Lax Colimits and Fibrations of Double Categories

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Grothendieck Constructions for Double Categories

This talk is based on two papers:

- G.S.H. Cruttwell, M.J. Lambert, D.A. Pronk, M. Szyld, Double fibrations, *Theory and Applications of Categories*, Vol. 38, 2022, No. 35, pp 1326-1394.
- M. Bayeh, D.A. Pronk, M. Szyld, A Grothendieck construction for double categories, in progress.

The Grothendieck Construction / Category of Elements

For a pseudofunctor $F\colon \mathsf{C} \to \mathbf{Cat}$, the *Grothendieck category of elements*

$$\mathbf{El}F \to \mathsf{C}$$

can be characterized up to equivalence by either of the following two characterizations:

A. $\mathbf{El}F$ is the lax colimit of F in $\mathbf{Cat}$.

B1. $\mathbf{El}F$ is the value on objects of a 2-functor, which is an equivalence of 2-categories

$$\mathbf{El}\colon \mathrm{Hom}_p(\mathsf{C}, \mathbf{Cat}) \longrightarrow \mathrm{coFib}(\mathsf{C})$$

B2. For $F\colon \mathsf{C}^{\mathrm{op}} \to \mathbf{Cat}$, $\mathbf{El}F$ is the value on objects of a 2-functor, which is an equivalence of 2-categories

$$\mathbf{El}\colon \mathrm{Hom}_p(\mathsf{C}^{\mathrm{op}}, \mathbf{Cat}) \longrightarrow \mathrm{Fib}(\mathsf{C})$$

Fibrations

Let $P : \mathscr{E} \longrightarrow \mathscr{B}$ be a functor between categories.

- An arrow f of $\mathscr{E}$ is Cartesian if:

$$\begin{tikzcd} Z \arrow[d, dashed, "\hat{h}"'] \arrow[dr, bend left, "\forall g"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} PZ \arrow[d, "\forall h"'] \arrow[dr, bend left, "Pg"] & \\ PX \arrow[r, "Pf"'] & PY \end{tikzcd}$$

- P is a fibration when:

$$B^* \xrightarrow{u^*E} E \quad \leftsquigarrow \quad B \xrightarrow{u} PE$$

(Cartesian lift)

- A cleavage is a choice of a Cartesian lift for each arrow of $\mathscr{B}$.
  A cloven fibration is a fibration and a chosen cleavage.

-Any cloven fibration gives rise to an Indexed category $F : \mathscr{B}^{op} \to \mathbf{Cat}$.
-Any indexed category gives rise to a cloven fibration by its Grothendieck construction/category of elements.

Morphisms of Fibrations

Given cloven fibrations $P : \mathscr{E} \longrightarrow \mathscr{B}$ and $P' : \mathscr{E}' \longrightarrow \mathscr{B}'$,

- A **morphism** f between them is:
$$\begin{tikzcd} \mathscr{E} \arrow[r, "f^\top"] \arrow[d, "P"'] & \mathscr{E}' \arrow[d, "P'"] \\ \mathscr{B} \arrow[r, "f^\perp"'] & \mathscr{B}' \end{tikzcd}$$
  where $f^\top$ preserves the Cartesian arrows.
- f is said to be **cleavage-preserving** when $f^\top$ maps the arrows of the cleavage of P to arrows in the cleavage of P'.
- This defines 2-categories c**Fib** $\subseteq$ **Fib** $\subseteq$ $\mathbf{Arr}^s(\mathbf{Cat})$ (full on 2-cells, with objects the cloven fibrations).

The classical equivalence **Fib** $\simeq$ **ICat** (with pseudo transformations) restricts to c**Fib** $\simeq$ $\mathbf{ICat}_t$ (with strict natural transformations.)

Double Categories

- A **double category** is an internal category in **Cat**,

$$C_1 \times_{C_0} C_1 \xrightarrow{\otimes} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0.$$

- It has
  - **objects** (objects of C_0);
  - **inner/horizontal arrows** (arrows of C_0), $d_0(f) \xrightarrow{f} d_1(f)$;
  - **outer/vertical arrows** (objects of C_1), $s(v) \xrightarrow{v} t(v)$;
  - **double cells** (arrows of C_1), denoted

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "w"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

where $d_0(\alpha) = v$, $d_1(\alpha) = w$, $s(\alpha) = f$, and $t(\alpha) = f'$.

Examples

1. For any 2-category $\mathcal{C}$, $\mathbb{Q}(\mathcal{C})$ is the double category of quintets in $\mathcal{C}$, with double cells

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, "v", "\bullet" marking] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

for each $\alpha \colon vf \Rightarrow gu$ in $\mathcal{C}$.

2. For any 2-category $\mathcal{C}$, $\mathbb{H}(\mathcal{C})$ is the double category with double cells

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "1_A"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, "1_B", "\bullet" marking] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

for each $\alpha \colon f \Rightarrow g$ in $\mathcal{C}$.

3. The double category $\mathbb{V}(\mathcal{C})$ is defined analogously.

More Examples

- For any 2-category $\mathcal{C}$ with a distinguished family of arrows Σ that forms a sub-category, we can define $\mathbb{Q}^{\Sigma}(\mathcal{C}) \subseteq \mathbb{Q}(\mathcal{C})$ by requiring the inner/horizontal arrows to be in Σ:

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, "f"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g", "\bullet" marking] \\ C \arrow[r, "n"'] & D \end{tikzcd}$$

for each $\alpha \colon gm \Rightarrow nf$ in $\mathcal{C}$; $m, n \in \Sigma$.

Many examples of double categories are not exactly like this but have this *flavor*: $\mathbb{R}$el: functions and relations; $\mathbb{P}$rof: functors and profunctors; $\mathbb{S}$pan(Cat): functions and spans; $\mathbb{R}$ing: ring homomorphisms and bimodules; etc...

But note: except in $\mathbb{R}$el, vertical composition is no longer strict!

(Pseudo) Double Categories

- A **(pseudo) double category** is an internal **pseudo** category in **Cat**,

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\otimes"] & C_1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & C_0 \arrow[l, "y" description] \end{tikzcd} .$$

  The pull-back is still the same 2-pull-back, but instead of associativity and unit axioms we have invertible 2-cells (natural transformations)

$$\begin{tikzcd} C_1 \times_{C_0} \times C_1 \times_{C_0} C_1 \arrow[r, "1\times\otimes"] \arrow[d, "\otimes\times 1"'] \arrow[dr, phantom, "\overset{\mathfrak{a}}{\cong}"] & C_1 \times_{C_0} C_1 \arrow[d, "\otimes"] \\ C_1 \times_{C_0} C_1 \arrow[r, "\otimes"'] & C_1 \end{tikzcd}$$

$$\begin{tikzcd} C_1 \arrow[r, "\langle y,1\rangle"] \arrow[dr, equal, bend right] \arrow[dr, phantom, "\overset{\mathfrak{l}}{\cong}"] & C_1 \times_{C_0} C_1 \arrow[d, "\otimes"] & C_1 \arrow[l, "\langle 1,y\rangle"'] \arrow[dl, equal, bend left] \arrow[dl, phantom, "\overset{\mathfrak{r}}{\cong}"] \\ & C_1 & \end{tikzcd}$$

(Pseudo) Double Categories

- A **double category** (Grandis-Paré, 1999) is a **pseudo** category in **Cat**,

$$C_1 \times_{C_0} C_1 \xrightarrow{\otimes} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0 \;.$$

with $y\colon C_0 \to C_1$.

- Informally, this means that inner (horizontal) composition remains strict, but external (vertical) composition is pseudo.
- There is a 2-category **DblCat** of pseudo (double=internal) categories, pseudo (double=internal) functors to be defined on the next slide, and (horizontal=internal) transformations.

We have now all the examples from before (and more!)

Double Functors as Internal Functors

Internal pseudo categories can be considered in any 2-category $\mathcal{K}$ with 2-pullbacks instead of **Cat** (Martins-Ferreira, 2006).

A **lax double functor** $F\colon \mathbb{C} \to \mathbb{D}$ consists then of two arrows $F_0\colon C_0 \to D_0$ and $F_1\colon C_1 \to D_1$ and comparison 2-cells (+ axioms)

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\otimes"] \arrow[d, "F_1 \times_{F_0} F_1"'] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & C_1 \arrow[d, "F_1"] \\ D_1 \times_{D_0} D_1 \arrow[r, "\otimes"'] & D_1 \end{tikzcd} \qquad \begin{tikzcd} C_0 \arrow[r, "y"] \arrow[d, "F_0"'] \arrow[dr, phantom, "\overset{\iota}{\Rightarrow}"] & C_1 \arrow[d, "F_1"] \\ D_0 \arrow[r, "y"'] & D_1 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "F_1"'] & C_0 \arrow[d, "F_0"] \\ C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & C_0 \end{tikzcd}$$

If the comparison cells are invertible, F is a **pseudo double functor**.

Note that the interaction with s and t is required to be **stricter** than that with y and $\otimes$.

The category **DblCat** - Definition

The category **DblCat** of double categories has:

- **objects:** double categories $\mathbb{C}, \mathbb{D}, \ldots$;
- **arrows:** double functors $F, G, \ldots$;
- **transformations:** these come in two *flavors*:
  - a **horizontal transformation** $\gamma \colon F \Rightarrow G$ is given by

$$\begin{tikzcd} FA \arrow[r, "\gamma_A"] \arrow[d, "Fv"', "\bullet" marking] \arrow[dr, phantom, "\gamma_v"] & GA \arrow[d, "Gv", "\bullet" marking] \\ FB \arrow[r, "\gamma_B"'] & GB \end{tikzcd} \quad \text{for each } A \text{ in } \mathrm{dom}(F)$$

    pseudo functorial in the vertical direction and natural in the horizontal direction.
  - **vertical transformations** $\nu \colon F \Rightarrow\!\!\bullet\, G$ are defined dually, pseudonatural in the vertical direction and functorial in the horizontal direction;
- **modifications** given by a family of double cells.

The category **DblCat** - Properties

- **DblCat** is not a double category;
- a double category has two types of arrows, and **DblCat** has only one;
- a double category has one type of 2-cell, and **DblCat** has two;
- there are 2-categories $\mathbf{DblCat}_h$ and $\mathbf{DblCat}_v$;
- **DblCat** is *enriched* in double categories: $\mathbf{DblCat}(\mathbb{C}, \mathbb{D})$ is a double category for each pair of double categories $\mathbb{C}, \mathbb{D}$;
- so we need to replace **DblCat** by a double category as codomain for the indexing functors.

Replacements for **DblCat**, Option 1: $\mathbb{S}\mathrm{pan}(\mathbf{Cat})$

- A **double 2-category** is a pseudo category in the 2-category of 2-categories, 2-functors and 2-natural transformations.
- There is a double 2-category $\mathbb{S}\mathrm{pan}(\mathcal{K})$ for any 2-category $\mathcal{K}$ with double cells and 2-cells between them:

$$\begin{tikzcd} X \arrow[r, "f"] & X' \\ S \arrow[u, "r"] \arrow[d, "\ell"'] \arrow[r, "s"'] & S' \arrow[u, "r'"'] \arrow[d, "\ell'"] \\ Y \arrow[r, "g"'] & Y' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, bend left, "f'"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\varphi"] & X' \\ S \arrow[u, "r"] \arrow[d, "\ell"'] \arrow[r, bend left, "s'"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\sigma"] & S' \arrow[u, "r'"'] \arrow[d, "\ell'"] \\ Y \arrow[r, bend left, "g'"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\psi"] & Y' \end{tikzcd}$$

A lax double functor from the terminal double category to $\mathbb{S}\mathrm{pan}(\mathbf{Cat})$ is precisely a double category.

Replacements for **DblCat**, Option 2: $\mathbb{Q}\mathbf{DblCat}_v$

When considering colimits of double categories we would like to have a double category that has double categories as objects. There are six double categories

- $\mathbb{V}\mathbf{DblCat}_{h/v}$,
- $\mathbb{H}\mathbf{DblCat}_{h/v}$
- $\mathbb{Q}\mathbf{DblCat}_{h/v}$.

We will work with $\mathbb{Q}\mathbf{DblCat}_v$.

Diagrams Indexed by a Double Category

These observations lead us to two types of "double indexing functors":

- **When aiming for double fibrations:** A double indexing functor is a *contravariant lax pseudo double functor*,

$$\mathbb{D}^{\mathsf{op}} \to \mathbb{S}\mathsf{pan}(\mathbf{Cat})$$

  where $\mathbb{S}\mathsf{pan}(\mathbf{Cat})$ is a double 2-category (as we are considering **Cat** here as a 2-category).

- **When aiming for doubly lax colimits:** An indexing double functor is a double functor

$$\mathbb{D} \to \mathbb{Q}(\mathbf{DblCat}_v),$$

  also referred to as a *vertical double functor*

$$\mathbb{D} \xrightarrow{\;v\;} \mathbf{DblCat}.$$

Grothendieck for $F\colon \mathbb{D} \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})$

A lax double pseudo functor $F\colon \mathbb{D}^{\mathrm{op}} \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})$ gives rise to pseudo functors

$$F_0\colon \mathbb{D}_0^{\mathrm{op}} \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})_0 = \mathbf{Cat} \text{ and } F_1\colon \mathbb{D}_1^{\mathrm{op}} \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})_1 \overset{\mathrm{apx}}{\to} \mathbf{Cat}$$

The Grothendieck category of elements gives us cloven fibrations

$$\mathbb{E}\mathrm{l}(F)_0 \to \mathbb{D}_0 \quad \text{and} \quad \mathbb{E}\mathrm{l}(F)_1 \to \mathbb{D}_1.$$

Now, $\mathbb{E}\mathrm{l}(F)_0$ and $\mathbb{E}\mathrm{l}(F)_1$ form the category of objects and arrows respectively of the double category $\mathbb{E}\mathrm{l}(F)$ with a *double fibration*

$$\mathbb{E}\mathrm{l}(F) \to \mathbb{D}$$

The Double Fibration $\mathbb{E}\mathrm{l}(F) \to \mathbb{D}$

Notation

For $F\colon \mathbb{D} \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})$, and an outer arrow $m\colon A \longrightarrow\!\!\!\!\!\bullet\!\!\!\!\rightarrow B$ of $\mathbb{D}$, we denote its image by

$$\begin{tikzcd} & Fm \arrow[dl, "L_m"'] \arrow[dr, "R_m"] & \\ FA & & FB \end{tikzcd}$$

Then $\mathbb{E}\mathrm{l}(F)$ has

- *Inner arrows* $(A, X) \xrightarrow{(f,\overline{f})} (C, Z)$ with $f\colon A \to C$ in $\mathbb{D}$ and $\overline{f}\colon X \to f^*Z$ in FA;
- *Outer arrows* $(m, \overline{m})\colon (A, X) \longrightarrow\!\!\!\!\!\bullet\!\!\!\!\rightarrow (B, Y)$ with $m\colon A \longrightarrow\!\!\!\!\!\bullet\!\!\!\!\rightarrow B$ in $\mathbb{D}$ and $\overline{m} \in Fm$ such that $L_m\overline{m} = X$ and $R_m\overline{m} = Y$

The Double Fibration $\mathbb{E}\mathrm{l}(F) \to \mathbb{D}$

- $\mathbb{E}\mathrm{l}(F)$ has squares of the form

$$\begin{tikzcd} (A,X) \arrow[r, "(f,\overline{f})"] \arrow[d, "(m,\overline{m})"', "\bullet" marking] \arrow[dr, phantom, "(\theta,\overline{\theta})"] & (C,Z) \arrow[d, "(n,\overline{n})", "\bullet" marking] \\ (B,Y) \arrow[r, "(g,\overline{g})"'] & (D,W) \end{tikzcd} \quad \text{for} \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"', "\bullet" marking] \arrow[dr, phantom, "\theta"] & C \arrow[d, "n", "\bullet" marking] \\ B \arrow[r, "g"'] & D \end{tikzcd} \text{ in } \mathbb{D}$$

  and $\overline{\theta}\colon \overline{m} \to \theta^*\overline{n}$ in Fm such that $L_m\overline{\theta} = \overline{f}$ and $R_m\overline{\theta} = \overline{g}$.
- The projection double functor $\mathbb{E}\mathrm{l}(F) \to \mathbb{D}$ is a double fibration.

What is a Double Fibration?

Suggestion

Take an internal category in **Fib**.

Problem

Fib doesn't have all the 2-pullbacks we would need.

Also, the *fibrational strictness* of s and t would the same as that of y and $\otimes$, which is not in line with what we know about pseudo double functors.

The solution

A double fibration is a pseudo category in **Fib** such that s and t are in c**Fib** (that is, they preserve the chosen cleavages).

This translates into:

Definition of a Double Fibration

A **double fibration** as defined on the previous slide is the same as a (strict) double functor $P : \mathbb{E} \to \mathbb{B}$ between (pseudo) double categories

$$\begin{tikzcd}
\mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 \arrow[r, "\otimes_{\mathbb{E}}"] \arrow[d, "P_1 \times_{P_0} P_1"'] & \mathbb{E}_1 \arrow[r, shift left=2, "s_{\mathbb{E}}"] \arrow[r, shift right=2, "t_{\mathbb{E}}"'] \arrow[d, "P_1"] & \mathbb{E}_0 \arrow[l, "y_{\mathbb{E}}" description] \arrow[d, "P_0"] \\
\mathbb{B}_1 \times_{\mathbb{B}_0} \mathscr{B}_1 \arrow[r, "\otimes_{\mathbb{B}}"] & \mathbb{B}_1 \arrow[r, shift left=2, "s_{\mathbb{B}}"] \arrow[r, shift right=2, "t_{\mathbb{B}}"'] & \mathbb{B}_0 \arrow[l, "y_{\mathbb{B}}" description]
\end{tikzcd}$$

such that

1. P_0 and P_1 are fibrations,
2. they admit a cleavage such that $s_{\mathbb{E}}$ and $t_{\mathbb{E}}$ are cleavage-preserving, and
3. $y_{\mathbb{E}}$ and $\otimes_{\mathbb{E}}$ are Cartesian-morphism preserving.

Some Examples

- When $\mathbb{E}_0 = \mathbb{B}_0 = 1$, we recover monoidal fibrations [1];
- For any 2-functor $P : \mathsf{E} \to \mathsf{B}$, we have that P is a 2-fibration [2] if and only if $\mathbb{Q}P : \mathbb{Q}\mathsf{E} \to \mathbb{Q}\mathsf{B}$ is a double fibration;
- When P_0 and P_1 are discrete fibrations, we recover discrete double fibrations [3];
- The double Grothendieck construction in Definition 5.3 of [4] is also a double fibration.

[1] Framed Bicategories and Monoidal Fibrations, Shulman (2008).
[2] Fibred 2-Categories and Bicategories, Buckley (2014).
[3] Discrete Double Fibrations, Lambert (2021).
[4] Double Categories of Open Dynamical Systems, Myers (2021).

More Examples

- The domain fibration: $\mathrm{dom}\colon \mathbb{D}^2 \to \mathbb{D}$,

$$\begin{tikzcd} \mathbb{D}_1^2 \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] \arrow[d, "\mathrm{dom}"'] & \mathbb{D}_0^2 \arrow[d, "\mathrm{dom}"] \\ \mathbb{D}_1 \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & \mathbb{D}_0 \end{tikzcd}$$

- $Im\colon \mathbb{S}\mathrm{pan} \to \mathbb{R}\mathrm{el}$ is a double opfibration.
- There is a split double fibration $\Pi\colon \mathbb{F}\mathrm{am}(\mathcal{C}) \to \mathbb{S}\mathrm{pan}$.
- There is a codomain fibration $\mathrm{cod}\colon \mathbb{D}^2 \to \mathbb{D}$ if
  - $\mathbb{D}_1$ and $\mathbb{D}_0$ have chosen finite limits,
  - these limits are preserved on the nose by s and t
  - and up to iso by y and $\otimes$.

Double Fibrations are Internal Fibrations

The notion of *internal fibration* for a 2-category was given by Street in 1974. Let **DblCat** be the 2-category of pseudo double categories, pseudo functors and horizontal/inner transformations.

Theorem [Cruttwell, Lambert, P., Szyld]

A *strict* double functor $P : \mathbb{E} \to \mathbb{B}$ is an internal fibration in **DblCat** if and only if it is a double fibration

Double Fibrations are Internal Fibrations

The notion of *internal fibration* for a 2-category was given by Street in 1974. Let **DblCat** be the 2-category of pseudo double categories, pseudo functors and horizontal/inner transformations.

Theorem [Cruttwell, Lambert, P., Szyld]

A *strict* double functor $P : \mathbb{E} \to \mathbb{B}$ is an internal fibration in **DblCat** if and only if it is a double fibration

In addition,

- A *pseudo* double functor P is an internal fibration in $\mathbf{DblCat}_\ell$ iff P_0 and P_1 are fibrations that admit cleavages preserved by $s_\mathbb{E}$ and $t_\mathbb{E}$
- It is an internal fibration in **DblCat** iff in addition, $y_\mathbb{E}$ and $\otimes_\mathbb{E}$ are Cartesian-morphism preserving.
- a strict double functor P is an internal fibration in $\mathbf{DblCat}_s$ iff P_0 and P_1 are fibrations that admit cleavages preserved by $s_\mathbb{E}$, $t_\mathbb{E}$, $y_\mathbb{E}$ and $\otimes_\mathbb{E}$.

The {Fibrations} $\overset{\simeq}{\leftarrow}$ {Indexed} Theorem

Let $\mathbf{I}\mathbb{S}\mathrm{pan}(\mathbf{Cat})$ be the category of contravariant lax pseudo double functors valued in the double 2-category $\mathbb{S}\mathrm{pan}(\mathbf{Cat})$.

Theorem [Cruttwell, Lambert, P., Szyld]

There is an equivalence of categories $\mathbf{DblFib} \simeq \mathbf{I}\mathbb{S}\mathrm{pan}(\mathbf{Cat})$

Idea for the proof: use pseudo monoids in double 2-categories.

$$\mathbf{Fib} \simeq \mathbf{ICat} \text{ restricts to } \mathrm{c}\mathbf{Fib} \simeq \mathbf{ICat}_t, \text{ so } \mathbb{S}\mathrm{pan}_c(\mathbf{Fib}) \simeq \mathbb{S}\mathrm{pan}_t(\mathbf{ICat}).$$

Now we lift:

$$\mathbf{DblFib} := \mathbf{PsMon}(\mathbb{S}\mathrm{pan}_c(\mathbf{Fib})) \simeq \mathbf{PsMon}(\mathbb{S}\mathrm{pan}_t(\mathbf{ICat})) \simeq \mathbf{I}\mathbb{S}\mathrm{pan}(\mathbf{Cat}) \quad \square$$

Restricting to monoidal or to discrete fibrations, we recover the results in (Moeller-Vasilakopoulou, 2020) and (Lambert, 2021). The right-to-left functor restricts to the construction spelled out in (Paré, 2011).

Option 2: Vertical Indexing Functors $F\colon \mathbb{D} \to \mathbb{Q}\mathbf{DblCat}_v$

We have so far only worked out the strict case, where both $\mathbb{D}$ and F are assumed to be strict, and are working on the pseudo case.

Some concerns you may have:

- Have we lost our ability to use horizontal transformations and modifications?
- Have we lost our ability to distinguish between horizontal and vertical arrows in the indexing double category $\mathbb{D}$?

No, they will show up in the notion of **doubly lax transformation**. Our lax colimits are lax with respect to a new notion of tranformation.

Intro to Doubly Lax Transformations

- We will introduce a **cylinder double category** $\mathrm{Cyl}_v(\mathbf{DblCat})$.
- There are vertical double functors

$$\mathrm{Cyl}_v(\mathbf{DblCat}) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \mathbf{DblCat}$$

- A **doubly lax transformation** $\alpha \colon F \Rightarrow G \colon \mathbb{D} \nrightarrow \mathbf{DblCat}$ is given by a double functor

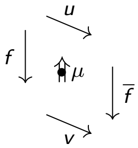
$$\alpha \colon \mathbb{D} \to \mathrm{Cyl}_v(\mathbf{DblCat})$$

  such that $d_0\alpha = F$ and $d_1\alpha = G$.

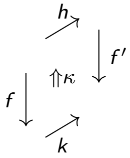
The Double Category of (Vertical) Cylinders

The double category $\mathrm{Cyl}_v(\mathbf{DblCat})$ of **vertical cylinders** is defined by:

- Objects are double functors, denoted by $\downarrow f$.
- Vertical arrows $f \xrightarrow{(u,\mu,v)} \overline{f}$ are given by vertical transformations,



- Horizontal arrows $f \xrightarrow{(h,\kappa,k)} f'$ are given by horizontal transformations,



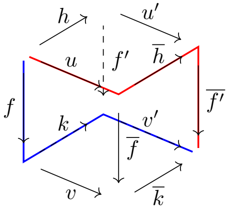
Double Cylinders

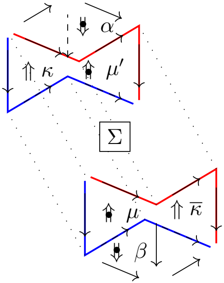
A double cell, $\begin{tikzcd} f \arrow[r, "{(h,\kappa,k)}"] \arrow[d, "{(u,\mu,v)}"'] \arrow[dr, phantom, "{(\alpha,\Sigma,\beta)}"] & f' \arrow[d, "{(u',\mu',v')}"] \\ \overline{f} \arrow[r, "{(\overline{h},\overline{\kappa},\overline{k})}"'] & \overline{f'} \end{tikzcd}$ consists of two vertical 2-cells,

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \cdot \arrow[d, "u'"] \\ \cdot \arrow[r, "\overline{h}"'] & \cdot \end{tikzcd}, \quad \begin{tikzcd} \cdot \arrow[r, "k"] \arrow[d, "v"'] \arrow[dr, phantom, "\Downarrow\beta"] & \cdot \arrow[d, "v'"] \\ \cdot \arrow[r, "\overline{k}"'] & \cdot \end{tikzcd}$$

and a modification Σ,

$$\begin{tikzcd} v'kf \arrow[r, Rightarrow, "v'\kappa"] \arrow[d, Rightarrow, "\beta f"'] & v'f'h \arrow[d, Rightarrow, "\mu' h"] \\ \overline{k}vf \arrow[d, Rightarrow, "\overline{k}\mu"'] \arrow[r, phantom, "\Sigma"] & \overline{f'}u'h \arrow[d, Rightarrow, "\overline{f'}\alpha"] \\ \overline{k}\,\overline{f}u \arrow[r, Rightarrow, "\overline{\kappa}u"'] & \overline{f'}\,\overline{h}u \end{tikzcd}$$





Cylinders and Transformations

- There are vertical double functors $d_0, d_1\colon \mathrm{Cyl}_v(\mathbf{DblCat}) \xrightarrow{v} \mathbf{DblCat}$, sending a cylinder to its top and bottom respectively;
- A **doubly lax transformation** $\theta\colon F \Rightarrow G$ between vertical double functors $F, G\colon \mathbb{D} \xrightarrow{v} \mathbf{DblCat}$ is given by a double functor

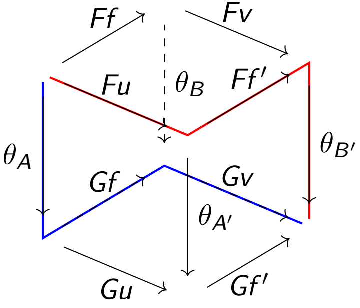
$$\theta\colon \mathbb{D} \to \mathrm{Cyl}_v(\mathbf{DblCat}),$$

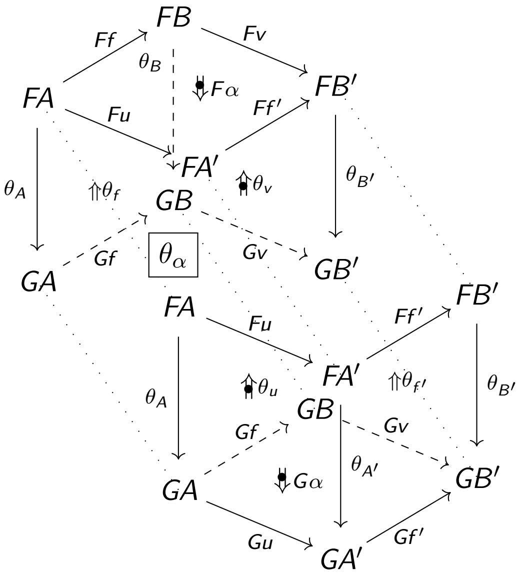
such that $d_0\theta = F$ and $d_1\theta = G$.

Doubly Lax Transformations $\theta\colon F \Rightarrow G$

For each double cell
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd},$$

$$\begin{tikzcd} {\color{blue}GvGf\theta_A} \arrow[r, Rightarrow, "Gv\theta_f"] \arrow[d, Rightarrow, "G\alpha\theta_A"'] \arrow[ddr, phantom, "\theta_\alpha"] & Gv\theta_B Ff \arrow[d, Rightarrow, "\theta_v Ff"] \\ Gf'Gu\theta_A \arrow[d, Rightarrow, "Gf'\theta_u"'] & \theta_{B'}FvFf \arrow[d, Rightarrow, "\theta_{B'}F\alpha"] \\ Gf'\theta_{A'}Fu \arrow[r, Rightarrow, "\theta_{f'}Fu"'] & {\color{red}\theta_{B'}Ff'Fu} \end{tikzcd}$$





FB
Ff
Fv
θ_B
$F\alpha$
FA
FB'
Fu
Ff'
FA'
θ_A
θ_f
θ_v
$\theta_{B'}$
GB
θ_α
Gf
Gv
GA
GB'
FA
FB'
Fu
Ff'
FA'
θ_A
θ_u
$\theta_{f'}$
$\theta_{B'}$
GB
Gf
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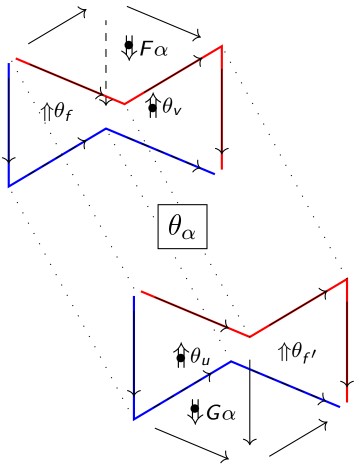
$G\alpha$
$\theta_{A'}$
GA
GB'
Gu
Gf'
GA'



$F\alpha$
θ_f
θ_v
θ_α
θ_u
$\theta_{f'}$
$G\alpha$

Doubly Lax Transformations

- Let $F, G\colon \mathbb{D} \xrightarrow{v} \mathbf{DblCat}$ be vertical double functors.
- Since doubly lax transformations $F \Rightarrow G$ are represented by double functors,

$$\mathbb{D} \to \mathrm{Cyl}_v(\mathbf{DblCat})$$

  they are the objects of a hom double category

$$\mathbb{H}\mathrm{om}_{d\ell}(F, G) \subset \mathbf{DblCat}(\mathbb{D}, \mathrm{Cyl}_v(\mathbf{DblCat})).$$

Lax Transformations Between 2-Functors

- By applying $\mathbb{Q}$ to the hom-categories of a 2-category $\mathcal{B}$, we can make it into a **DblCat**-enriched category $\widehat{\mathrm{Q}}(\mathcal{B})$.
- This allows us to view lax transformations between 2-functors as a special case of the new doubly lax transformations.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathcal{B} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathbb{Q}\mathcal{A} \arrow[r, bend left, "\mathbb{Q}F"] \arrow[r, bend right, "\mathbb{Q}G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \widehat{\mathrm{Q}}(\mathcal{B}) \end{tikzcd}$$

- By taking a restricted $\mathbb{Q}$ on the codomain, taking only a particular class Ω of 2-cells of $\mathcal{B}$ for the local horizontal arrows, we obtain Ω-transformations.
- By taking a restricted $\mathbb{Q}$ on the domain, we get Σ-transformations.

Doubly Lax Colimits

- A **doubly lax cocone** for a vertical double functor $F : \mathbb{D} \twoheadrightarrow \mathbf{DblCat}$ with vertex $\mathbb{E} \in \mathbf{DblCat}$ is a doubly lax transformation $F \overset{\theta}{\Rightarrow} \Delta\mathbb{E}$.
- There is a double category,

$$\mathbb{L}\mathsf{C}(F, \mathbb{E}) := \mathbb{H}\mathsf{om}_{d\ell}(F, \Delta\mathbb{E})$$

  of doubly lax cocones with vertex $\mathbb{E}$.
- A doubly lax cocone $F \overset{\lambda}{\Longrightarrow} \Delta\mathbb{L}$ is the **doubly lax colimit** of F if, for every $\mathbb{E} \in \mathbf{DblCat}$,

$$\mathbf{DblCat}(\mathbb{L}, \mathbb{E}) \xrightarrow{\lambda^*} \mathbb{L}\mathsf{C}(F, \mathbb{E})$$

  is an isomorphism of double categories.
- The doubly lax colimit can be obtained by a **double Grothendieck construction**, denoted by $\mathbb{G}\mathrm{r}\, F = \int_{\mathbb{D}} F$.

The Double Grothendieck Construction: Objects and Arrows

Let $\mathbb{D} \xrightarrow[v]{F} \mathbf{DblCat}$ be a vertical double functor. The **double category of elements**, $\mathbb{G}\mathrm{r}\, F = \int_{\mathbb{D}} F$, is defined by:

- Objects: (C, x) with C in $\mathbb{D}$ and x in FC,
- Vertical arrows:
$$(C, x) \xrightarrow{(u,\rho)} (C', x'),$$
where $C \xrightarrow{u} C'$ in $\mathbb{D}$ and $Fux \xrightarrow{\rho} x'$ in FC'.
- Horizontal arrows:
$$(C, x) \xrightarrow{(f,\varphi)} (D, y),$$
where $C \xrightarrow{f} D$ in $\mathbb{D}$, and $Ffx \xrightarrow{\varphi} y$ in FD.

The Double Grothendieck Construction: Double Cells

- Double cells: $$\begin{tikzcd} (C,x) \arrow[r, "{(f,\varphi)}"] \arrow[d, "{(u,\rho)}"'] \arrow[dr, phantom, "{(\alpha,\Phi)}"] & (D,y) \arrow[d, "{(v,\lambda)}"] \\ (C',x') \arrow[r, "{(f',\varphi')}"'] & (D',y') \end{tikzcd}$$, where $\alpha\colon \left(u\ {}^{f}_{f'}\ v\right)$ is a double cell in $\mathbb{D}$ and Φ is a double cell in FD':

$$\begin{tikzcd} FvFfx \arrow[r, "Fv\varphi"] \arrow[d, "(F\alpha)_x"'] & Fvy \arrow[dd, "\lambda"] \\ Ff'Fux \arrow[d, "Ff'\rho"'] \arrow[r, phantom, "\Phi"] & \\ Ff'x' \arrow[r, "\varphi'"'] & y' \end{tikzcd}$$

Factorization

- Any horizontal arrow (f,φ) can be factored as $(A,x) \xrightarrow{(f,1_{Ffx})} (B,Ffx) \xrightarrow{(1_B,\varphi)} (B,y)$.
- Any vertical arrow (u,ρ) can be factored as $(A,x) \overset{(u,1^\bullet_{Fux})}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} (A',Fux) \overset{(1^\bullet_{A'},\rho)}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} (A',x')$.
- And any double cell (α,Φ) can be factored as

$$
\begin{tikzcd}
(A,x) \arrow[r, "{(f,1_{Ffx})}"] \arrow[dd, "\bullet" marking, "{(u,1^\bullet_{Fux})}"'] & (B,Ffx) \arrow[r, "{(1_B,\varphi)}"] \arrow[d, "\bullet" marking, "{(v,1^\bullet_{F(vf)x})}"'] \arrow[dr, phantom, "{(1_v,1^\bullet_{Fv\varphi})}"] & (B,y) \arrow[d, "\bullet" marking, "{(v,1^\bullet_{Fvy})}"] \\
{} \arrow[r, phantom, "{(\alpha,1_{(F\alpha)_x})}"] & (B',FvFfx) \arrow[r, "{(1_{B'},Fv\varphi)}"] \arrow[d, "\bullet" marking, "{(1^\bullet_{B'},(F\alpha)_x)}"] \arrow[dr, phantom, "{(1^\square_{B'},\Phi)}"] & (B',Fvy) \arrow[dd, "\bullet" marking, "{(1^\bullet_{B'},\lambda)}"] \\
(A',Fux) \arrow[r, "{(f',1_{F(f'u)x})}"] \arrow[d, "\bullet" marking, "{(1^\bullet_{A'},\rho)}"'] \arrow[dr, phantom, "{(1^\bullet_{f'},1_{Ff'\rho})}"] & (B',Ff'Fux) \arrow[d, "\bullet" marking, "{(1^\bullet_{B'},Ff'\rho)}"] & \\
(A',x') \arrow[r, "{(f',1_{Ff'x'})}"'] & (B',Ff'x') \arrow[r, "{(1_{B'},\varphi')}"'] & (B',y')
\end{tikzcd}
$$

The Main Theorem

- There is a doubly lax cocone $F \overset{\lambda}{\Longrightarrow} \Delta \mathbb{G}\mathrm{r}\, F$ with the required universal property:

$$\lambda^* \colon \mathbf{DblCat}\left(\int_{\mathbb{D}} F, \mathbb{E}\right) \to \mathbb{L}\mathsf{C}\left(\int_{\mathbb{D}} F, \mathbb{E}\right)$$

  is an iso of double categories for all $\mathbb{E} \in \mathbf{DblCat}$.

- Furthermore, $\int_{\mathbb{D}}$ extends to a functor of **DblCat**-categories

$$\mathrm{Hom}_v(\mathbb{D}, \mathbf{DblCat})_{d\ell} \to \mathbf{DblCat}/\mathbb{D}$$

  which is locally an isomorphism of double categories

$$\mathbb{H}\mathrm{om}_{d\ell}(F, G) \cong (\mathbf{DblCat}/\mathbb{D})\left(\int_{\mathbb{D}} F \to \mathbb{D}, \int_{\mathbb{D}} G \to \mathbb{D}\right).$$

Application I: Tricolimits in **2-Cat**

- For a 2-category $\mathcal{A}$ and a 2-functor $F\colon \mathcal{A} \to \mathbf{2\text{-}Cat}$, we construct a double index functor as follows. First take

$$\mathcal{A} \xrightarrow{F} \mathbf{2\text{-}Cat} \xrightarrow{\mathbb{V}} \mathbf{DblCat}_v$$

  and then apply $\mathbb{V}$ to obtain:

$$\mathbb{V}(\mathcal{A}) \xrightarrow{\mathbb{V}(\mathbb{V}\circ F)} \mathbb{V}(\mathbf{DblCat}_v) \xrightarrow{\text{incl}} \mathbb{Q}(\mathbf{DblCat}_v).$$

- Applying the double Grothendieck construction gives us

$$\int_{\mathbb{V}\mathcal{A}} \mathbb{V}(\mathbb{V}\circ F) = \mathbb{V}\int_{\mathcal{A}} F$$

  (as defined by Bakovic and Buckley)
- The functor $\mathbb{V}\colon \mathbf{2\text{-}Cat} \to \mathbf{DblCat}_v$ induces an isomorphism of 3-categories between **2-Cat** and its image in $\mathbf{DblCat}_v$.
- It follows that $\int_{\mathcal{A}} F$ is the **lax tricolimit** of F in **2-Cat**.

Application II: Categories of Elements

- For a functor $F\colon \mathsf{A} \to \mathbf{Set}$,

$$\mathrm{colim} F = \pi_0 \mathbf{El}(dF),$$

  where

$$\mathsf{A} \xrightarrow{F} \mathbf{Set} \xrightarrow{d} \mathbf{Cat}$$

  and $\mathbf{El}(dF)$ has objects (A, x) with $x \in F(A)$ and arrows $f\colon (A, x) \to (A', x')$ where $f\colon A \to A'$ with $F(f)(x) = x'$.

- This follows from the universal property of the elements construction as lax colimit by applying it to cones with discrete categories as vertex and using the adjunction $\pi_0 \dashv d$.

- We can apply the same paradigm to a functor $F\colon \mathcal{A} \to \mathbf{Cat}$ and use

$$\mathbf{Cat} \underset{\mathbb{V}}{\overset{\pi_0}{\underset{\perp}{\leftrightarrows}}} \mathbf{DblCat}_v$$

where the π_0 is taken with respect to horizontal arrows and cells to obtain a quotient of the vertical category of a double category.

- It follows from our Main Theorem that $\pi_0 \int_{\mathbb{H}\mathsf{A}} \mathbb{Q}(\mathbb{V}\circ F)$ gives the **strict 2-categorical colimit** of F.
- $\int_{\mathbb{H}\mathsf{A}} \mathbb{Q}(\mathbb{V}\circ F)$ is actually $\mathbb{E}\mathrm{l}(F)$, introduced by Paré (1989): its double cells "(α, Φ)" are in this case given by 2-cells $\alpha\colon f \Longrightarrow f'$ in $\mathcal{A}$:

$$\begin{tikzcd}
(C,x) \arrow[r, "{(f,id)}"] \arrow[d, "{(id,\rho)}"'] \arrow[dr, phantom, "{(\alpha,id)}"] & (D,y) \arrow[d, "{(id,\lambda)}"] \\
(C,x') \arrow[r, "{(f',id)}"'] & (D,y')
\end{tikzcd}
\qquad
\begin{tikzcd}
Ffx \arrow[r, "id"] \arrow[d, "(F\alpha)_x"'] \arrow[dr, phantom, "id"] & Ffx \arrow[d, "\lambda"] \\
Ff'x \arrow[r, "Ff'\rho"'] & Ff'x' .
\end{tikzcd}$$

Application III: The double categorical wreath product

For a functor $F\colon \mathscr{A}^{\mathrm{op}} \to \mathbf{Cat}$, we consider:

$$\mathscr{A}^{\mathrm{op}} \xrightarrow{F} \mathbf{Cat} \xrightarrow{\mathbb{Q}} \mathbf{DblCat}_v \xrightarrow{(\ )^{\wedge}} \mathbf{DblCat}_v$$

where $\mathbb{E} \to \mathbb{E}^{\wedge}$ is the horizontal flip functor, and apply $\mathbb{Q}$ to all of this:

$$\int_{\mathbb{Q}\mathscr{A}} \mathbb{Q}((\mathbb{Q} \circ F)^{\wedge}) = F \wr F^{op}$$

as introduced by Myers (2020). In this case our Φ in (α, Φ) matches the basic diagram in his definition

$$\begin{tikzcd}
FvFfx \arrow[r, "Fv\varphi"] \arrow[d, "(F\alpha)_x"', "\bullet" marking] & Fvy \arrow[dd, "\lambda", "\bullet" marking] \\
Ff'Fux \arrow[d, "Ff'\rho"', "\bullet" marking] \arrow[ur, phantom, "\Phi"] & \\
Ff'x' \arrow[r, "\varphi'"] & y'
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
f_1^* E_3 \arrow[r, "f_1^\sharp"] \arrow[d, "f_1^* g_{2\sharp}"'] & E_1 \arrow[dd, "g_{1\sharp}"] \\
f_1^* g_2^* E_4 \arrow[d, equal] & \\
g_1^* f_2^* E_4 \arrow[r, "g_1^* f_2^\sharp"] & g_1^* E_2
\end{tikzcd}$$

$\mathbb{G}\mathrm{r}\, F \to \mathbb{D}$ is also a fibration

A double functor $P\colon \mathbb{E} \to \mathbb{B}$ is an **hv-split coop-fibration** if the following four induced functors are opfibrations of categories that admit cleavages that are suitably **compatible** and **hv-split**.

1v. *Opfibration on vertical arrows*: $P\colon \mathscr{V}\mathbb{E} \to \mathscr{V}\mathbb{B}$ between the categories of objects and vertical arrows,

1h. *Opfibration on horizontal arrows*: $P\colon \mathscr{H}\mathbb{E} \to \mathscr{H}\mathbb{B}$ between the categories of objects and horizontal arrows,

2h. *Opfibration on double cells with horizontal composition*: $P\colon \mathbb{E}_1^h \to \mathbb{B}_1^h$ between the categories which have vertical arrows as objects and double cells as arrows with horizontal composition, and, let $(\mathbb{E}_1^v)_f$ be the *fiber category* which has horizontal arrows $C \to D$ over $PC \xrightarrow{1_{PC}} PC$ as objects and double cells $\alpha : (u \,{}^{g}_{g'}\, v)$ over $1_{Pu} : (Pu \,{}^{1}_{1}\, Pu)$ as arrows, composed vertically,

2v.1 *Opfibration on the 2h-fibers with vertical composition*: $P_f\colon (\mathbb{E}_1^v)_f \to \mathscr{V}\mathbb{B}$; where P_f maps $C \to D$ as above to PC and α as above to Pu.

The connection with 2-fibrations

Proposition (Bayeh, P., Szyld)

Let $P\colon \mathsf{B} \to \mathsf{E}$ *be a 2-functor between 2-categories. Then* P *is a split-2-coop-fibration as in (Buckley, 2014) if and only if* $\mathbb{V}P : \mathbb{V}\mathsf{B} \to \mathbb{V}\mathsf{E}$ *is an hv-split coop-fibration.*

The Correspondence

Theorem (Bayeh, P., Szyld)

The double Grothendieck construction $\mathbb{G}r$ is the value on objects of a **Dblcat**-*functor*

$$\mathcal{H}\mathrm{om}_v(\mathbb{D}, \mathcal{D}\mathrm{blCat})_s \xrightarrow{\mathbb{G}r} \mathbf{coop}\mathcal{F}\mathbf{ib}_{hv\text{-}s}(\mathbb{D}),$$

which is an equivalence of **Dblcat**-*categories; that is, it is esentially surjective and locally an isomorphism of double categories*

$$\mathbb{H}\mathrm{om}_s(F, G) \xrightarrow{\mathbb{G}r} (\mathbf{coop}\mathcal{F}\mathbf{ib}_{hv\text{-}s}(\mathbb{D}))(\mathbb{G}rF, \mathbb{G}rG) \tag{5.2}$$

The double functor $S\colon \mathbb{Q}\mathbf{DblCat}_v \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})$

There is a double functor connecting the two codomain options we have explored:

$$S\colon \mathbb{Q}\mathbf{DblCat}_v \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})$$

defined as follows:

- *On objects*: $\mathbb{X} \mapsto \mathbb{X}_0$;
- *On inner arrows*: $(\mathbb{X} \xrightarrow{f} \mathbb{Y}) \mapsto (\mathbb{X}_0 \xrightarrow{f_0} \mathbb{Y}_0)$;
- *On outer arrows*:
$$\begin{tikzcd} \mathbb{X} \arrow[d, "\chi", "\bullet" marking] & & \mathbb{X}_0 \\ \mathbb{Y} & \mapsto & \mathbb{X}_0 \times_{\mathbb{Y}_0} \mathbb{Y}_1 \arrow[u, "\pi_0"'] \arrow[d, "t\pi_1"] \\ & & \mathbb{Y}_0 \end{tikzcd}$$
where the pullback is taken with respect to χ_0 and s.

The double functor $S\colon \mathbb{Q}\mathrm{DblCat}_v \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})$

- *On double cells*:

$$
\begin{tikzcd}
\mathbb{X} \arrow[r, "f"] \arrow[d, "\chi"'] \arrow[dr, phantom, "\theta"] & \mathbb{X}' \arrow[d, "\chi'"] \\
\mathbb{Y} \arrow[r, "g"'] & \mathbb{Y}'
\end{tikzcd}
\qquad \mapsto \qquad
\begin{tikzcd}
\mathbb{X}_0 \arrow[r, "f_0"] & \mathbb{X}'_0 \\
\mathbb{X}_0 \times_{\mathbb{Y}_0} \mathbb{Y}_1 \arrow[u, "s\pi_0"'] \arrow[d, "t\pi_1"] \arrow[r, "(f_1\pi_0, g_1\pi_1 \otimes \theta t\pi_0)"] & \mathbb{X}'_0 \times_{\mathbb{Y}'_0} \mathbb{Y}'_1 \arrow[u, "s\pi_0"'] \arrow[d, "t\pi_1"] \\
\mathbb{Y}_0 \arrow[r, "g_0"'] & \mathbb{Y}'_0
\end{tikzcd}
$$

- $\mathbb{E}\mathrm{l}(S \circ F) = \mathbb{G}r(F)$ for any indexing functor $F\colon \mathbb{D} \to \mathbb{Q}\mathrm{DblCat}_v$.
- Work in progress: can we view $\mathbb{E}\mathrm{l}(F)$ as a double colimit for more general indexing functors into $\mathbb{S}\mathrm{pan}(\mathbf{Cat})$?