Monoidal Kleisli double categories

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Outline

1. Double categories and monoidal double categories
2. Double monads and monoidal double monads
3. The monoidal Kleisli double construction
4. Arithmetic product of coloured symmetric sequences

Idea: generalise and formalise the so-called 'arithmetic product' of symmetric sequences or species – whose interchange with substitution provides a duoidal structure – using double categorical machinery

$$(F \boxtimes G)(m) = \int^{m_1,m_2} S(m, m_1 \cdot m_2) \times F(m_1) \times G(m_2)$$

Fibrant double categories

- A double category $\mathbb{D}$ consists of
  - object category $\mathbb{D}_0$ (0-cells & vertical 1-cells)
  - arrow category $\mathbb{D}_1$ (horizontal 1-cells & 2-morphisms) $\begin{array}{ccc} X & \overset{A}{\nrightarrow} & Y \\ f\downarrow & \Downarrow\alpha & \downarrow g \\ Z & \underset{B}{\nrightarrow} & W \end{array}$
  - $\mathbb{D}_0 \xrightarrow{\mathbf{1}} \mathbb{D}_1$, $\mathbb{D}_1 \underset{\mathfrak{t}}{\overset{\mathfrak{s}}{\rightrightarrows}} \mathbb{D}_0$, $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\circ} \mathbb{D}_1$ plus coherent isomorphisms.

0-cells, horizontal 1-cells, *globular* 2-morphisms make bicategory $\mathcal{H}(\mathbb{D})$.

- In *fibrant* double categories, vertical 1-cells turn to horizontal ones coherently : each $f\colon X \to Y$ gives $\widehat{f}\colon X \nrightarrow Y$ and $\check{f}\colon Y \nrightarrow X$ with

$$\begin{array}{ccc} X & \overset{\widehat{f}}{\nrightarrow} & X' \\ f\downarrow & \Downarrow & \| \\ X' & \underset{1}{\nrightarrow} & X' \end{array} \qquad \begin{array}{ccc} X & \overset{1}{\nrightarrow} & X \\ \| & \Downarrow & \downarrow f \\ X & \underset{\widehat{f}}{\nrightarrow} & X' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X' & \overset{\check{f}}{\nrightarrow} & X \\ \| & \Downarrow & \downarrow f \\ X' & \underset{1}{\nrightarrow} & X' \end{array} \qquad \begin{array}{ccc} X & \overset{1}{\nrightarrow} & X \\ \downarrow f & \Downarrow & \| \\ X' & \underset{\check{f}}{\nrightarrow} & X \end{array}$$

Running example: enriched profunctors

Suppose $\mathcal{V}$ is a braided monoidal cocomplete category, with $\otimes$ cocontinuous in each variable.

Fibrant double category $\mathbb{P}\mathrm{rof}_{\mathcal{V}}$ where

- object category is $\mathsf{Cat}_{\mathcal{V}}$
- a $\mathcal{V}$-profunctor $M\colon X \nrightarrow Y$ is a $\mathcal{V}$-functor $M\colon Y^{\mathrm{op}} \otimes X \to \mathcal{V}$
- a 2-morphism is a $\mathcal{V}$-natural transformation $M \Rightarrow N \circ (G^{\mathrm{op}} \otimes F)$, where $M\colon Y^{\mathrm{op}} \otimes X \to \mathcal{V}$, $G^{\mathrm{op}} \otimes F\colon Y^{\mathrm{op}} \otimes X \to Z^{\mathrm{op}} \otimes W$, $N\colon Z^{\mathrm{op}} \otimes W \to \mathcal{V}$
- horizontal composition is $(N \circ M)(z,x) = \int^{y} N(z,y) \otimes M(y,x)$
- each $\mathcal{V}$-functor $F\colon X \to Y$ gives $\mathcal{V}$-profunctors $\widehat{F}(y,x) = Y(y,Fx)$ and $\check{F}(x,y) = Y(Fx,y)$

Maps of double categories

- For double cats $\mathbb{C}$ and $\mathbb{D}$, there is a double cat $\mathsf{DblCat}[\mathbb{C},\mathbb{D}]$ where

$$\begin{array}{ccc} F & \overset{\beta}{\Longrightarrow} & G \\ {\scriptstyle\sigma}\Downarrow & \Downarrow\gamma & \Downarrow{\scriptstyle\tau} \\ H & \underset{\delta}{\Longrightarrow} & K \end{array}$$

  - F, G, H, K are double functors
  - σ, τ are vertical transformations
  - β, δ are horizontal transformations
  - γ is modification (...)

A vertical transformation σ has companion $\widehat{\sigma}$ if and only if it is *special*.

Namely each $\sigma_X\colon FX \to HX$ has $\widehat{\sigma_X}$ in $\mathbb{D}$ and each transpose is invertible

$$\begin{array}{ccccc} FX & \xrightarrow{FM} & FY & \xrightarrow{\widehat{\sigma}_Y} & HY \\ \| & & \Downarrow\widehat{\sigma}_M & & \| \\ FX & \xrightarrow[\widehat{\sigma}_X]{} & HX & \xrightarrow[HM]{} & HY \end{array} = \begin{array}{ccccccc} FX & \xrightarrow{1} & FX & \xrightarrow{FM} & FY & \xrightarrow{\widehat{\sigma_Y}} & HY \\ \| & \Downarrow & \sigma_X\downarrow & \Downarrow\sigma_M & \downarrow\sigma_Y & \Downarrow & \| \\ FX & \xrightarrow[\widehat{\sigma_X}]{} & HX & \xrightarrow[HM]{} & HY & \xrightarrow[1]{} & HY \end{array}$$

Monoidal double categories

- A double category $\mathbb{D}$ is monoidal when
  - $\mathbb{D}_0$ and $\mathbb{D}_1$ are monoidal categories, $\mathfrak{s}, \mathfrak{t}$ are strict monoidal functors
  - $(N \circ M) \otimes (N' \circ M') \cong (M \otimes M') \circ (N \otimes N')$ and $1_X \otimes 1_{X'} \cong 1_{X \otimes X'}$
  - coherence axioms are satisfied

- A double category is *oplax* monoidal with only comparison maps

$$(N \circ M) \otimes (N' \circ M') \to (M \otimes M') \circ (N \otimes N'), \quad 1_X \otimes 1_{X'} \to 1_{X \otimes X'}$$
$$I_1 \to I_1 \circ I_1, \quad I_1 \to 1_I$$

and *oplax* double functor axioms are required.

> An oplax monoidal double category with a single object and vertical arrow is precisely a duoidal category.

⋆ There is a *normality* condition that reduces to normal duoidal structure.

Maps of monoidal double categories

■ For monoidal double $\mathbb{C}$ and $\mathbb{D}$, there is double cat $\mathrm{MonDblCat}[\mathbb{C},\mathbb{D}]$

$$\begin{array}{ccc} F & \overset{\beta}{\Longrightarrow\!\!\!\!\!\!\!\!|\;\;\;\;} & G \\ {\scriptstyle\sigma}\Big\Downarrow & \Downarrow\!\!\Downarrow{\scriptstyle\gamma} & \Big\Downarrow{\scriptstyle\tau} \\ H & \underset{\delta}{\Longrightarrow\!\!\!\!\!\!\!\!|\;\;\;\;} & K \end{array}$$

- $F, .., K$ are (lax) monoidal double functors
- σ, τ are *pseudomonoidal* vertical transfs
- β, δ are monoidal horizontal transfs
- γ is monoidal modification (...)

$\star$ Pseudomonoidality is dictated by desired application: the "free symmetric monoidal category" monad is only *pseudocommutative*!

$$\begin{array}{ccc} FX\otimes FY & \longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\; & FX\otimes FY \\ \downarrow & & \downarrow{\scriptstyle\sigma_X\otimes\sigma_Y} \\ F(X\otimes Y) & \Downarrow & HX\otimes HY \\ {\scriptstyle\sigma_{X\otimes Y}}\downarrow & & \downarrow \\ H(X\otimes Y) & \longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\; & H(X\otimes Y) \end{array} \qquad \begin{array}{ccc} I & \longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\; & I \\ \downarrow & & \\ FI & \Downarrow & \Big\downarrow \\ {\scriptstyle\sigma_I}\downarrow & & \\ HI & \longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\; & HI \end{array}$$

A pseudomonoidal vertical transf has companion if and only if it is special.

Monoids in monoidal double categories

- A *vertical monoid* in a mon double cat $\mathbb{C}$ is a monoid (A, m, e) in $\mathbb{C}_0$
- A *horizontal monoid* in a mon double cat $\mathbb{C}$ is an object A with $\mu\colon A \otimes A \nrightarrow A, \eta\colon I \nrightarrow A$ and coherent

$$\begin{array}{ccccc} (A \otimes A) \otimes A & \xrightarrow{\mu\otimes 1} & A \otimes A & \xrightarrow{\mu} & A \\ {\scriptstyle\alpha}\downarrow & & \Downarrow & & \| \\ A \otimes (A \otimes A) & \xrightarrow[1\otimes\mu]{} & A \otimes A & \xrightarrow[\mu]{} & A \end{array} \quad + \text{two 2-morphisms}$$

⋆ Correspond to pseudomonoids in $\mathcal{H}(\mathbb{C})$ under mild assumptions.

A vertical monoid (A, m, e) produces a horizontal monoid $(A, \widehat{m}, \widehat{e})$ when these companions exist.

Double monads

■ For double $\mathbb{C}$, the double $\mathsf{DblCat}[\mathbb{C},\mathbb{C}]$ is monoidal with composition.

▶ A *vertical double monad* is a vertical monoid therein ; a *horizontal double monad* is a horizontal monoid therein.

For a vertical double monad $T\colon \mathbb{C}\to\mathbb{C}$ with special $m\colon TT\Rightarrow T$ and $e\colon 1\Rightarrow T$, $(T,\widehat{m},\widehat{e})$ is a horizontal double monad.

· components $m_X\colon TTX\to TX$ give $\widehat{m}_X\colon TTX\to\!\!\!\!\!+\; TX$

· naturality gets $\begin{array}{ccc} TTX & \overset{\widehat{m}_X}{\to\!\!\!\!\!+} & TX \\ {\scriptstyle TTf}\downarrow & \Downarrow & \downarrow{\scriptstyle Tf} \\ TTY & \underset{\widehat{m}_Y}{\to\!\!\!\!\!+} & TY \end{array}$ · m_M gets $\begin{array}{ccccc} TTX & \overset{TTM}{\to\!\!\!\!\!+} & TTY & \overset{\widehat{m}_Y}{\to\!\!\!\!\!+} & TY \\ \| & & \Downarrow\widehat{m}_M & & \| \\ TTX & \underset{\widehat{m}_X}{\to\!\!\!\!\!+} & TX & \underset{TM}{\to\!\!\!\!\!+} & TY \end{array}$

· vertical axioms give associativity and unit structure 2-maps

Monoidal double monads

■ For monoidal $\mathbb{C}$, MonDblCat$[\mathbb{C},\mathbb{C}]$ is monoidal with composition.

Namely the composite of two monoidal double functors $F, G \colon \mathbb{C} \to \mathbb{C}$ becomes monoidal with $GFX \otimes GFY \to G(FX \otimes FY) \to GF(X \otimes Y)$ and for two pseudomonoidal vertical $\sigma \colon F \Rightarrow F', \tau \colon G \Rightarrow G'$ have $\tau\sigma$ with

$$
\begin{array}{ccc}
GFX \otimes GFY & \xrightarrow{\quad\vert\quad} & GFX \otimes GFY \\
\downarrow & & \downarrow{\scriptstyle \tau\otimes\tau} \\
G(FX \otimes FY) & \Downarrow & G'FX \otimes G'FY \\
{\scriptstyle \tau}\downarrow & & \downarrow \\
G'(FX \otimes FY) & \xrightarrow{\quad\vert\quad} & G'(FX \otimes FY) \\
\downarrow & & \downarrow{\scriptstyle G'(\sigma\otimes\sigma)} \\
G'F(X \otimes Y) & \Downarrow & G'(F'X \otimes F'X) \\
{\scriptstyle G'\sigma}\downarrow & & \downarrow \\
G'F'(X \otimes Y) & \xrightarrow{\quad\vert\quad} & G'F'(X \otimes Y)
\end{array}
$$

► *Pseudomonoidal vertical* and *monoidal horizontal* double monads are vertical/horizontal monoids in $(\mathsf{MonDblCat}[\mathbb{C},\mathbb{C}],\circ)$.

> For pseudomonoidal vertical double $T\colon \mathbb{C}\to\mathbb{C}$ with special $m\colon TT\Rightarrow T$ and $e\colon 1\Rightarrow T$, $(T,\widehat{m},\widehat{e})$ is monoidal horizontal double monad.

E.g. from pseudomonoidality of vertical $m\colon TT\Rightarrow T$ get monoidality of horizontal $\widehat{m}\colon TT \Rrightarrow T$ via induced

$$
\begin{array}{ccc}
TTX\otimes TTY & \xrightarrow{\widehat{m}\otimes\widehat{m}} & TX\otimes TY \\
\downarrow & & \\
T(TX\otimes TY) & \Downarrow & \\
\downarrow & & \downarrow \\
TT(X\otimes Y) & \xrightarrow[\widehat{m}]{} & T(X\otimes Y)
\end{array}
$$

Horizontal Kleisli double category

- For horizontal double monad $T\colon \mathbb{C} \to \mathbb{C}$, there is a double $\mathbb{K}\mathrm{l}(T)$ with
  - $\mathbb{C}_0$ the category of objects
  - $M\colon X \rightsquigarrow Y$ are horizontal $M\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow TY$ in $\mathbb{C}$
  - 2-morphisms $\begin{array}{ccc} X & \overset{M}{\rightsquigarrow} & Y \\ {\scriptstyle f}\big\downarrow & \Downarrow & \big\downarrow{\scriptstyle g} \\ Z & \overset{N}{\rightsquigarrow} & W \end{array}$ are $\begin{array}{ccc} X & \overset{M}{\relbar\joinrel\mapstochar\joinrel\rightarrow} & TY \\ {\scriptstyle f}\big\downarrow & \Downarrow & \big\downarrow{\scriptstyle Tg} \\ Z & \overset{N}{\relbar\joinrel\mapstochar\joinrel\rightarrow} & TW \end{array}$ in $\mathbb{C}$
  - horizontal composition is $X \overset{M}{\relbar\joinrel\mapstochar\joinrel\longrightarrow} TY \overset{TN}{\relbar\joinrel\mapstochar\joinrel\longrightarrow} TTZ \overset{m_Z}{\relbar\joinrel\mapstochar\joinrel\longrightarrow} TZ$

For $\mathbb{C}$ monoidal and T monoidal horizontal double monad, if $I \to TI$ and $TX \otimes TY \xrightarrow{\tau} T(X \otimes Y)$ have companions then $\mathbb{K}\mathrm{l}(T)$ is oplax monoidal.

Induced tensor is $M \boxtimes N = X \otimes Z \overset{M\otimes N}{\relbar\joinrel\mapstochar\joinrel\longrightarrow} TY \otimes TW \overset{\widehat{\tau}}{\relbar\joinrel\mapstochar\joinrel\longrightarrow} T(Y \otimes W)$

⋆ For pseudomonoidal vertical double monad, this holds for $(T, \widehat{m}, \widehat{e})$. If moreover, *(co)strength* of T are special, then $\mathbb{K}\mathsf{l}(T)$ is normal oplax:

$$\begin{array}{ccc}
X \otimes TZ & \xrightarrow{M \otimes TN} & Y \otimes TW \\
\downarrow & \Downarrow e \otimes 1 & \downarrow \\
TX \otimes TZ & \xrightarrow{TM \otimes TN} & TY \otimes TW \\
\downarrow & \Downarrow \tau & \downarrow \\
T(X \otimes Z) & \xrightarrow[T(M \otimes N)]{} & T(Y \otimes W)
\end{array}$$

Results extended to the case of (the horizontal) bicategories: when $\mathbb{C}$ has (some) companions, $\mathcal{H}(\mathbb{C})$ is normal oplax monoidal (new!)

For $\mathbb{C}$ monoidal and T pseudomonoidal vertical double monad, under above assumptions $\mathcal{H}(\mathbb{K}\mathsf{l}(T))$ is a normal oplax monoidal bicategory.

The case of profunctors and sequences

Let $\mathcal{V}$ be cocomplete cartesian (...) monoidal closed.

- $\mathbb{P}\mathrm{rof}_{\mathcal{V}}$ is monoidal with $(M \otimes N)((y,y'),(x,x')) = M(y,x) \otimes N(y',x')$
- Free symmetric strict monoidal 2-category monad $S\colon \mathrm{Cat}_{\mathcal{V}} \to \mathrm{Cat}_{\mathcal{V}}$ via $S_n(X)((x_1,\dots,x_n),(y_1,\dots,y_n)) = \bigsqcup \prod X(x_{\sigma(i)},y_i)$ and $SX = \bigsqcup S_n(X)$
- S extends to vertical double monad on $\mathbb{P}\mathrm{rof}_{\mathcal{V}}$, with Kleisli double category $\mathbb{C}\mathrm{at}\mathbb{S}\mathrm{ym}_{\mathcal{V}}$ of *categorical symmetric sequences* $M\colon SY^{\mathrm{op}} \times X \to \mathcal{V}$
- In particular, 'discrete' case $\mathbb{S}\mathrm{ym}_{\mathcal{V}}$ of *colored symmetric sequences*

$\star$ Horizontal composition is many-object generalisation of substitution monoidal structure of symmetric sequences

$$(N \circ M)(\vec{z},x) = \int^{SZ,SY} SZ[\vec{z}, \bigotimes_i \vec{w}^i] \times \prod N(\vec{w}^i, y_i) \times M(\vec{y},x)$$

Arithmetic product

- Vertical double monad $S\colon \mathbb{P}\mathrm{rof}_{\mathcal{V}} \to \mathbb{P}\mathrm{rof}_{\mathcal{V}}$ is pseudomonoidal

Strength looks like $X \times SY \to S(X \times Y)$ via $(x, \vec{y}) \mapsto \overbrace{((x, y_1), \ldots, (x, y_n))}^{\text{cartesianness}}$

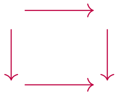
- All other relevant conditions are satisfied

The double categories – and respective bicategories – $\mathbb{C}\mathrm{at}\mathbb{S}\mathrm{ym}_{\mathcal{V}}$ and $\mathbb{S}\mathrm{ym}_{\mathcal{V}}$ admit normal oplax monoidal structure given by the arithmetic product

$$(M \boxtimes N)(\vec{a}, (x, z)) = \int^{\vec{y}, \vec{w}} S(Y \times W)(\vec{a}, \vec{y} \boxtimes \vec{w}) \times M(\vec{y}, x) \times N(\vec{w}, z)$$

$\star$ Future work: Boardman-Vogt tensor of bimodules of symmetric coloured operads...

Thank you for your attention!



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