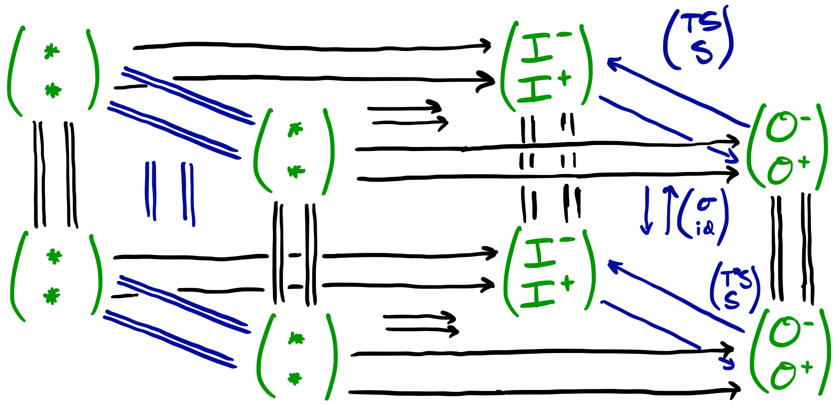
The Para Construction as a Distributive Law



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Plan

0) Why the Para Construction?

- To separate inputs from parameters / control variables

1) What the Para Construction?

- ~~Monoidal~~ Actegory Cat $\longrightarrow$ Double Cat

2) How the Para Construction?

- A distributive law between pseudomonads in fSpan(Cat)

3) Where the Para Construction?

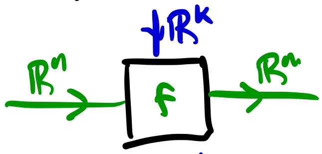
- Any suitably complete 2-cat $\mathbb{K}$ (e.g. DblCat !)

Why the Para Construction?

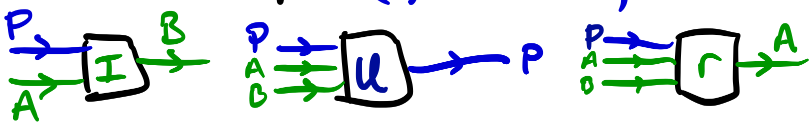
- Fong, Spivak, and Tuyeras first defined "para" as

  parameters

  obs: $\mathbb{R}^n$, maps: $(k, f) : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathcal{C}^\circ$ $f : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^m$



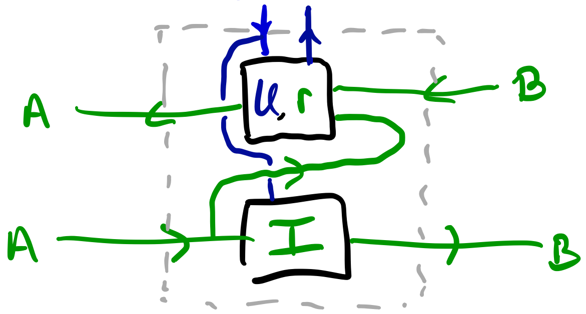
  and Learners as triples $(P, (I, U, r)) : A \to B$



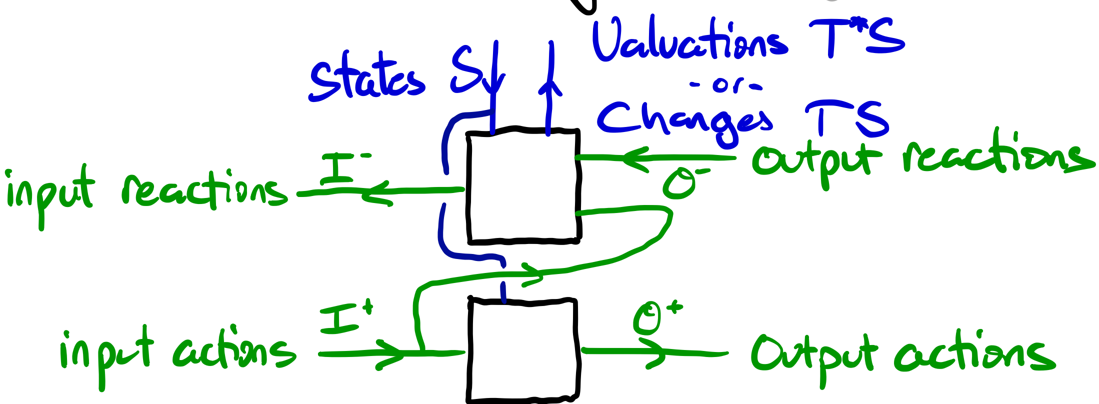
- Note, Learners are Parametrized Lenses

$$\begin{pmatrix} P \\ P \end{pmatrix} \times \begin{pmatrix} A \\ A \end{pmatrix} \underset{\longrightarrow}{\overset{\begin{pmatrix} (U,r) \\ I \end{pmatrix}}{\longleftarrow}} \begin{pmatrix} B \\ B \end{pmatrix}$$

parameters



Para-Lenses as controlled systems (cybernetics)



- Capucci, Gavranović, Hedges, Rischel: "cybernetic systems"
- Shapiro, Spivak: Dynamic Categories / Operads

$$\frac{\dfrac{S \longrightarrow [p, q](S)}{Sy^S \longrightarrow [p, q]}}{Sy^S \otimes p \longrightarrow q}$$

Lenses and Charts

- Def (Spivak): If $\mathcal{E} : \mathcal{B} \to \mathrm{CAT}$ is "an arena" an indexed cat, then

$$\mathcal{L}ens_{\mathcal{E}} := \int^{B \in \mathcal{B}} \mathcal{E}(B)^{op}, \qquad \begin{pmatrix} f^\# \\ f \end{pmatrix} : \begin{pmatrix} E_1 \\ B_1 \end{pmatrix} \leftrightarrows \begin{pmatrix} E_2 \\ B_2 \end{pmatrix} \text{ is } \begin{array}{l} f^\# : f^* E_2 \to E_1 \\ f : B_1 \to B_2 \end{array}$$

- Def (M.): The Grothendieck Double construction of $\mathcal{E}$ is

$$\mathrm{Arena}_{\mathcal{E}} := \begin{tikzcd} \begin{pmatrix} E_1 \\ B_1 \end{pmatrix} \arrow[r, shift left, "\binom{g_1^b}{g_1}"] \arrow[r, shift right] \arrow[d, shift left] \arrow[r, phantom, "\text{"both squares commute"}", yshift=-30pt] & \begin{pmatrix} E_2 \\ B_2 \end{pmatrix} \arrow[d, shift left] \\ \begin{pmatrix} E_3 \\ B_3 \end{pmatrix} \arrow[u, shift left, "\binom{f_1^\#}{f_1}"] \arrow[r, shift left] \arrow[r, shift right, "\underbrace{\binom{g_2^b}{g_2}}_{\text{"charts"}}"'] & \begin{pmatrix} E_4 \\ B_4 \end{pmatrix} \arrow[u, shift left, "\binom{f_2^\#}{f_2}"'] \end{tikzcd}$$

is

$$\begin{tikzcd} B_1 \arrow[r, "f_1"] \arrow[d, "g_1"'] & B_3 \arrow[d, "g_2"] \\ B_2 \arrow[r, "f_2"'] & B_4 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} f_1^* E_3 \arrow[rr, "f_1^\#"] \arrow[d, "f_1^* g_2^b"'] & & E_1 \arrow[dd, "g_1^b"] \\ f_1^* g_2^* E_4 \arrow[d, equal] & & \\ g_1^* f_2^* E_4 \arrow[rr, "g_1^* f_2^\#"'] & & g_1^* E_2 \end{tikzcd}$$

Systems and Behaviors

- Def (M.): A parameter-setting monoidal systems theory is an indexed cat monoidal $\mathcal{E} : \mathcal{B}^{op} \to \mathrm{CAT}$ and a pseudomonoidal section $T : \mathcal{B} \to \int^{b \in \mathcal{B}} \mathcal{E}(b)$

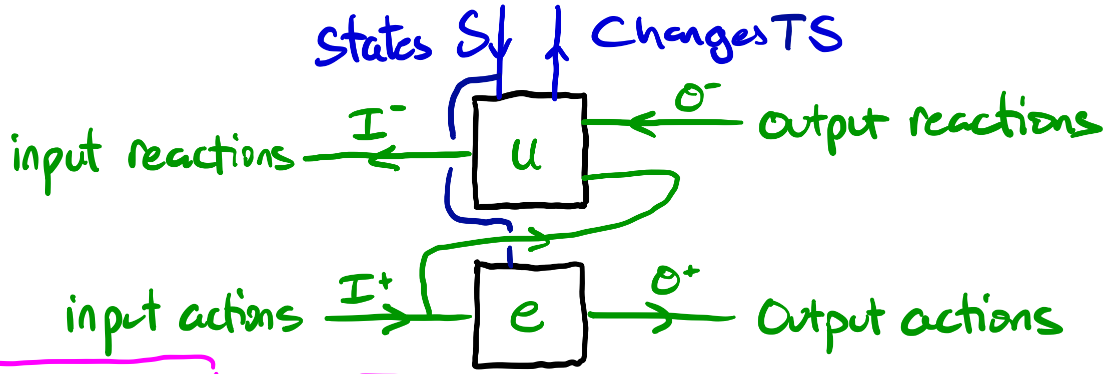
- A system in the systems theory $(\mathcal{E}, T)$ is a lens

$$\begin{pmatrix} u \\ e \end{pmatrix} : \begin{pmatrix} TS \\ S \end{pmatrix} \leftrightarrows \begin{pmatrix} I \\ O \end{pmatrix} \qquad \begin{array}{l} e : S \to O \\ u : e^* I \to TS \end{array}$$

- A control system is a para-lens (in a monoidal systems theory)

$$\begin{pmatrix} u \\ e \end{pmatrix} : \begin{pmatrix} TS \\ S \end{pmatrix} \otimes \begin{pmatrix} I^- \\ I^+ \end{pmatrix} \leftrightarrows \begin{pmatrix} O^- \\ O^+ \end{pmatrix} \qquad \begin{array}{l} e : S \otimes I^+ \to O^+ \\ u : e^* O^- \to TS \otimes I^- \end{array}$$

Systems and Behaviors



"Charting the behavior"

Given $\overbrace{i^+\in I^+,\ i^-\in I^-,\ o^+\in O^+,\ o^-\in O^-}$, a steady state is $s\in S$

st: $e(s, i^+) = o^+,\quad u(s, i^+, o^-) = (Ts, i^-)$

"Para(Arena)"

$$
\begin{tikzcd}
\binom{*}{*} \arrow[r, shift left, "\binom{i^-}{i^+}"] \arrow[r, shift right] & \binom{I^-}{I^+} \\
\binom{*}{*} \arrow[u, shift left] \arrow[from=u, shift left, "\binom{*}{*},\binom{id}{id}"'] \arrow[r, shift left] \arrow[r, shift right, "\binom{o^-}{o^+}"'] \arrow[ur, phantom, "\binom{Ts}{S}"] & \binom{O^-}{O^+} \arrow[u, shift left] \arrow[from=u, shift left, "\binom{S}{S},\binom{u}{e}"]
\end{tikzcd}
:=
\begin{tikzcd}
\binom{*}{*}\otimes\binom{*}{*} \arrow[r, shift left, "\binom{Ts}{S}\otimes\binom{i^-}{i^+}"] \arrow[r, shift right] & \binom{S}{S}\otimes\binom{I^-}{I^+} \\
\binom{*}{*} \arrow[u, shift left] \arrow[from=u, shift left] \arrow[r, shift left] \arrow[r, shift right, "\binom{o^-}{o^+}"'] & \binom{O^-}{O^+} \arrow[u, shift left] \arrow[from=u, shift left]
\end{tikzcd}
\text{ in } \mathbf{Arena}_{\mathcal{E}}
$$

Maps of Dynamic Operads

- For a monoidal systems theory $(\mathcal{E}, T)$, define $\mathbf{Org}(\mathcal{E},T)$ to be the monoidal double category

a variant of Shapiro-Spivak

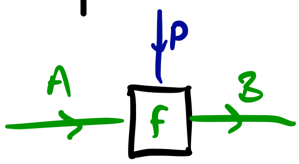
$$
\begin{tikzcd}
\binom{A_1^-}{A_1^+} \arrow[r, shift left] \arrow[r, shift right] & \binom{A_2^-}{A_2^+} \\
\binom{A_3^-}{A_3^+} \arrow[u, shift left] \arrow[from=u, shift left, "\binom{TS_1}{S_1},\binom{u_1}{e_1}"'] \arrow[r, shift left] \arrow[r, shift right] \arrow[ur, phantom, "\binom{T\phi}{\phi}"] & \binom{A_4^-}{A_4^+} \arrow[u, shift left] \arrow[from=u, shift left, "\binom{TS_2}{S_2},\binom{u_2}{e_2}"]
\end{tikzcd}
$$

- An operad enriched in $\mathbf{Org}$ is a dynamic operad
- Eg for a set X, $\binom{\Delta_+\Omega}{\Delta_+\Omega}\otimes\binom{X}{\Delta_+X}^{\otimes n} \leftleftarrows \binom{X}{\Delta_+X}$ is the prediction market dynamic operad.
- The assignment $X \mapsto$ prediction market on X is functorial into this variant of $\mathbf{Org}$.

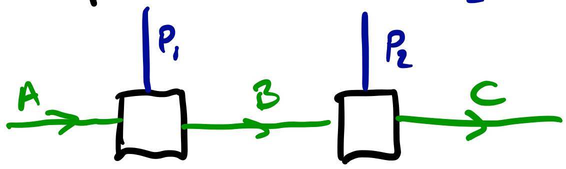
What the Para Construction?

- Given a monoidal cat $\mathcal{M}$, get a cat $\mathrm{Para}(\mathcal{M})$

  objects: Those of $\mathcal{M}$, maps: $(P, f) : A \to B$ is $f : P \otimes A \to B$



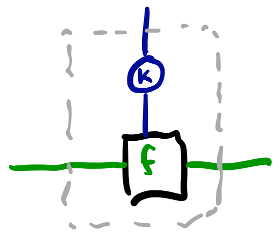
  Id: $\mathbb{1} \otimes A \xrightarrow{\sim} A$, Comp: $(P_2 \otimes P_1) \otimes A \xrightarrow{\sim} P_2 \otimes (P_1 \otimes A) \xrightarrow{P_2 \otimes f} P_2 \otimes B \xrightarrow{g} C$



- Problem: not associative!

  Solutions: Quotient by reparametrization

  ↳ any κ or just iso κ.



Using an Actegory

- An **actegory** is an action $\odot : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ of a monoidal category $\mathcal{M}$ on a category $\mathcal{C}$.

- $\mathrm{Para}(\odot)$ has maps $P \odot A \to B$

Laws

$$\left[\begin{array}{c} \rho : (M_1 \otimes M_2) \odot C \cong M_1 \odot (M_2 \odot C) \\ \eta : \mathbb{1} \odot C \cong C \end{array}\right]$$

Reparameterization and the Double Cat Para

$$\begin{tikzcd} A_1 \arrow[r, "\alpha"] \arrow[d, "P_1, f_1"'] \arrow[dr, phantom, "\overset{\kappa}{\Rightarrow}"] & A_2 \arrow[d, "P_2, f_2"] \\ B_1 \arrow[r, "\beta"'] & B_2 \end{tikzcd} \quad \text{means} \quad \begin{tikzcd} P_1 \odot A_1 \arrow[r, "\kappa \odot \alpha"] \arrow[d, "f_1"'] & P_2 \odot A_2 \arrow[d, "f_2"] \\ B_1 \arrow[r, "\beta"'] & B_2 \end{tikzcd}$$

Warning: This is a different double category than Para(Arena)

How the Para Construction?

- So $\mathbf{Para} : \mathbf{Actegory} \longrightarrow \mathbf{Double\ Category}$

  $\qquad\qquad\mathrm{Act}(\mathrm{Cat}) \qquad\qquad \mathrm{Cat}(\mathrm{Cat})$

- How to construct this? Thinking about the maps...

$$\begin{tikzcd}
& \odot \downarrow \mathcal{C} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & \\
\mathcal{M} \times \mathcal{C} \arrow[dr, "\odot"'] & & \mathcal{C}^{\downarrow} \arrow[dl, "\mathrm{dom}"] \\
& \mathcal{C} &
\end{tikzcd}$$

objects here are verticals in $\mathbf{Para}(\odot)$!

maps are the correct squares:

$$\begin{tikzcd}
P_1 \odot A_1 \arrow[r, "\kappa \odot \alpha"] \arrow[d, "f_1"'] & P_2 \odot A_2 \arrow[d, "f_2"] \\
B_1 \arrow[r, "\beta"'] & B_2
\end{tikzcd}$$

How the Para Construction?

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$$\begin{tikzcd}
& & \odot \downarrow \mathcal{C} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & \\
& \mathcal{M} \times \mathcal{C} \arrow[dl, "\pi"'] \arrow[dr, "\odot"'] & & \mathcal{C}^{\downarrow} \arrow[dl, "\mathrm{dom}"] \arrow[dr, "\mathrm{cod}"] & \\
\mathcal{C} & & \mathcal{C} & & \mathcal{C}
\end{tikzcd}$$

$\underbrace{\mathcal{C} \leftarrow \mathcal{M}\times\mathcal{C} \rightarrow \mathcal{C}}_{\text{Cayley Dbl Cat of } \odot}$ $\qquad$ $\underbrace{\mathcal{C} \leftarrow \mathcal{C}^{\downarrow} \rightarrow \mathcal{C}}_{\text{DblCat of Squares in } \mathcal{C}}$

objects here are verticals in $\mathbf{Para}(\odot)$!

maps are the correct squares:

$$\begin{tikzcd}
P_1 \odot A_1 \arrow[r, "\kappa \odot \alpha"] \arrow[d, "f_1"'] & P_2 \odot A_2 \arrow[d, "f_2"] \\
B_1 \arrow[r, "\beta"'] & B_2
\end{tikzcd}$$

Composing Monads in Span(Cat) !?

Def: $fSpan(Cat)$ is the tricategory of categories, spans

$$\begin{tikzcd} & \mathcal{E} \arrow[dl] \arrow[dr] & \\ B_1 & & B_2 \end{tikzcd}$$

whose left leg is a cloven cartesian fibration.

2-cells

$$\begin{tikzcd} B_1 \arrow[rr, equal] & & B_1 \\ & & \\ \mathcal{E}_1 \arrow[uu] \arrow[dd] \arrow[rr, "f"] \arrow[uurr, phantom, "="] & & \mathcal{E}_2 \arrow[uu] \arrow[dd] \arrow[ddll, phantom, "\Downarrow \phi \, \sim"] \\ & & \\ B_2 \arrow[rr, equal] & & B_2 \end{tikzcd}$$

$f : \mathcal{E}_1 \to \mathcal{E}_2$ is cartesian

3-cells are those from Span(Cat)

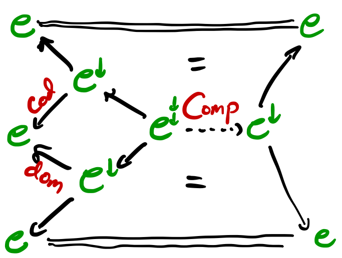
Composition is (strict) pullback.

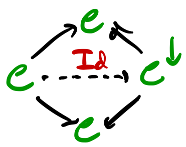
- For any cat $\mathcal{C}$,

$$\begin{tikzcd} & \mathcal{C}^{\downarrow} \arrow[dl, "dom"'] \arrow[dr, "cod"] & \\ \mathcal{C} & & \mathcal{C} \end{tikzcd}$$

is a pseudomonad in $fSpan(Cat)$.

$$\begin{tikzcd} c_1 \arrow[r] \arrow[d] & c_2 \arrow[d] \\ c_3 \arrow[r] & c_4 \end{tikzcd}$$



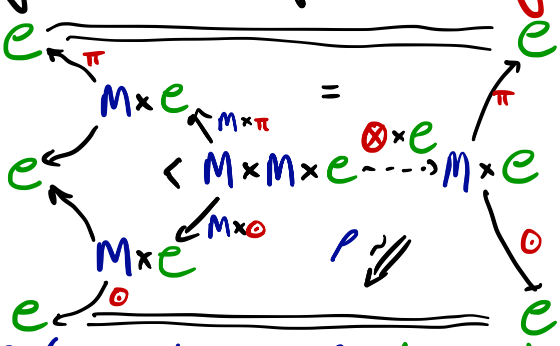


The Cayley Double Category

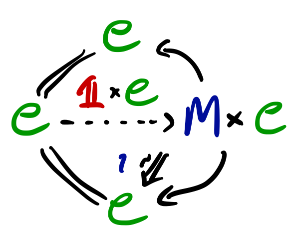
$$\begin{tikzcd} & M \times \mathcal{C} \arrow[dl, "\pi"'] \arrow[dr, "\odot"] & \\ \mathcal{C} & & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} c_1 \arrow[r] \arrow[d, "M_1"'] & c_2 \arrow[d, "M_2"] \\ M_1 \odot c_1 \arrow[r] & M_2 \odot c_2 \end{tikzcd}$$

- A categorification of the Cayley Graph



$\rho : (M_1 \otimes M_2) \odot c \cong M_1 \odot (M_2 \odot c)$

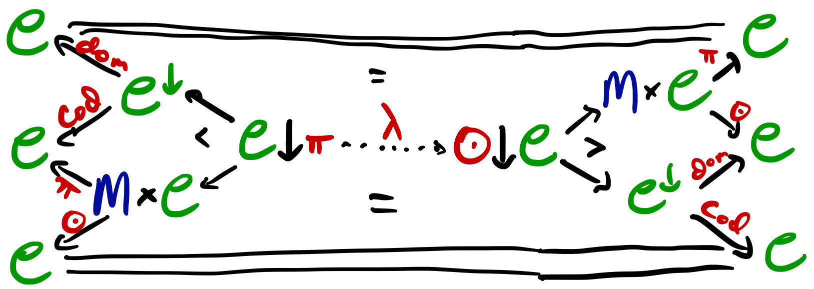


$\eta : \mathbb{1} \odot c \cong c$

- Note: The pseudomonad structure on the Cayley DblCat is exactly the structure of the actegory.

Thm: A pseudomonad in $fSpan(Cat)$ with left leg a product projection is precisely an actegory.

The Distributive Law

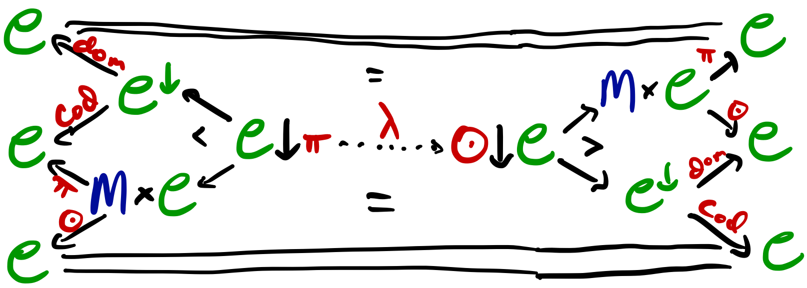


$$\left(\begin{array}{c} c' \\ f\downarrow \\ c \end{array}, m, c\right) \xmapsto{\lambda} \left(m, c', \begin{array}{c} m\odot c' \\ m\odot f \downarrow \\ m\odot c \end{array}\right)$$

+ 4 axioms and 9 coherence conditions due to Marmolejo.

$$\left(\begin{array}{c} m_1\odot c_1 \\ f\downarrow \\ c_2 \end{array}, \begin{array}{c} m_2\odot c_2 \\ g\downarrow \\ c_3 \end{array}\right) \longmapsto \begin{array}{c} (m_2\otimes m_1)\odot c_1 \xrightarrow{\rho \atop \sim} m_2\odot(m_1\odot c_1) \\ \underbrace{\quad m_2\odot f\downarrow \quad}_{\lambda(f, m_2, c_2)} \\ m_2\odot c_2 \\ \downarrow \\ c_3 \end{array}$$

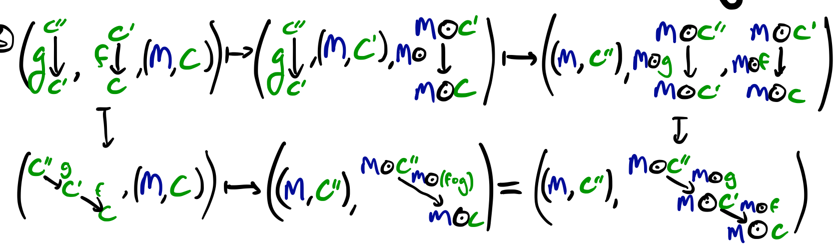
The Distributive Law

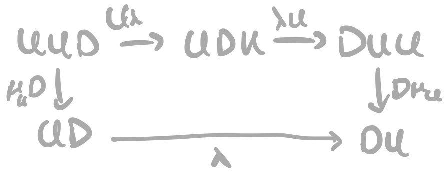


$$\left(\begin{array}{c} c' \\ f\downarrow \\ c \end{array}, m, c\right) \xmapsto{\lambda} \left(m, c', \begin{array}{c} m\odot c' \\ m\odot f \downarrow \\ m\odot c \end{array}\right)$$

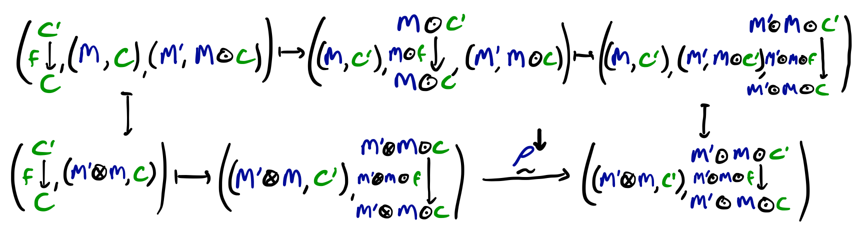
① Cartesian lift of identity is identity ② $\mathbb{1}$ pullsback to $\mathbb{1}$

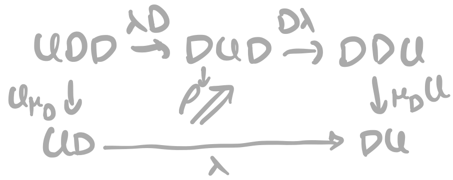
③





④





Aside: Dependent Actegories

Note: λ was the universal cartesian lift of $\pi : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$.

Def: A dependent actegory is a pseudomonad in $f\mathrm{Span}(\mathrm{Cat})$.

$$\begin{tikzcd} & \mathcal{M} \arrow[dl, "\pi"'] \arrow[dr, "\odot"] & \\ \mathcal{C} & & \mathcal{C} \end{tikzcd}$$

i.e. for $C \in \mathcal{C}$, $M \in \mathcal{M}_C \mapsto M \odot C \in \mathcal{C}$.

Distribute:

$$\begin{tikzcd} & & & & \mathcal{M} \arrow[dr, "\odot"] & \\ \mathcal{C}\downarrow\pi \arrow[r, "\mathrm{lift}"] \arrow[rr, bend right, "\lambda"'] & \mathcal{M}^{\downarrow} \arrow[r, dashed] \arrow[urrr, bend left, "\mathrm{dom}"] \arrow[drrr, bend right, "\odot\downarrow"'] & \odot\downarrow\mathcal{C} \arrow[urr] \arrow[drr] & & & \mathcal{C} \\ & & & & \mathcal{C}^{\downarrow} \arrow[ur, "\mathrm{dom}"'] & \end{tikzcd}$$

Eg: If $\mathcal{C}$ has pullbacks, $\mathrm{Para}(\mathcal{C} \xleftarrow{\mathrm{cod}} \mathcal{C}^{\downarrow} \xrightarrow{\mathrm{dom}} \mathcal{C}) = \mathrm{Span}(\mathcal{C})$.

$$\mathcal{C}\downarrow \mathrm{cod} = \mathcal{C}^{\lrcorner} \xrightarrow{\mathrm{pull}} \mathcal{C}^{\square} \dashrightarrow \mathcal{C}^{\ulcorner} = \mathrm{dom}\downarrow\mathcal{C}$$

$$\begin{tikzcd} & S \arrow[dl, "s"'] \arrow[dr, "f"] & \\ C_1 & & C_2 \end{tikzcd}$$

$x : C_1 \vdash S(x)$ type

$\vdash f : \sum_{x:C_1} S(x) \to C_2$

Where the Para Construction?

- $f\mathrm{Span}(\mathrm{Cat})$ used only pullbacks of isofibrations in Cat
- To define $\mathcal{C}^{\downarrow}$ we need powers by the arrow.
- So if $\mathbb{K}$ is any "1-cosmos"

  A 2-category with a pullback stable class of isofibrations and powered by finitely presentable categories where precomposition along injective functors is an isofibration.

  Then we can define $f\mathrm{Span}(\mathbb{K})$ and Para.

  $\pi : \mathcal{M} \to \mathcal{C}$ is a cartesian fibration in $\mathbb{K}$ if the induced map $\pi_* : \mathcal{M}^{\downarrow} \to \mathcal{C}\downarrow\pi$ has a right adjoint right inverse
- In particular,
  - if $\mathbb{K} = \mathrm{MonCat}_{\mathrm{pseudo}}$, get the monoidal structure on Para.
  - if $\mathbb{K} = \mathrm{sDbl}_v$ is the 2-cat of strict Dbl cats and vertical transformations,

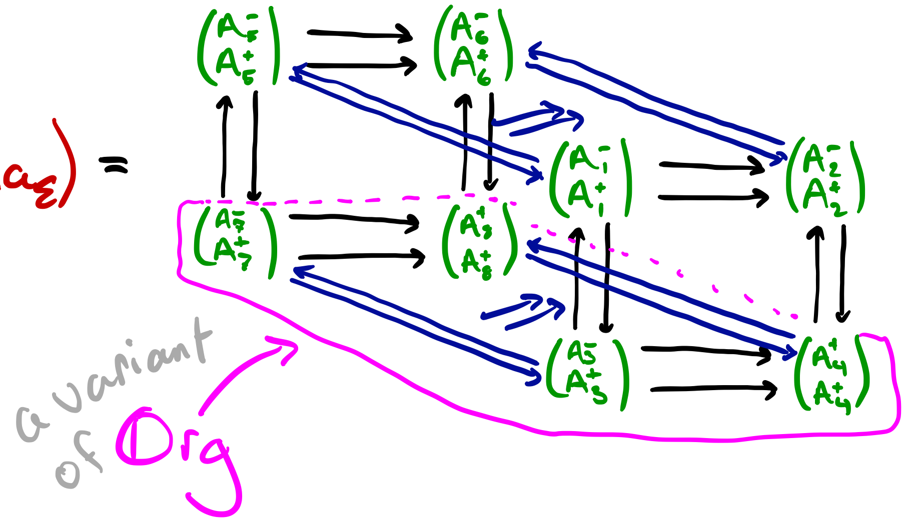
    we get a triple cat $\mathrm{Para}(\mathrm{Arena})$

The Triple Cat $\mathbf{Para}(\mathbf{Arena}_\mathcal{E})$

Let $\mathcal{E} : B \to \mathbf{Cat}$, $T : B \to \int^{b \in B} \mathcal{E}(b)$ be a monoidal systems theory

Then $hB \times \mathbf{Arena}_\mathcal{E} \xrightarrow{T \times id} \mathbf{Arena}_\mathcal{E} \times \mathbf{Arena}_\mathcal{E} \xrightarrow{\otimes} \mathbf{Arena}_\mathcal{E}$ gives an actegory in $\mathbf{sDbl}_v$.

$\mathbf{Para}(\mathbf{Arena}_\mathcal{E}) =$



Thanks!

References

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- Shapiro, Spivak - Dynamic Categories, Dynamic Operads
- M. - Categorical Systems Theory
- Marmolejo - Distributive Laws for Pseudomonads