# Gray-type monoidal product and Bifunctor Theorem for double categories

**Bojana Femić**

**Virtual Double Categories Workshop**

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  - Gray-type tensor product on (strict-lax) double categories
  - Bifunctor Theorem
  - "(Un)currying 2-functors
  - application to monads in double categories

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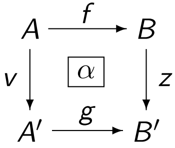


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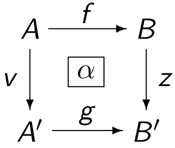


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$C_0$: 0-cells and 1v-cells, $C_1$: 1h-cells and 2-cells.

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$$s, t : C_1 \longrightarrow C_0, \quad u : C_0 \longrightarrow C_1 \quad \text{and } c : C_1 \times_{C_0} C_1 \longrightarrow C_1$$

- natural transformations (2-cells in $\mathrm{Cat}_2$)

$$\alpha : c \otimes (id_{C_1} \times_{C_0} c) \Rightarrow c \otimes (c \times_{C_0} id_{C_1})$$

$$\lambda : c \otimes (u \times_{C_0} id_{C_1}) \Rightarrow id_{C_1}$$

$$\rho : c \otimes (id_{C_1} \times_{C_0} u) \Rightarrow id_{C_1}$$

which satisfy a pentagon and a triangle.

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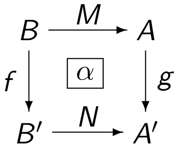


- 2: bimodule morphisms

$\alpha : M \longrightarrow N$ $A$-$B$-bimodule morphism

$$a \cdot n \cdot b := g(a) \cdot n \cdot f(b)$$

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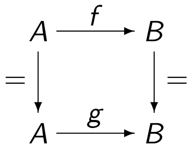


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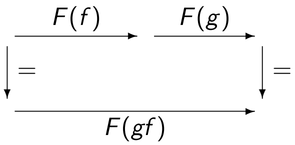
- Each bicategory can be embedded into a pseudodouble category.

• Every pseudodouble category is double-equivalent to a double category [Grandis-Paré: "Limits in double categories" (1999)].

# Functors of double categories

- (strict) double functors,
- pseudo double functors,
- (co)lax double functors.



**Why**

**the 2-category $\mathrm{Mnd}(\mathcal{K})$**

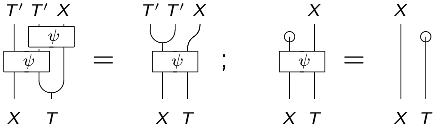
**of 2-monads**

**isn't enough**

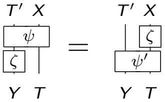
# The 2-category Mnd($\mathcal{K}$) of monads in $\mathcal{K}$

<u>0-cells:</u>
2-monads $(\mathcal{A},\ T\colon \mathcal{A}\to\mathcal{A},\ \mu_T\colon TT\to T,\ \eta_T\colon \mathrm{Id}_{\mathcal{A}}\to T)$

<u>1-cells:</u> pairs $(X,\psi):(\mathcal{A},T)\to(\mathcal{A}',T')$ where $X:\mathcal{A}\to\mathcal{A}'$ is a 1-cell and $\psi: T'X\Rightarrow XT$ a 2-cell s.t.




<u>2-cells:</u> $(X,\psi)\Rightarrow(Y,\psi')$ are given by 2-cells $\zeta: X\to Y$ in $\mathcal{K}$ satisfying:



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BUT:

• monad morphisms between monads on the bicategories $Span_d(\mathcal{V})$ and $\mathcal{V}$-$Mat$ are not functors of categories internal in $\mathcal{V}$, resp. of categories enriched over $\mathcal{V}$.

# The double category Mnd($\mathbb{D}$) of (double) monads

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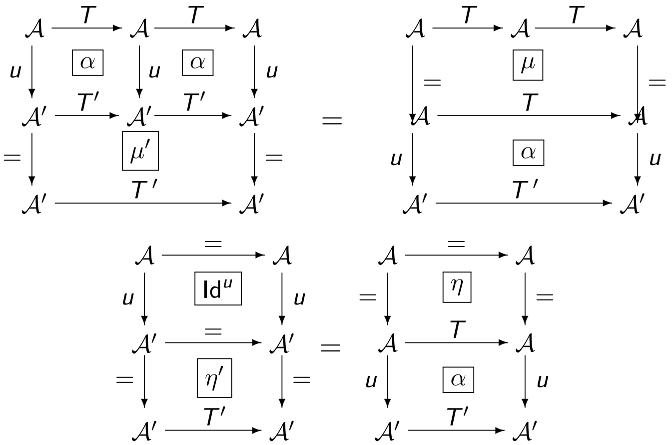

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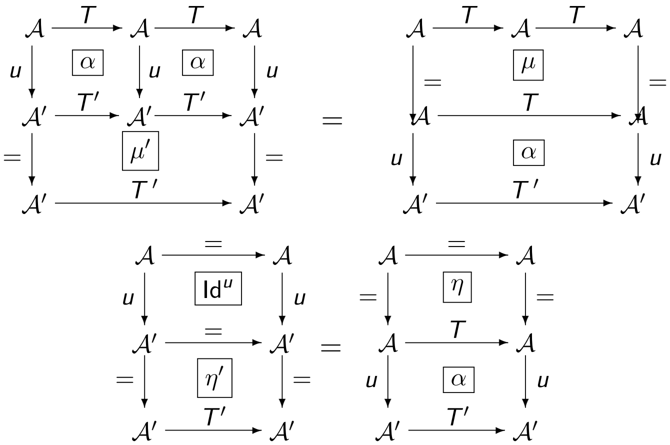

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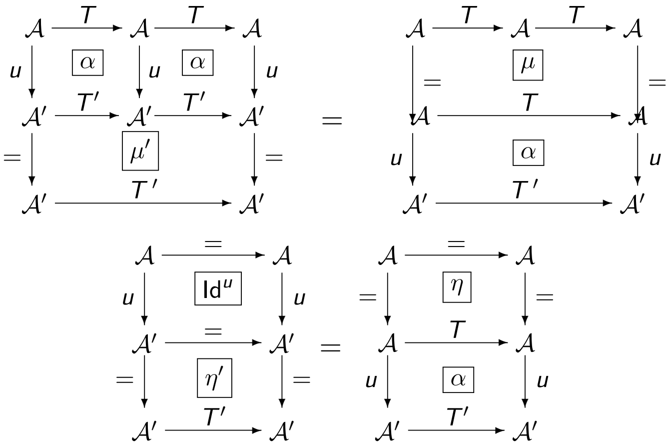


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and the interchange does not hold in general on lax transformations (strictness of transformations is needed).

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One looks for a 2-category $\mathcal{A} \otimes \mathcal{B}$ s.t.:

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Gray proved that $\mathcal{A} \otimes \mathcal{B}$ yields a monoidal product on 2-Cat.

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[G. Böhm: "The Gray Monoidal Product of Double Categories" (2020)]
monoidal structure in $(Dbl_{st}^{st}, \otimes)$ is obtained from:

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In
[B. Femić: "Enrichment and internalization in tricategories, the case of tensor categories and alternative notion to intercategories"]:

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[MacLane:]

Given functors $L_C : \mathcal{B} \longrightarrow \mathcal{D}$ and $M_B : \mathcal{C} \longrightarrow \mathcal{D}$ so that $L_C(B) = M_B(C), \forall B \in \mathcal{B}, C \in \mathcal{C}$.

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Let $\sigma \in \text{Dist}(\mathcal{B}, \mathcal{C}, \mathcal{D})$ be a distributive law between families of lax functors $L_C : \mathcal{B} \longrightarrow \mathcal{D}$ and $M_B : \mathcal{C} \longrightarrow \mathcal{D}$ $\forall B \in \mathcal{B}, C \in \mathcal{C}$.

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Bifunctor Theorem as a generalization of Beck's result on the composition of monads:

$$\begin{array}{ccc}
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\cong \big\downarrow & & \big\downarrow \cong \\
\mathsf{Mnd}(\mathsf{Mnd}(\mathcal{H}(\mathbb{D}))) & \xrightarrow[\mathrm{Comp}(\mathcal{H}(\mathbb{D}))]{} & \mathsf{Mnd}(\mathcal{H}(\mathbb{D}))
\end{array}$$