The Free Cornering of a Monoidal Category

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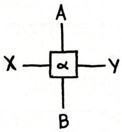
Virtual Double Categories Workshop 2022

In this talk, double categories are *strict*.

We work with *single-object* double categories. These have:

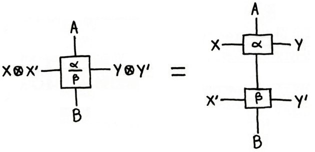
- Vertical and horizontal *monoids* $\mathbb{D}_H$ and $\mathbb{D}_V$.
- Cells:

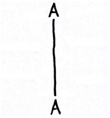


$$\leftrightsquigarrow \qquad \begin{array}{ccc} * & \xrightarrow{A} & * \\ X \big\downarrow & \alpha & \big\downarrow Y \\ * & \xrightarrow[B]{} & * \end{array}$$

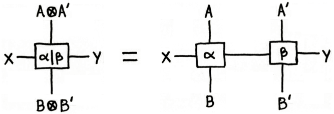
where $A, B \in \mathbb{D}_H$ and $X, Y \in \mathbb{D}_V$.

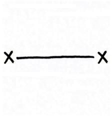
Vertical composition with string diagrams:



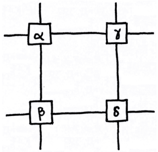


Horizontal composition with string diagrams:



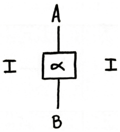


Interchange makes the following diagram unambiguous:

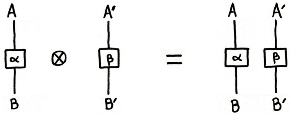


Let $\mathbb{D}$ be a single-object double category.

A cell of $\mathbb{D}$ is *vertical* in case it has trivial horizontal boundaries:



The vertical cells form a monoidal category $\mathbf{V}\mathbb{D}$:



Similarly, the *horizontal cells* form a monoidal category $\mathbf{H}\mathbb{D}$.

Let $\mathbb{A}$ be a monoidal category.

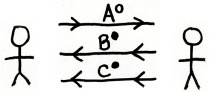
Then $\mathbb{A}^{\circ\bullet}$ is the free monoid on the set of polarised objects of $\mathbb{A}$:

$$\mathbb{A}^{\circ\bullet} = (\mathbb{A}_0 \times \{\circ, \bullet\})^*$$

The binary operation is written $\otimes$, and $I \in \mathbb{A}^{\circ\bullet}$ denotes the unit.

Interpret elements of $\mathbb{A}^{\circ\bullet}$ as $\mathbb{A}$-*valued exchanges*.

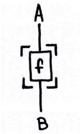
For example, if $A, B, C \in \mathbb{A}_0$ then $A^\circ \otimes B^\bullet \otimes C^\bullet \in \mathbb{A}^{\circ\bullet}$ as in:

`Alice` $\rightsquigarrow$  $\leftrightsquigarrow$ `Bob`

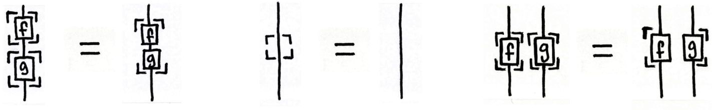
Let $\mathbb{A}$ be a monoidal category.

We define the *free cornering of* $\mathbb{A}$, written $\ulcorner\mathbb{A}\lrcorner$, to be the free single-object double category generated by:

- horizontal monoid $(\mathbb{A}_0, \otimes, I)$
- vertical monoid $\mathbb{A}^{\circ\bullet}$
- for each $f : A \to B$ of $\mathbb{A}$ a cell $\ulcorner f \lrcorner$:



subject to equations:



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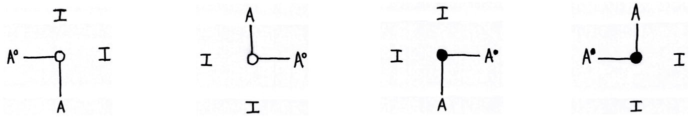
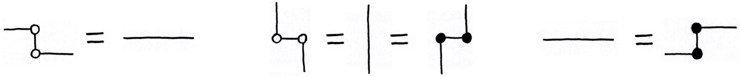
- for each object A of $\mathbb{A}$, cells:

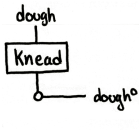


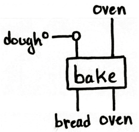
subject to equations:



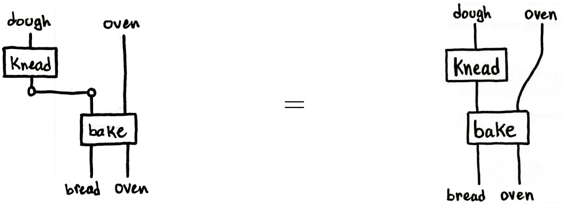
Many names for this. Companion and conjoint structure. Proarrow equipment. Framed bicategory. Corner structure?

If arrows of $\mathbb{A}$ are understood *processes*, then cells of $\ulcorner \mathbb{A} \urcorner$ are processes that interact with their environment.

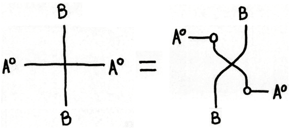


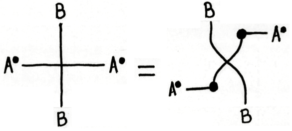


When composed horizontally, they interact with each other:

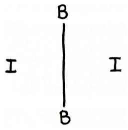


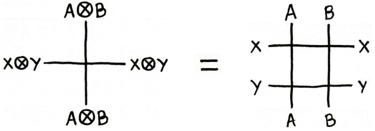
If $\mathbb{A}$ is a *symmetric* monoidal category, define *crossing cells* as in:





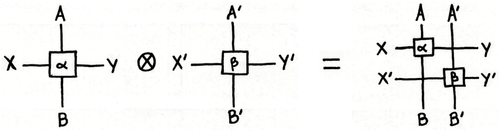
for the base cases, and for the inductive cases as in:





These *look* interesting, but are they?

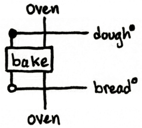
Crossing cells allow us to define the *tensor product* of cells:

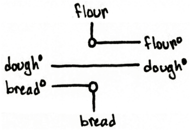


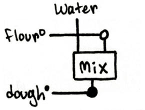
This makes $\ulcorner \mathbb{A} \urcorner$ into a *monoidal double category*.

That is, a pesudomonoid in the 2-category of double categories, double functors, and vertical transformations (Shulman 2010).

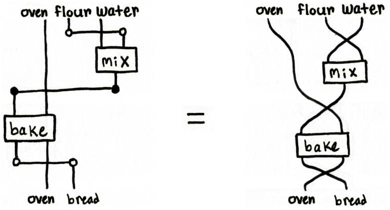
Consider the cells:





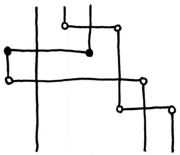


Composing these, we obtain:



The crossing cells are convienient.

These diagrams are invariant under deformation (Myers 2018).



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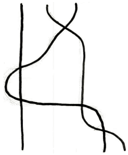

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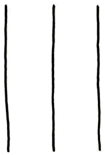
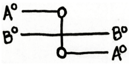


Just like string diagrams for monoidal categories!

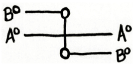
Consequently, $\mathbf{V}\,\ulcorner\!\!\llcorner\mathbb{A}\lrcorner\!\!\urcorner \cong \mathbb{A}$. So we understand the vertical cells!

Isomorphism in $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ is equivalence of exchanges.

For example $A^\circ \otimes B^\circ \cong B^\circ \otimes A^\circ$ via:



and



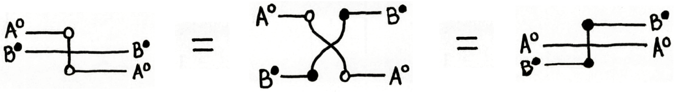
Similarly $A^\bullet \otimes B^\bullet \cong B^\bullet \otimes A^\bullet$.

We also have:

- $A^\circ \otimes B^\circ \cong (A \otimes B)^\circ$ and $A^\bullet \otimes B^\bullet \cong (A \otimes B)^\bullet$
- $I^\circ \cong I$ and $I^\bullet \cong I$

$\mathbf{H}\ulcorner\mathbb{A}\lrcorner$ need not be symmetric monoidal.

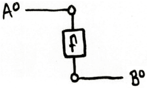
We have an arrow $A^\circ \otimes B^\bullet \to B^\bullet \otimes A^\circ$:



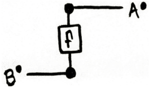
But there need not be any arrow $B^\bullet \otimes A^\circ \to A^\circ \otimes B^\bullet$.

Can't send things we don't have. Causal structure?

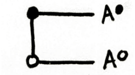
Both $\mathbb{A}$ and $\mathbb{A}^{\mathsf{op}}$ occur as full subcategories of $\mathbf{H}\llcorner\!\ulcorner\mathbb{A}\urcorner\!\lrcorner$ via:



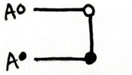
and



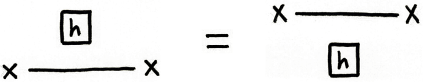
And of course A° is formally left adjoint to $A^{\bullet}$ in $\mathbf{H}\llcorner\!\ulcorner\mathbb{A}\urcorner\!\lrcorner$ via:



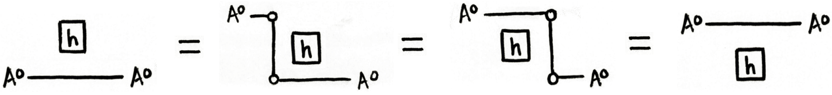
and



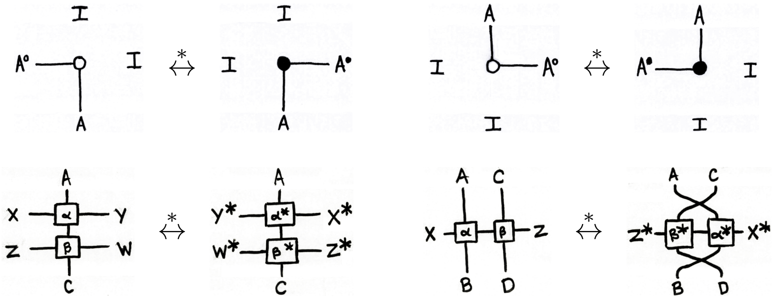
A monoidal category is *spatial* in case:



If $\mathbb{A}$ is spatial then so is $\mathbf{H}\ulcorner\mathbb{A}\urcorner$:



We define an involution on the cells of $\ulcorner \mathbb{A} \urcorner$:



This restricts to a contravariant involution $* : \mathbf{H}\ulcorner \mathbb{A} \urcorner^{\mathsf{op}} \to \mathbf{H}\ulcorner \mathbb{A} \urcorner$.

Further, $(f \otimes g)^* = f^* \otimes g^*$. "Contravariant monoidal involution"?

In general $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ doesn't seem to inherit much structure from $\mathbb{A}$.

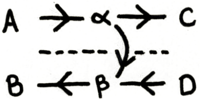
Except ...

If $\mathbb{A}$ is compact closed, then there is an equivalence of categories:

$$\mathbb{A} \simeq \mathbf{H}\ulcorner\mathbb{A}\urcorner$$

As a monoidal category, $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ is weird.

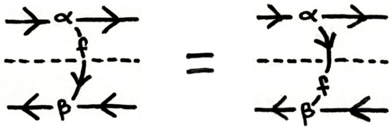
Optics $\langle \alpha \mid \beta \rangle_M : (A, B) \to (C, D)$ in a monoidal category $\mathbb{A}$ consist of arrows $\alpha : A \to M \otimes C$ and $\beta : M \otimes D \to B$ in $\mathbb{A}$.



For example, *lenses* are optics in a cartesian monoidal category.

Optics are subject to "sliding equations" of the following form:

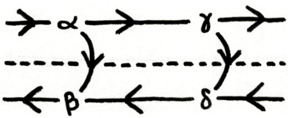
$$\langle \alpha(f \otimes 1_C) \mid \beta \rangle_N = \langle \alpha \mid (f \otimes 1_C)\beta \rangle_M$$



where $f : M \to N$, $\alpha : A \to M \otimes C$, $\beta : N \otimes D \to B$ in $\mathbb{A}$.

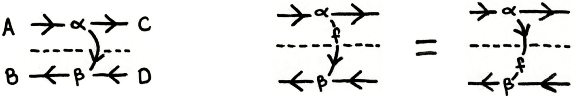
Optics in $\mathbb{A}$ form a category $\mathsf{Optic}_{\mathbb{A}}$. Composition is given by:

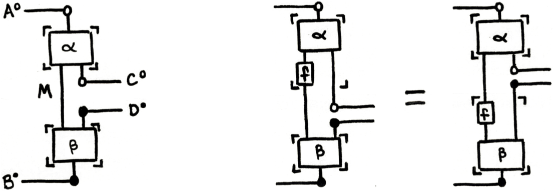
$$\langle \alpha \mid \beta \rangle_M \langle \gamma \mid \delta \rangle_N = \langle \alpha(1_M \otimes \gamma) \mid (1_M \otimes \delta)\beta \rangle_{M \otimes N}$$



and identities are given by $\langle 1_A \mid 1_A \rangle_I : (A, A) \to (A, A)$.

$\mathrm{Optic}_{\mathbb{A}}$ embeds into $\mathbf{H}\ulcorner\mathbb{A}\lrcorner$:

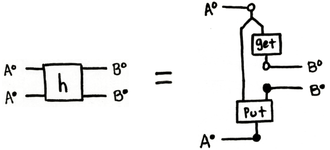




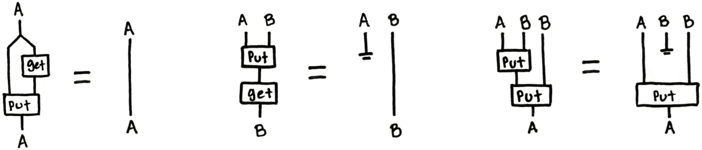
In fact, $\mathrm{Optic}_{\mathbb{A}}$ is the full subcategory of $\mathbf{H}\ulcorner\mathbb{A}\lrcorner$ on objects $A^\circ \otimes B^\bullet$.

Lenses are optics in a cartesian monoidal category

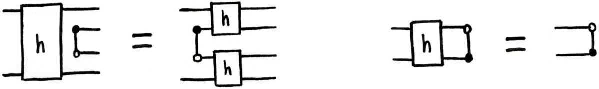
Lenses $h : (A, A) \to (B, B)$ have a special form:



A lens is said to satisfy the *lens laws* in case:

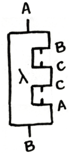


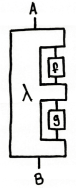
Say that an optic $h : (A, A) \to (B, B)$ is *lawful* in case:

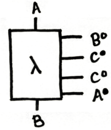


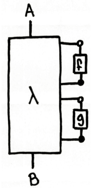
Lenses $h : (A, A) \to (B, B)$ are lawful if and only if they satisfy the lens laws (Riley 2018).

Comb Diagrams:

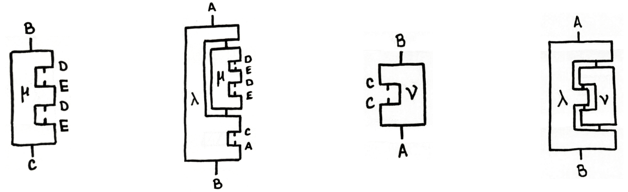


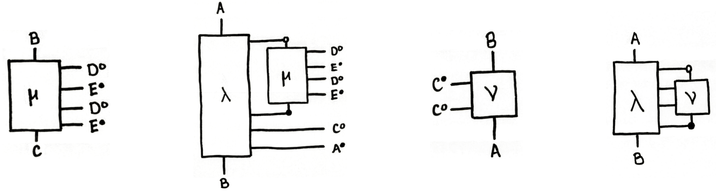






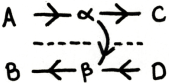
Exotic operations on comb diagrams:



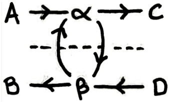


Comb diagrams form a multicategory (above left). There is also a polycategory of comb diagrams (Hefford & Comfort 2022).

The notion of optic in a monoidal category is asymmetric:



A notion of symmetric lens has been proposed. Intuitively:



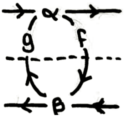
Proposal: A *symmetric optic* $\langle \alpha \mid \beta \rangle_M^P : (A, B) \to (C, D)$ in a symmetric monoidal category $\mathbb{A}$ consists of arrows $\alpha : A \otimes P \to M \otimes C$ and $\beta : M \otimes D \to B \otimes P$ of $\mathbb{A}$.

Symmetric optics are subject to equations:

$$\langle \alpha(f \otimes 1_C) \mid \beta(1_B \otimes g) \rangle_N^Q = \langle (1_A \otimes g)\alpha \mid (f \otimes 1_D)\beta \rangle_M^P$$

for all $f : M \to N$ and $g : P \to Q$ of $\mathbb{A}$.

Intuitively:



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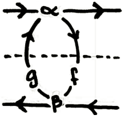
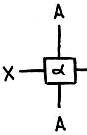
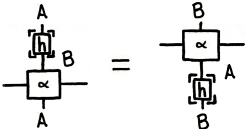


This defines a category of symmetric optics in $\mathbb{A}$.

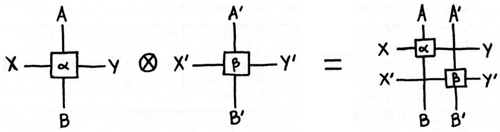
Define a monoidal category $\mathcal{I}(\mathbb{A})$ as follows:

- **objects** are elements of $\mathbb{A}^{\circ\bullet}$.
- **arrows** $(A, \alpha) : X \to Y$ are given by cells of $\ulcorner\mathbb{A}\urcorner$:

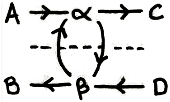


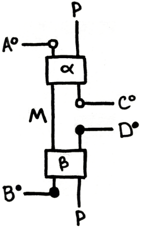


  subject to "sliding equivalence", pictured above right.
- **composition** and **identities** as in $\mathbf{H}\ulcorner\mathbb{A}\urcorner$.
- **tensor product** given by the tensor product of cells in $\ulcorner\mathbb{A}\urcorner$:



The category of symmetric optics in $\mathbb{A}$ is the full subcategory of $\mathcal{I}(\mathbb{A})$ on objects of the form $A^\circ \otimes B^\bullet$:

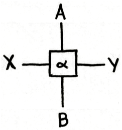




The self-duality of the category of symmetric optics factors through the involution $*$ on the cells of $\ulcorner\!\!\!\llcorner \mathbb{A} \lrcorner\!\!\!\urcorner$.

Let $\mathbb{A}$ be symmetric monoidal. Define a category $\langle\mathbb{A}\rangle$ by:

- **objects** are objects of $\mathbb{A}$
- **arrows** $\alpha : A \to B$ are cells of $\ulcorner\mathbb{A}\urcorner$ as in:



- **composition** and **identities** are vertical composition and identities in $\ulcorner\mathbb{A}\urcorner$.

This category plays an important role in the notion of *situated transition system* (Proceedings of ACT 2021).

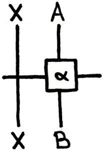
A *premonoidal category* is like a monoidal category, but instead of $f \otimes g$, we have $f \ltimes X$ and $X \rtimes f$ for all arrows f, objects X.

On objects $X \ltimes Y = X \rtimes Y$, and this operation forms a monoid.

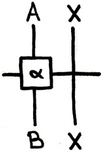
An arrow f of a premonoidal category is *central* in case for all g:

The central arrows form a monoidal category, called the *center*.

If $\mathbb{A}$ is symmetric monoidal then $\langle \mathbb{A} \rangle$ is symmetric premonoidal:

$X \rtimes \alpha =$ 

and

$\alpha \ltimes X =$ 

The center of $\langle \mathbb{A} \rangle$ is precisely $\mathbf{V}\ulcorner\!\!\llcorner\mathbb{A}\lrcorner\!\!\urcorner \cong \mathbb{A}$.

See also Freyd categories. Effectful categories. $\mathbb{A} \hookrightarrow \langle \mathbb{A} \rangle$.

Model of simple I/O for programming languages. Future work.

`https://www.ioc.ee/~cneste/`

The Structure of Concurrent Process Histories
COORDINATION 2021

Situated Transition Systems
ACT 2021

Cornering Optics
ACT 2022
with Guillaume Boisseau and Mario Román.

Concurrent Process Histories and Resource Transducers
LMCS (to appear)

Soon: Chris Heunen's ACT adjoint school proposal for 2023.