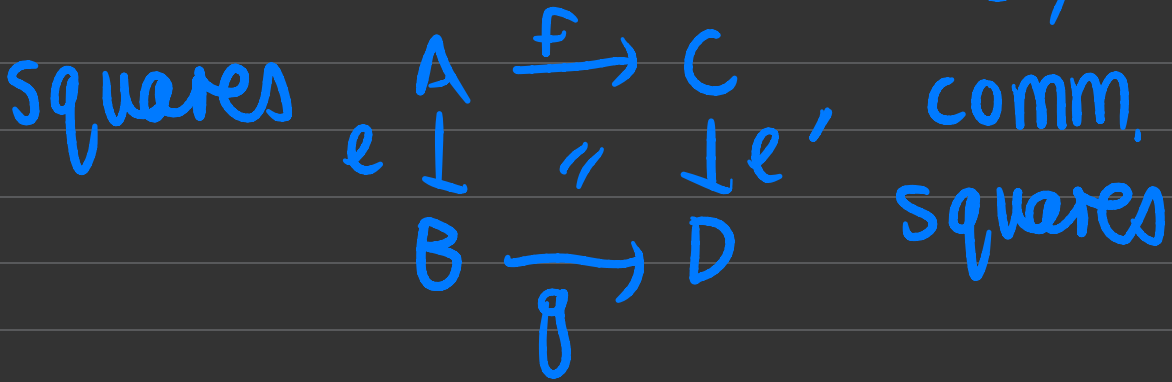
$$\mathcal{E} \hookrightarrow \text{Arr}(\mathcal{C})$$

of arrow category

- Both cats assoc. to \mathcal{E}
(or \mathcal{M}) are useful.
Can we put them
in 1 structure?

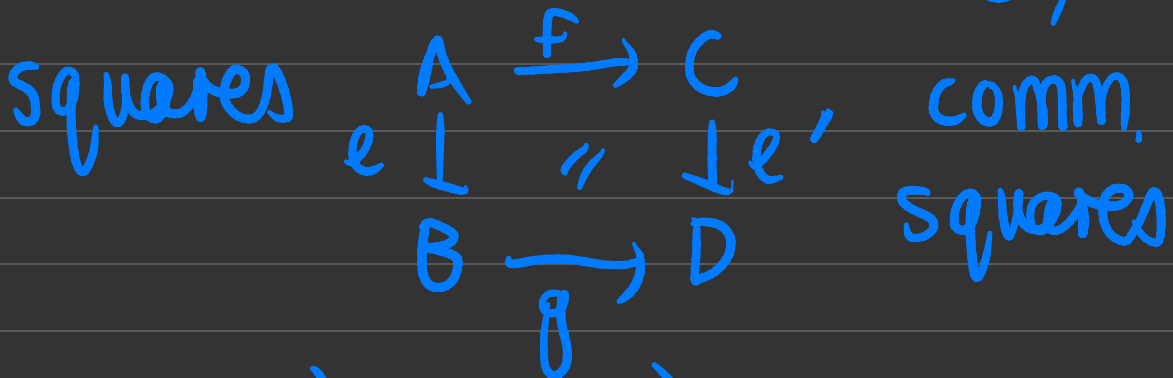
Yes - into a double cat

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- $Sq(\mathcal{C}) = D(\mathcal{C})$ -

horizontal, vert. arrows as in \mathcal{C} , all squares

Forgetful double Fun

$$U: D(\mathcal{E}) \longrightarrow Sq(\mathcal{C})$$

$$\begin{array}{ccc} \mathcal{E} & & \\ \downarrow & & \\ e \downarrow & \begin{array}{ccc} A & \xrightarrow{F} & C \\ \parallel & & \parallel \\ B & \xrightarrow{\quad} & D \end{array} & \xrightarrow{\quad} & \begin{array}{ccc} A & \xrightarrow{F} & C \\ \parallel & & \parallel \\ B & \xrightarrow{\quad} & D \end{array} \\ e' \in \mathcal{E} & & e' \end{array}$$

forgets U . maps in \mathcal{E} .

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Forgetful double Fun

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forgets v. maps in \mathcal{E} .

- Encodes $\mathcal{C}_{\mathcal{E}} \hookrightarrow \mathcal{C}$ via cats of vert. morphs
- Encodes $\mathcal{E} \hookrightarrow \text{Mor}(\mathcal{C})$ via cats of vert arrows & squares.

Double cats of maps

- Think of $U: \mathcal{A} \rightarrow \mathcal{S}_q(\mathcal{C})$
as double cat of maps over \mathcal{C}

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- Think of $U: \mathcal{A} \rightarrow Sq(\mathcal{C})$ as double cat of maps over \mathcal{C}
- In practice, U usually id. on horiz. cats, faithful on squares - concrete double cat of maps.
- Eg. $U: D(\mathcal{E}) \rightarrow Sq(\mathcal{C})$

Examples

$$\textcircled{1} D(\text{Surj}) \longrightarrow \text{Sq}(\text{Set})$$

e.g.

$$\begin{array}{ccc} A & \xrightarrow{r} & C \\ f \downarrow & \parallel & \downarrow g \\ B & \xrightarrow{s} & D \end{array}$$

ie. double cat of surjections

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ie. double cat of surjections

Encodes property
(being surjection)
but can encode
structure too.

Examples

$$\textcircled{2} U: \text{SplEpi} \longrightarrow \text{Sq}(\text{Set})$$

U. arrows: $A \xrightarrow{(f, u)} B$
section of f

squares:
$$\begin{array}{ccc} A & \xrightarrow{r} & C \\ u \uparrow \downarrow f & & v \uparrow \downarrow g \\ B & \xrightarrow{s} & D \end{array}$$

- Vert. comp : compose sections

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Maps w' structure -
U faithful \checkmark (full \times) on squares

Examples

- On Cat , conc. double cats
- of - adjunctions
 - equivalences
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- any flavour of map

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- On Cat , conc. double cats of
 - adjunctions
 - equivalences
 - fibrations & any flavour of map
- In connection with hott - structured fibrations on simplicial/cubical sets.

Useful guiding example

$$\textcircled{1} \text{ SplRef} \longrightarrow \text{Sq}(\text{cat})$$

vert. maps "split reflection":

$$A \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{\perp} \end{array} B$$

$$+ (f \dashv u, \eta: 1 \Rightarrow uf, \epsilon = \text{id})$$

Useful guiding example

① $\text{SplRef} \rightarrow \text{Sq}(\text{Cat})$
vert. maps "split reflection":

$$A \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{u} \\ \end{array} B$$

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② $\text{SplFib} \rightarrow \text{Sq}(\text{Cat})$
vert maps "split fibrations"

So far :

- replaced classes of maps

$\mathcal{E}, \mathcal{M} \subseteq \text{Mor}(\mathcal{C})$ by

double cats of maps

$$\mathbb{U} \xrightarrow{u} \text{Sq}(\mathcal{C}) \xleftarrow{v} \text{IR}$$

So far :

- replaced classes of maps
 $\mathcal{E}, \mathcal{M} \subseteq \text{Mor}(\mathcal{C})$ by

double cats of maps

$$\mathbb{L} \xrightarrow{u} \text{Sq}(\mathcal{C}) \xleftarrow{v} \text{IR}$$

- Can we understand
"fact. systems" in this
setting approp. to
structured maps?

Orthogonality via double cats?

- Given OFS $(\mathcal{E}, \mathcal{M})$
consider

$$D(\mathcal{E}) \xrightarrow{u} S_q(\mathcal{E}) \xleftarrow{v} D(\mathcal{M}).$$

Orthogonality via double cats?

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$$D(\mathcal{E}) \xrightarrow{u} S_q(\mathcal{E}) \xleftarrow{v} D(\mathcal{M}).$$

- Given $f \in D(\mathcal{E}), g \in D(\mathcal{M})$ vert.

$$\& \quad uA \xrightarrow{r} vC$$

$$\begin{array}{ccc} uf \downarrow & \exists! \nearrow & \downarrow vg \\ uB & \xrightarrow{s} & vD \end{array}$$

Thoughts

- For structured maps, unique diagonals too strong.

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→ Structured liftings
satisfying compatibilities

Lifting operations

Consider $\mathbb{L} \xrightarrow{u} S_q(\mathbb{C}) \xleftarrow{v} \mathbb{R}$

(\mathbb{L}, \mathbb{R}) -lifting operation φ

consists of :

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(\mathbb{L}, \mathbb{R})-lifting operation \mathcal{U}

consists of :

@ $f \in \mathbb{L}, g \in \mathbb{R}$ vertical

& each square

$$uA \xrightarrow{r} vC$$

$$uF \downarrow$$

$$\downarrow vG$$

$$uB$$

$$\xrightarrow{s} vD$$

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$$\begin{array}{ccc} \mathbb{U}A & \xrightarrow{r} & \mathbb{V}C \\ \mathbb{U}f \downarrow & \nearrow \varphi(f, g) & \downarrow \mathbb{V}g \\ \mathbb{U}B & \xrightarrow{s} & \mathbb{V}D \end{array}$$

a diagonal filler
such that

the fillers

$$\begin{array}{ccc} UA & \xrightarrow{r} & VC \\ UF \downarrow & \nearrow \varphi \circ \eta \circ (1, s) & \downarrow Vg \\ UB & \xrightarrow{s} & VD \end{array}$$

are compatible with
double category
structure on
 \mathbb{L} & \mathbb{R} ;

① φ is natural in squares:

(ll) At $\alpha: F' \rightarrow F$ a square

$$\begin{array}{ccc} Ua \xrightarrow{(U\alpha)} Ua \xrightarrow{r} Vc & & Ua' \xrightarrow{r \cdot U\alpha} Vc \\ \downarrow U\alpha' & \searrow \varphi & \downarrow U\alpha' \\ Ub' \xrightarrow{(U\alpha)'} Ub \xrightarrow{s} Vd & = & Ub' \xrightarrow{s \cdot U\alpha} Vd \end{array}$$

(lr) & sim. on the right;

① φ is natural in squares:

(1l) At $\alpha: F' \rightarrow F$ a square

$$\begin{array}{ccc}
 Ua \xrightarrow{(\alpha)_a} Ua & \xrightarrow{r} & Vc \\
 \downarrow Uf & \searrow \varphi & \downarrow Vg \\
 Ub \xrightarrow{(\alpha)_b} Ub & \xrightarrow{s} & Vd
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 Ua' \xrightarrow{r \cdot U\alpha} Vc & & \\
 \downarrow Uf' & \searrow \varphi & \\
 Ub' \xrightarrow{s \cdot U\alpha} Vd & &
 \end{array}$$

(1r) & sim. on the right;

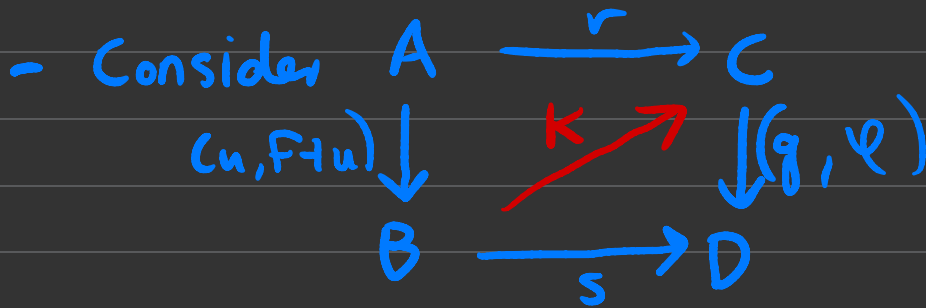
② φ respects vertical comp:

$$\begin{array}{ccc}
 Ua_1 \xrightarrow{r} Vc & & Ua_1 \xrightarrow{r} Vc \\
 \downarrow Uf_1 & \searrow \varphi_{F_1} & \downarrow Uf_1 \\
 Ua_2 & & Ua_2 \\
 \downarrow Uf_2 & \searrow \varphi_{F_2} & \downarrow Uf_2 \\
 Ua_3 \xrightarrow{s} Vd & & Ua_3 \xrightarrow{s} Vd
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 \end{array}$$

(2r) & on the right.

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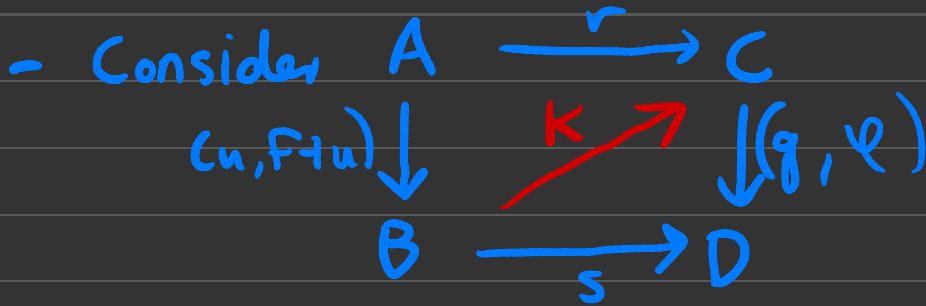
- Consider
$$\begin{array}{ccc} A & \xrightarrow{r} & C \\ \downarrow (u, F \dashv u) & \nearrow K & \downarrow (g, \psi) \\ B & \xrightarrow{s} & D \end{array}$$

- $f \dashv u$ w' unit $b \xrightarrow{\pi_b} uFb$

so $sb \xrightarrow{s\pi_b} suFb = grFb$

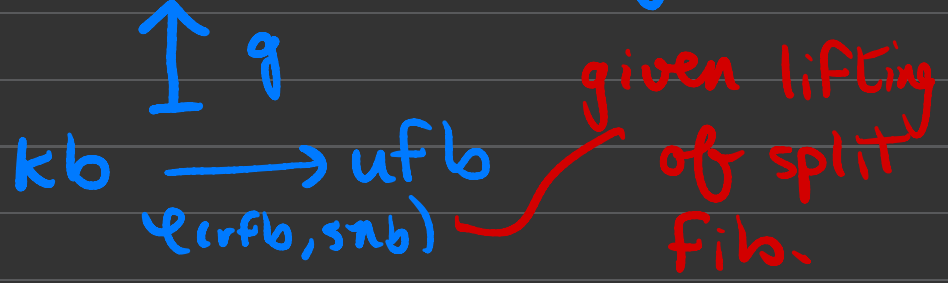
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- $f+u$ w' unit $b \xrightarrow{\eta_b} ufb$

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Example

- $(\mathcal{L}, \mathcal{R}) = (\text{split reflections, split fibrations})$

- Consider
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so $sb \xrightarrow{s\pi_b} suf_b = grfb$

$\uparrow g$

$kb \xrightarrow{\psi(rfb, s\pi_b)} ufb$

given lifting of split fib.

- Object part of lifting operation - exercise!

I call a triple

$$\mathbb{L} \xrightarrow{u} \mathcal{S}_q(\mathcal{C}), \mathbb{R} \xrightarrow{v} \mathcal{S}_q(\mathcal{C})$$

& (\mathbb{L}, \mathbb{R}) -lifting op. ψ

a lifting structure

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a lifting structure

Def) An algebraic weak
fact. system is a
lifting structure $(\mathbb{L}, \psi, \mathbb{R})$
sat. axioms of
orthogonality &
factorisation.

Axiom of orthogonality

(\mathbb{L}, \mathbb{R}) -liftings $\varphi \sim$

① $\mathbb{R} \xrightarrow{\varphi_r} \underline{RLP}(\mathbb{L})$ where

$RLP(\mathbb{L})$ has vert. arrow

(g, φ) where

$$\begin{array}{ccc} UA & \xrightarrow{r} & C \\ UF \perp & \searrow^{(\varphi_r)} & \downarrow g \in \mathcal{C} \\ UB & \xrightarrow{s} & D \end{array}$$

compat. w' squares & v. comp.
in \mathbb{L} .

② $\mathbb{L} \xrightarrow{\varphi_l} \underline{LLP}(\mathbb{R})$

where $LLP(\mathbb{R})$ dually defined.

$(\mathbb{L}, \psi, \mathbb{R})$ satisfies
axiom of orthogonality
if $\psi_L: \mathbb{L} \rightarrow \text{LLP}(\mathbb{R})$
& $\psi_r: \mathbb{R} \rightarrow \text{RLP}(\mathbb{L})$
are invertible.

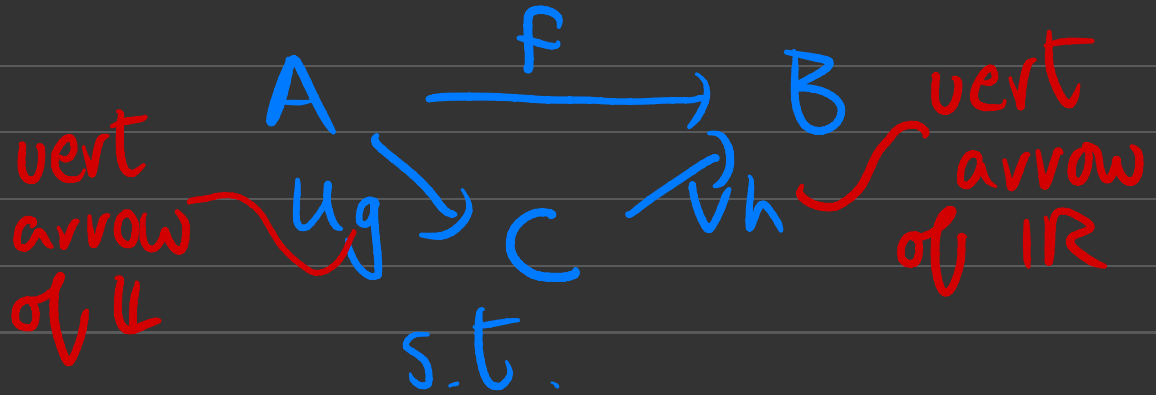
Axiom of Factorisation

Lifting str $(\mathcal{U}, \varphi, \mathbb{R})$

satisfies axiom

if each $A \xrightarrow{f} B \in \mathcal{C}$

admits fact:



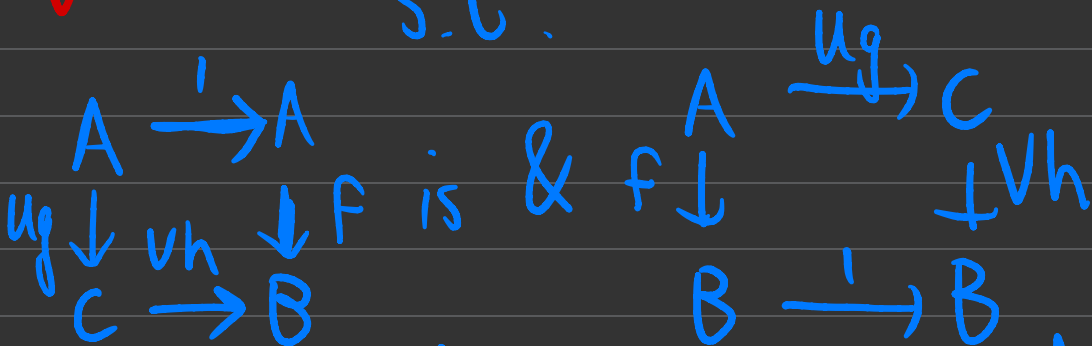
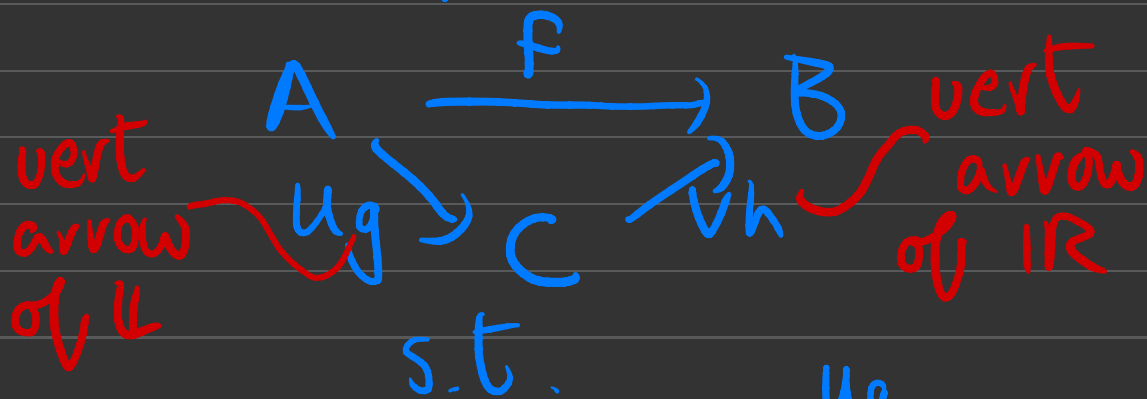
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\mathcal{U} -couniversal
in $\text{Mor}(\mathcal{C})$

\mathcal{V} -universal
in $\text{Mor}(\mathcal{C})$

Example

$(\text{SplRef}, \mathcal{U}, \text{SplFib})$
is awfs on cat -

$$A \xrightarrow{f} B$$

$$\begin{array}{ccc} & & \nearrow \\ & \searrow & \\ & B/F & \\ & \nearrow & \end{array}$$

comma cat:
 $\text{obs}(b \xrightarrow{f} fa, a)$

Awfs classically

• Awfs on \mathcal{C} involves!

- funct. fact $\mathcal{C}^{\rightarrow} \xrightarrow{K} \mathcal{C}^{\rightarrow \rightarrow}$

- comonad & monad L, R on $\mathcal{C}^{\rightarrow}$
related by a distrib. law.

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- Arose to rectify non-funct of wfs (Grandis-Tholen)
- Making small obs. argument converge (Garner)

Applications

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- Higher cats (Nikolaus, Garner.)

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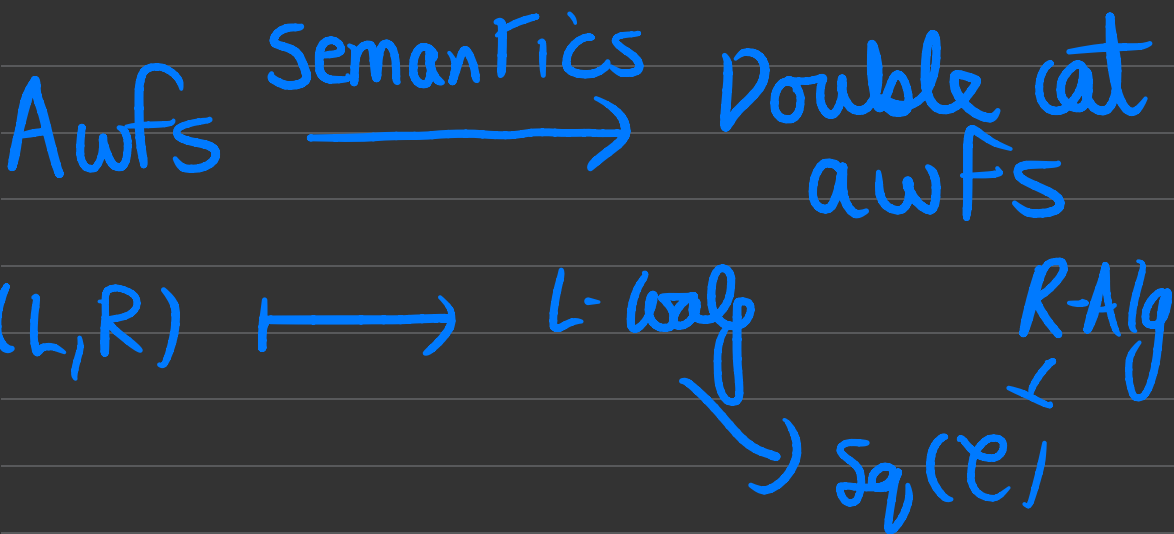
- Homotopy Theory (Riehl ...)
- Higher cats (Nikolaus, Garner..)
- Models of hott (Coquand, Gambino, van der Berg...)

Value of double cats

- Equiv. to double cat. def
(JB22)

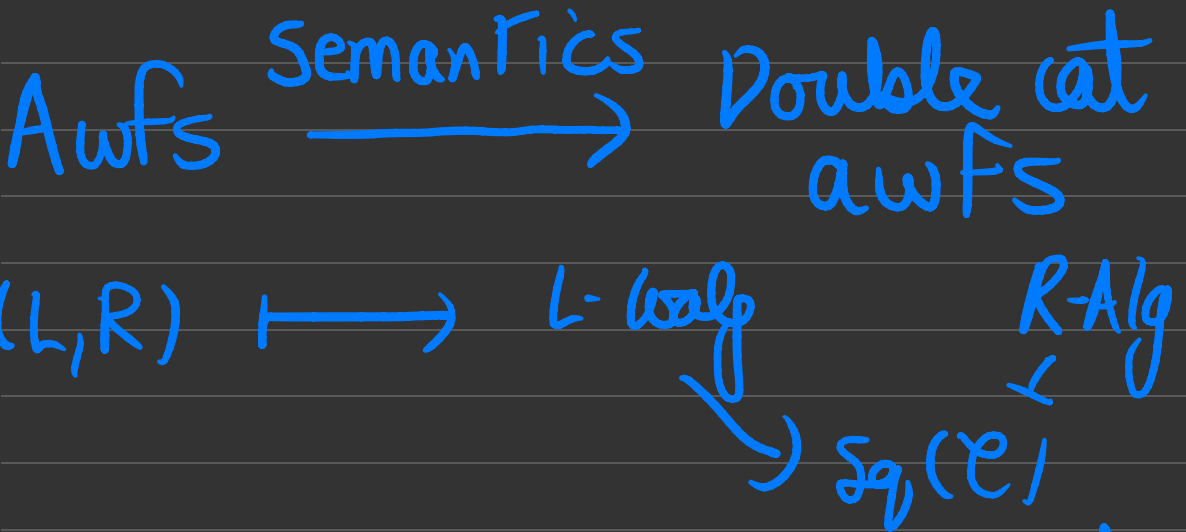
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Awfs in fact determined
by left or right
double cat of maps
(Recognition thm BG16)

- Constructions
(eg. limits) of awfs
often hard - lots of
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structure,

- Much easier to do
construction on semantic
(double-cat) side &
apply recognition theorem

See BG16.

MORAL

Whenever you are
struggling with a
problem about awfs,

think

double-categorically ✓

Some refs

Grandis-Tholen:

Natural weak Fact. systems

Garner: Understanding
small ob. arg.

Riehl - Algebras
model structures

Bourke-Garner: Awfs I -
accessible awfs

Bourke: An orthogonal
approach to awfs

Is this madness?

$$\mathbb{L} \xrightarrow{u} \cancel{S_q(\mathbb{C})} \xleftarrow{v} \mathbb{R}$$

\mathbb{C}

(L, IR)-lifting op :

$$UA \xrightarrow{v} VC$$

$$UA \xrightarrow{v} VC$$

$$\begin{array}{ccc} u_f \downarrow & A & \downarrow v_g \\ UB & \xrightarrow{s} & VD \end{array} = \begin{array}{ccc} u_f \downarrow & \varphi_1 A & \downarrow 1 \\ UB & \xrightarrow{\varphi_A} & VC \\ \downarrow & \varphi_2 A & \downarrow v_g \\ UB & \xrightarrow{s} & VD \end{array}$$

"Awfs"-like structures
on more gen double cats?