

THE FREE TABULATOR COMPLETION OF A DOUBLE CATEGORY

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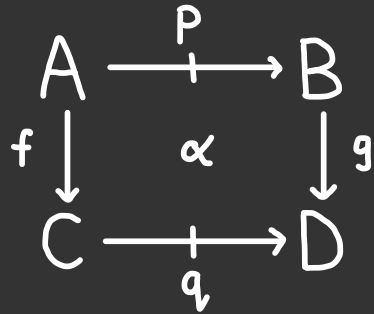
ItaCa Fest

19 May 2026

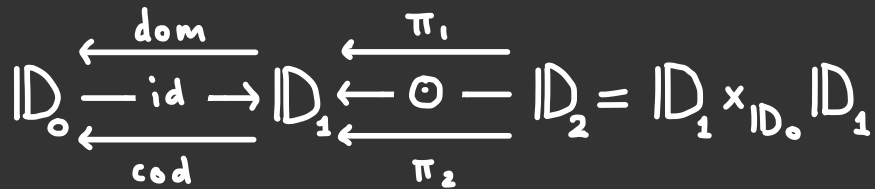
MOTIVATION & OVERVIEW

A **double category** ID consists of:

- objects A, B, C, D, \dots
- tight morphisms $A \rightarrow B$
- loose morphisms $A \rightrightarrows B$
- cells



Same as a **pseudocategory** in CAT :



- How can we reduce limits indexed by double categories

$$J \xrightarrow{F} ID$$

to limits indexed by categories?

$$I \xrightarrow{\hat{F}} ID_0$$

- Can we understand the span construction as a right adjoint?

$$ID \xrightarrow{\text{lax}} \mathcal{S}pan$$

GOAL: Answer these questions via the free tabulator completion!

TWO-DIMENSIONAL CATEGORICAL STRUCTURES

Double categories \mathbb{D}

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ f \downarrow & \alpha & \downarrow g \\ C & \xrightarrow{q} & D \end{array}$$

2-categories $\mathbb{T}_i(\mathcal{K})$

$$\begin{array}{ccc} A & \xrightarrow{id_A} & A \\ f \downarrow & \alpha & \downarrow g \\ B & \xrightarrow{id_B} & B \end{array}$$

Bicategories $\mathbb{L}_o(\mathcal{B})$

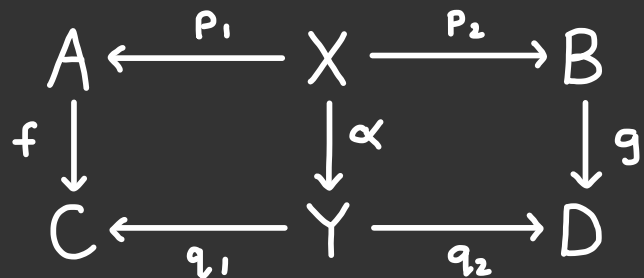
$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ 1_A \downarrow & \alpha & \downarrow 1_B \\ A & \xrightarrow{q} & B \end{array}$$

Monoidal categories $\mathbb{L}_o(\mathcal{V}, \otimes, I)$

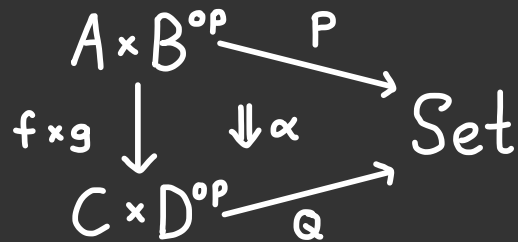
$$\begin{array}{ccc} * & \xrightarrow{A} & * \\ 1_* \downarrow & f & \downarrow 1_* \\ * & \xrightarrow{B} & * \end{array}$$

EXAMPLES

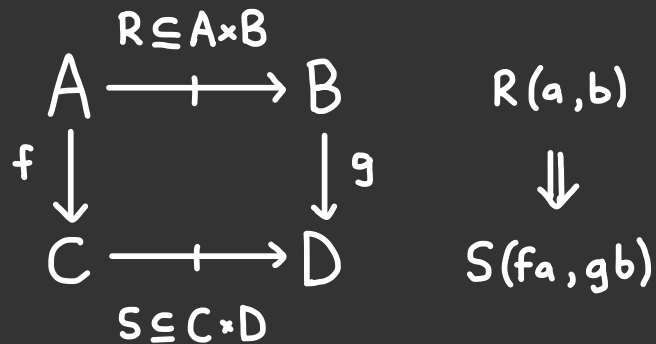
$\text{Span}(\mathcal{C})$, \mathcal{C} - has chosen pullbacks



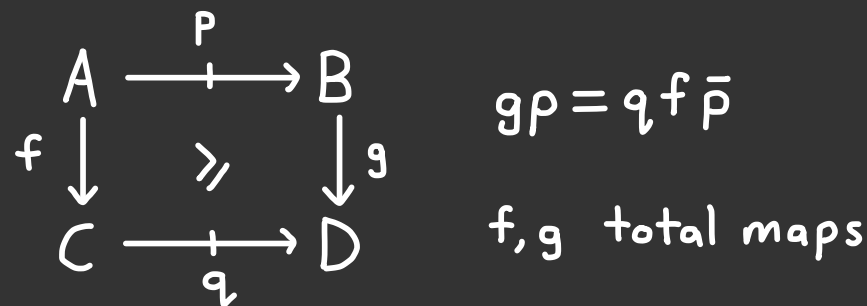
Dist



IRel

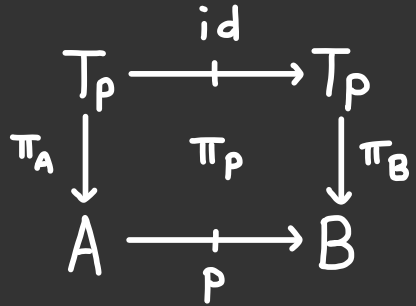


$\text{IR}(\mathcal{C})$, \mathcal{C} - restriction category



TABULATORS

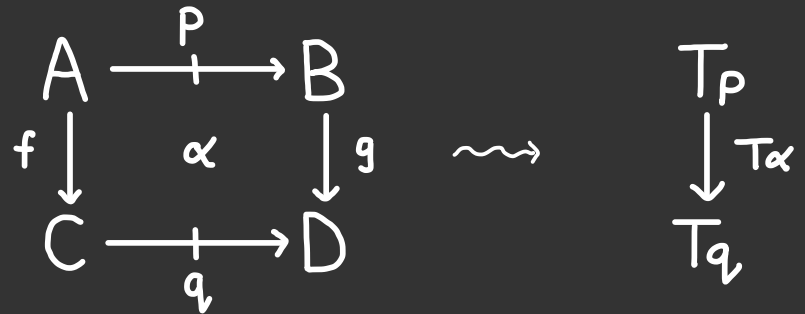
tabulator of a loose morphism $p: A \rightrightarrows B$



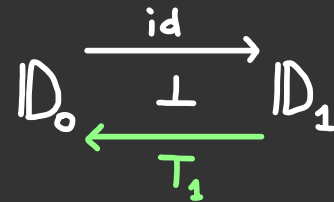
universal property



- Tabulators are limits of $\mathcal{D}_t \longrightarrow \mathbb{D}$
- Cotabulators are dual via tight opposite \mathbb{D}^{opt}
- Natural with respect to cells:



- \mathbb{D} admits all tabulators if and only if



EXAMPLES OF TABULATORS

2-categories. powers by 2

$$\begin{array}{ccc}
 A^2 & \xrightarrow{id} & A^2 \\
 \pi_1 \downarrow & \pi_A & \downarrow \pi_2 \\
 A & \xrightarrow{id_A} & A
 \end{array}$$

universal property

$$\begin{array}{ccc}
 X & \xrightarrow{id_X} & X \\
 f \downarrow & \alpha & \downarrow g \\
 A & \xrightarrow{id_A} & A
 \end{array}
 \iff
 \begin{array}{ccc}
 X & & \\
 \downarrow \langle f, g, \alpha \rangle & & \\
 A^2 & &
 \end{array}$$

bicategories: unique 2-cell from identity

$$\begin{array}{ccc}
 A & \xrightarrow{id_A} & A \\
 1_A \downarrow & \exists! & \downarrow 1_A \\
 A & \xrightarrow{p} & A
 \end{array}$$

monoidal categories: unique morphism from unit

$$\begin{array}{ccc}
 * & \xrightarrow{I} & * \\
 1_* \downarrow & \exists! & \downarrow 1_* \\
 * & \xrightarrow{A} & *
 \end{array}$$

MORE EXAMPLES OF TABULATORS

Span(C)

$$\begin{array}{ccccc}
 X & \xleftarrow{1_x} & X & \xrightarrow{1_x} & X \\
 \downarrow p_1 & & \downarrow 1_x & & \downarrow p_2 \\
 A & \xleftarrow{p_1} & X & \xrightarrow{p_2} & B
 \end{array}$$

Rel

$$\begin{array}{ccc}
 R & \xrightarrow{R \subseteq R \times R} & R \\
 \downarrow \pi_A & & \downarrow \pi_B \\
 A & \xrightarrow{R \subseteq A \times B} & B
 \end{array}
 \quad
 \begin{array}{c}
 (R(a,b), R(a,b)) \\
 \Downarrow \\
 R(a,b)
 \end{array}$$

$\mathcal{R}(c)$, \mathcal{C} -restriction category

$$\begin{array}{ccc}
 X & \xrightarrow{1_x} & X \\
 \downarrow \pi & & \downarrow p\pi \\
 A & \xrightarrow{p} & B
 \end{array}
 \quad
 \pi, p\pi \text{ total maps}$$

If the restriction idempotent \bar{p} splits (i.e. $\pi u = \bar{p}$ and $u\pi = 1_x$), then the tabulator of p exists + satisfies 2D universal property.

(CO)TABULATORS IN IDist

Tabulator of a distributor $P: A^{op} \times B \rightarrow \text{Set}$

$$\begin{array}{ccc}
 T_P & \xrightarrow{\text{id}} & T_P \\
 \pi_A \downarrow & \pi_P & \downarrow \pi_B \\
 A & \xrightarrow{p} & B
 \end{array}$$

- objects $(a \in A, b \in B, x \in P(a, b))$
- morphisms $(f, g): (a, b, x) \longrightarrow (a', b', x')$

$$\begin{array}{ccc}
 a & \overset{x}{\rightsquigarrow} & b \\
 f \downarrow & & \downarrow g \\
 a' & \overset{x'}{\rightsquigarrow} & b'
 \end{array}$$

$$P(f, 1_{b'}) (x') = P(1_a, g) (x)$$

Cotabulator of a distributor $P: A^{op} \times B \rightarrow \text{Set}$

$$\begin{array}{ccc}
 A & \xrightarrow{p} & B \\
 \mu_A \downarrow & \mu_P & \downarrow \mu_B \\
 \perp P & \xrightarrow{\text{id}} & \perp P
 \end{array}$$

- set of objects $ob(A) + ob(B)$
- Hom-sets

$$\perp P(x, y) = \begin{cases} A(x, y) & \text{if } x, y \in A \\ B(x, y) & \text{if } x, y \in B \\ P(x, y) & \text{if } x \in A \text{ and } y \in B \\ \phi & \text{otherwise} \end{cases}$$

BINARY TABULATORS

binary tabulator of $A \xrightarrow{p} B \xrightarrow{q} C$

$$\begin{array}{ccccc}
 T(p,q) & \xrightarrow{\text{id}} & T(p,q) & \xrightarrow{\text{id}} & T(p,q) \\
 \pi_A \downarrow & \pi_p & \downarrow \pi_B & \pi_q & \downarrow \pi_C \\
 A & \xrightarrow{p} & B & \xrightarrow{q} & C
 \end{array}$$

Lemma: Suppose \mathcal{D} has tabulators. Then binary tabulators \iff pullbacks of left projections along right projections.

$$\begin{array}{c}
 T(p,q) \\
 \swarrow \quad \searrow \\
 T_p \quad T_q \\
 \swarrow \quad \searrow \quad \swarrow \quad \searrow \\
 \pi_p \quad \pi_q \quad \pi_p \quad \pi_q \\
 \swarrow \quad \searrow \quad \swarrow \quad \searrow \\
 A \xrightarrow{p} B \xrightarrow{q} C
 \end{array}$$

The following are equivalent:

- \mathcal{D} admits all binary tabulators
- limits of normal lax $\exists_l \rightarrow \mathcal{D}$ exist

$$\begin{array}{ccccc}
 A & \xrightarrow{p} & B & \xrightarrow{q} & C \\
 1_A \downarrow & & \alpha & & \downarrow 1_C \\
 A & \xrightarrow{r} & & & C
 \end{array}$$

- There exists right adjoints

$$\begin{array}{ccc}
 \mathcal{D}_2 & \xleftarrow{T} & [\exists_l, \mathcal{D}]_{nl} \\
 \swarrow & & \nearrow \\
 \mathcal{D}_0 & & \mathcal{D}_0
 \end{array}$$

T₂ (dashed green arrow from \mathcal{D}_0 to \mathcal{D}_2)
 T₂ (dashed green arrow from $[\exists_l, \mathcal{D}]_{nl}$ to \mathcal{D}_0)

THE SPAN CONSTRUCTION

adequate triple $(\mathcal{C}, \mathcal{L}, \mathcal{R})$

- \mathcal{C} - category
- \mathcal{L}, \mathcal{R} - wide subcategories of \mathcal{C}
- Pullbacks of \mathcal{L} -morphisms along \mathcal{R} -morphisms exist in \mathcal{C} and the classes are stable

AdTrip: 2-category of adequate triples

DbITab: 2-category of double categories with tabulators and binary tabulators

$$\text{DbITab}_{\mathcal{L}} \xrightarrow{(-)_o} \text{AdTrip}$$

$$\text{ID} \mapsto (\text{ID}_o, \mathcal{L}P, \mathcal{R}P)$$

left projections \swarrow right projections \nwarrow

$$\text{AdTrip} \xrightarrow{\text{Span}} \text{DbITab}_{ps}$$

$$(\mathcal{C}, \mathcal{L}, \mathcal{R}) \mapsto \text{Span}(\mathcal{C}, \mathcal{L}, \mathcal{R})$$

- objects and tight morphisms from \mathcal{C}
- loose morphisms $A \xleftarrow{\ell \in \mathcal{L}} X \xrightarrow{r \in \mathcal{R}} B$

$$\text{AdTrip} \begin{array}{c} \xleftarrow{(-)_o} \\ \perp \\ \xrightarrow{\text{Span}} \end{array} \text{DbITab}_{\mathcal{L}}$$

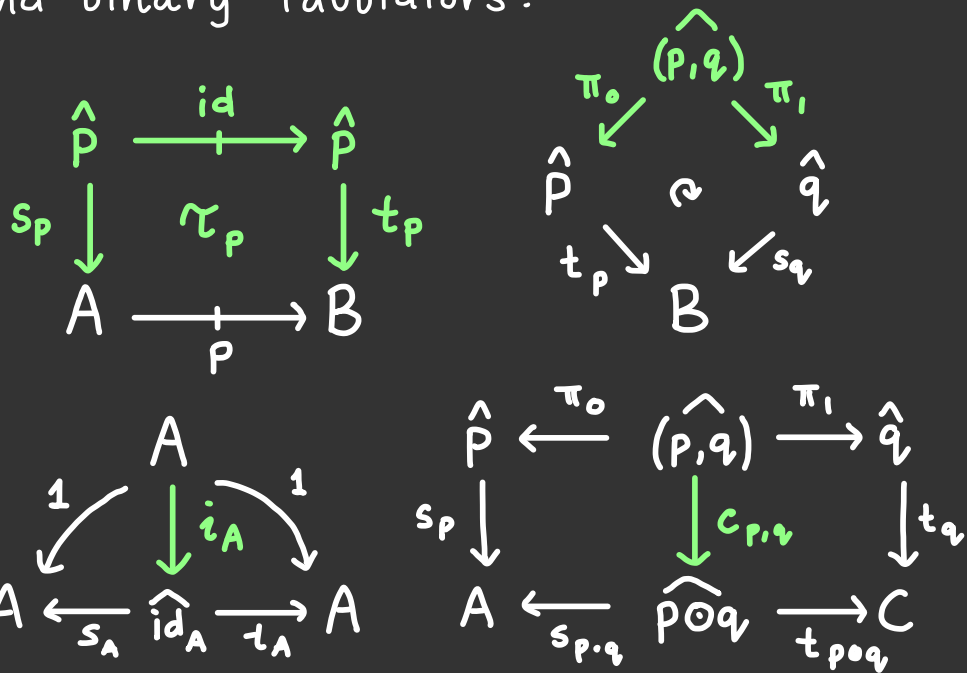
$$\text{unit} \quad \text{ID} \xrightarrow{\eta_{\text{ID}}} \text{Span}(\text{ID}_o, \mathcal{L}P, \mathcal{R}P)$$

$$A \xrightarrow{P} B \mapsto A \xleftarrow{\tau_A} T_p \xrightarrow{\pi_B} B$$

THE FREE COMPLETION UNDER TABULATORS

$$\text{Db|Tab}_\perp \begin{array}{c} \xleftarrow{IF} \\ \xrightarrow[u]{\perp} \end{array} \text{Db|}_\perp$$

Given ID, we may freely add tabulators and binary tabulators:



$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ f \downarrow & \alpha & \downarrow g \\ C & \xrightarrow{q} & D \end{array} \rightsquigarrow \begin{array}{ccc} A & \xleftarrow{s_p} \hat{p} \xrightarrow{t_p} & B \\ f \downarrow & \hat{\alpha} & \downarrow g \\ C & \xleftarrow{s_q} \hat{q} \xrightarrow{t_q} & D \end{array}$$

... modulo several equations.

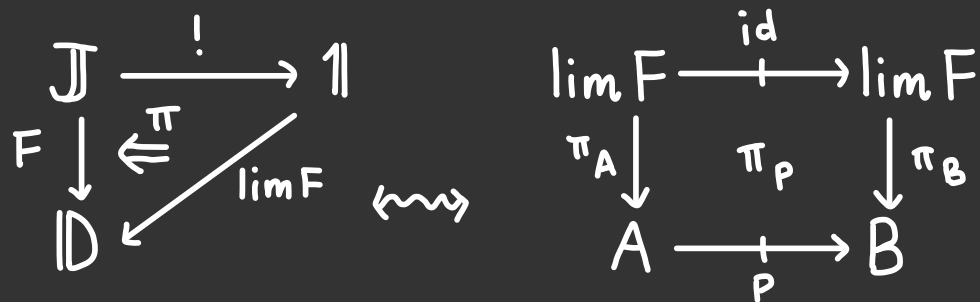
Write $\mathcal{LS}(ID) = IF(ID)_\circ$, the loose subdivision of a double category.

$$ID \xrightarrow{\text{lax}} \mathcal{Span}(C, \mathcal{L}, \mathcal{R})$$

$$(\mathcal{LS}(ID), \mathcal{LP}, \mathcal{RP}) \longrightarrow (C, \mathcal{L}, \mathcal{R})$$

REDUCTION OF LIMITS IN DOUBLE CATEGORIES

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Theorem (Grandis & Paré, 1999): A double category admits limits indexed by any small double category \mathbb{J} if and only if it admits small products, equalisers, & tabulators.

Proof strategy: Given a diagram $F: \mathbb{J} \rightarrow \mathbb{D}$, construct new diagram $\mathbb{I} \xrightarrow{G} \mathbb{D}_0$ into the underlying category of \mathbb{D} using tabulators and binary tabulators.

Then show that $\text{Cone}(F) \cong \text{Cone}(G)$.
 By the classical result, the limit of any $G: \mathbb{I} \rightarrow \mathbb{D}_0$ exists iff \mathbb{D}_0 has products and equalisers. \square

Original proof is 5 pages long and very explicit; maybe a more abstract proof is possible?

$$\mathbb{J} \xrightarrow{F} \mathbb{D} \quad \rightsquigarrow \quad F(\mathbb{J}) \xrightarrow{F'} \mathbb{D}$$

Show that $\Pi_i(\mathcal{L}S(\mathbb{J})) \rightarrow F(\mathbb{J})$ is an initial functor of double cats.

THE NERVE OF A DOUBLE CATEGORY

$$\Delta \xrightarrow{\mathbb{L}_0} \text{Dbl}_{ne}, \quad \mathbb{L}_0(k) = \underline{k}_\ell$$

$$\{0 \rightrightarrows 1 \rightrightarrows \dots \rightrightarrows k-1\}$$

$$\text{Dbl}_{ne} \xrightarrow{N} [\Delta^{\text{op}}, \text{CAT}]$$

$$N(\text{ID})_k = [\underline{k}_\ell, \text{ID}]_{ne}$$

For example:

- $N(\text{ID})_0 \cong \text{ID}_0$ objects & tight morphisms

- $N(\text{ID})_1 \cong \text{ID}_1$ loose morphisms & cells

- $N(\text{ID})_2 \rightsquigarrow$ objects are
$$\begin{array}{ccccc} A & \xrightarrow{p} & B & \xrightarrow{q} & C \\ 1 \downarrow & & \alpha & & \downarrow 1 \\ A & \xrightarrow{r} & & & C \end{array}$$

Theorem: The nerve is 2-fully faithful.

Proof: Follows Lack & Paoli (2008).

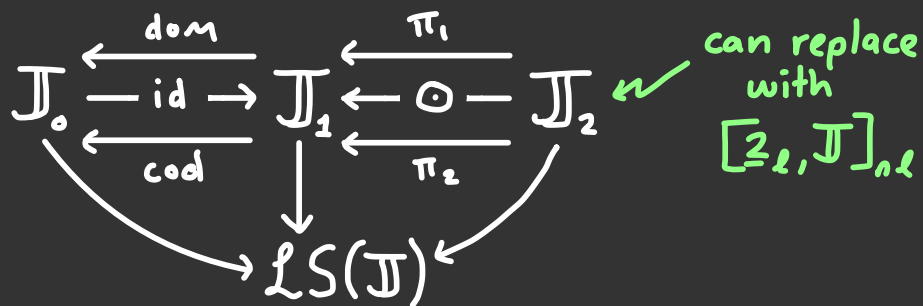
$$D := \{0 \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} 1 \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} 2\} \rightsquigarrow \Delta$$

Lemma: $\text{Dbl}_{ne} \xrightarrow{N} [\Delta^{\text{op}}, \text{CAT}]$ is locally fully faithful.

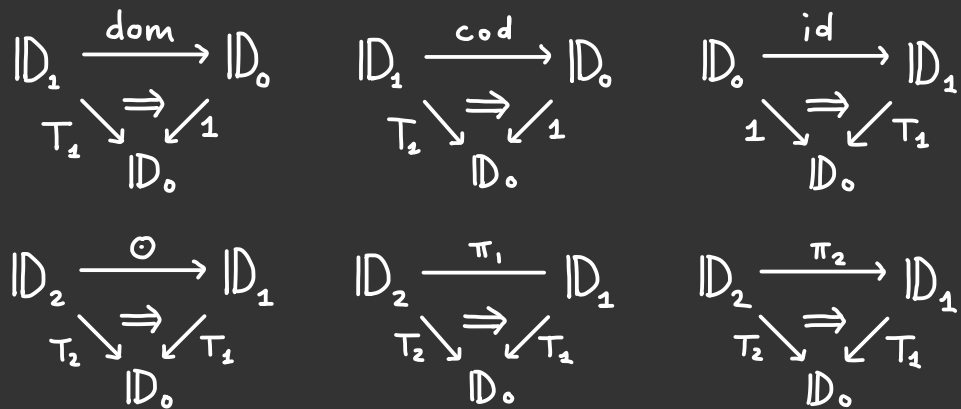
$$\begin{array}{ccccc} \mathbb{J}_0 & \begin{array}{c} \xleftarrow{\text{dom}} \\ \xrightarrow{\text{id}} \\ \xleftarrow{\text{cod}} \end{array} & \mathbb{J}_1 & \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} & [\underline{2}_\ell, \mathbb{J}] \\ F_0 \downarrow \begin{array}{c} \Rightarrow \\ \alpha_0 \end{array} & G_0 & F_1 \downarrow \begin{array}{c} \Rightarrow \\ \alpha_1 \end{array} & G_1 & F_2 \downarrow \begin{array}{c} \Rightarrow \\ \alpha_2 \end{array} & G_2 \\ \text{ID}_0 & \begin{array}{c} \xleftarrow{\text{dom}} \\ \xrightarrow{\text{id}} \\ \xleftarrow{\text{cod}} \end{array} & \text{ID}_1 & \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} & [\underline{2}_\ell, \text{ID}] \end{array}$$

LOOSE SUBDIVISION AS A COLAX COLIMIT

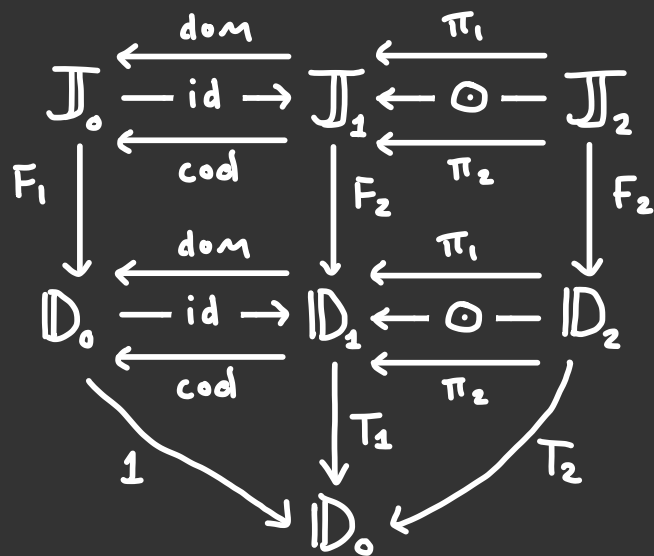
The loose subdivision $\mathcal{L}S(\mathcal{J})$ is the colax colimit of the "truncated nerve":



If \mathcal{D} has tabulators and binary tabulators:



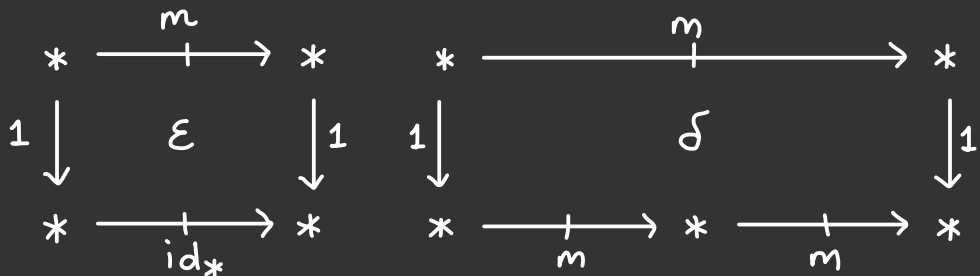
By universal property of colax colimits, each normal lax functor $F: \mathcal{J} \rightarrow \mathcal{D}$ induces a unique $\hat{F}: \mathcal{L}S(\mathcal{J}) \rightarrow \mathcal{D}_0$.



By local fully faithfulness of truncated nerve, we may show that $\text{Cone}(F) \cong \text{Cone}(\hat{F})$.

EXAMPLE: COMPUTING THE EM-CATEGORY OF A COMONAD

Consider Mnd^{opt} freely generated by cells

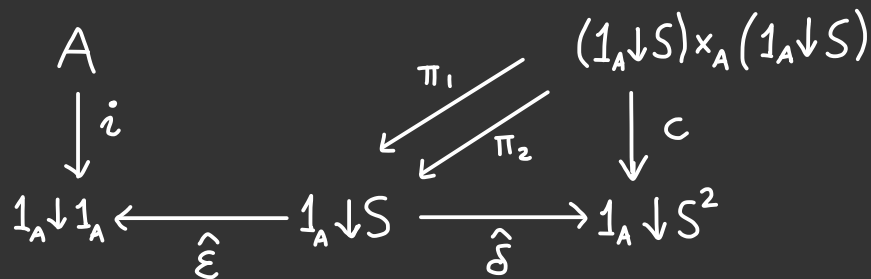


modulo the equations for a comonad.

Let $Mnd^{opt} \xrightarrow{F} Dist$ send $* \xrightarrow{m} *$ to the distributor $A(-, S-): A^{op} \times A \rightarrow Set$ for an endofunctor $S: A \rightarrow A$.

This diagram determines a comonad (S, ε, δ) on A , and its limit is the EM-category.

The functor $LS(Mnd^{opt}) \xrightarrow{\hat{F}} Cat$ is determined by the diagram



$$i(x) = (x, x, 1_x: x \rightarrow x)$$

$$\hat{\varepsilon}(x, y, x \xrightarrow{f} S y) = (x, y, x \xrightarrow{\varepsilon_y \circ f} y)$$

$$\hat{\delta}(x, y, x \xrightarrow{f} S y) = (x, y, x \xrightarrow{\mu_y \circ f} S^2 y)$$

$$c(x, y, z, x \xrightarrow{f} S y, y \xrightarrow{g} S z) = (x, z, x \xrightarrow{Sg \circ f} S z)$$

The objects of $\lim(\hat{F})$ are $(x, y, x \xrightarrow{f} S y)$ such that $x = y$, $\varepsilon_x \cdot f = 1_x$, and $\mu_f \cdot f = S f \cdot f$.

LOOSE MORPHISMS BETWEEN TABULATORS

Tight morphisms between tabulators come from naturality of limits:

$$\begin{array}{ccc}
 A & \xrightarrow{p} & B \\
 f \downarrow & \alpha & \downarrow g \\
 C & \xrightarrow{q} & D
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 T_p \\
 \downarrow T\alpha \\
 T_q
 \end{array}$$

Loose morphisms between tabulators are limits indexed by a loose distributor:

⚡
parallel tabulators

$$\begin{array}{ccc}
 \mathbb{Z}_\ell & \xrightarrow{\text{Hom}} & \mathbb{Z}_\ell \\
 p \downarrow & \Phi & \downarrow q \\
 \text{ID} & \xrightarrow{\text{Hom}} & \text{ID}
 \end{array}$$

Gives 2D universal property of tabulators.

$\lim \Phi: T_p \dashrightarrow T_q$ determined by data

$$\begin{array}{ccccc}
 A & \xrightarrow{p} & B & \xrightarrow{r_1} & Y & & A & \xrightarrow{r_2} & X & \xrightarrow{q} & Y \\
 1 \downarrow & & \alpha & & \downarrow 1 & & 1 \downarrow & & \beta & & \downarrow 1 \\
 A & \xrightarrow{r_3} & Y & & & & A & \xrightarrow{r_3} & Y & & Y
 \end{array}$$

If ID admits parallel tabulators and parallel binary tabulators, the limits indexed by $\mathbb{J}: \mathbb{S} \dashrightarrow \mathbb{T}$ reduce to:

$$\begin{array}{ccccc}
 \mathcal{L}\mathbb{S}(\mathbb{S}) & \longleftarrow & \mathcal{L}\mathbb{S}(\mathbb{J}) & \longrightarrow & \mathcal{L}\mathbb{S}(\mathbb{T}) \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{ID}_0 & \xleftarrow{\text{dom}} & \text{ID}_1 & \xrightarrow{\text{cod}} & \text{ID}_0
 \end{array}$$

SUMMARY & RELATED WORK

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- We introduced the free completion $\mathbb{F}(\mathbb{I}\mathbb{D})$ of $\mathbb{I}\mathbb{D}$ under tabulators and binary tabulators.

- Showed that limits of double functors

$$\mathbb{J} \xrightarrow{\mathbb{F}} \mathbb{I}\mathbb{D}$$

may be reduced to limits of functors

$$\mathbb{F}(\mathbb{I}\mathbb{D})_0 = \mathbb{L}\mathbb{S}(\mathbb{I}\mathbb{D}) \xrightarrow{\hat{\mathbb{F}}} \mathbb{I}\mathbb{D}_0$$

if $\mathbb{I}\mathbb{D}$ has tabulators & binary tabulators.

- Characterised $\mathbb{S}\text{pan}$ as a right adjoint.

$$\text{Ad Trip} \begin{array}{c} \xleftarrow{(-)_0} \\ \xleftarrow{\perp} \\ \xleftarrow{\mathbb{S}\text{pan}} \end{array} \text{Db|Tab}_\ell \begin{array}{c} \xrightarrow{\mathbb{F}} \\ \xrightarrow{\perp} \\ \xrightarrow{\quad} \end{array} \text{Db|}_\ell$$

- Dawson, Paré, & Pronk show that $\mathbb{S}\text{pan}$ is a left adjoint (TAC, 2010) and a right adjoint (unpublished) using NcoVDCs .

- Niefield (2012) and Grandis & Paré (2017) study $\mathbb{I}\mathbb{D} \rightarrow \mathbb{S}\text{pan}(\mathbb{I}\mathbb{D}_0)$ when $\mathbb{I}\mathbb{D}$ has tabulators.

- Mac Lane (CW, Ch. IX, S.5) and Loregian (2021) use the subdivision category \mathcal{C}^\S to show that ends of $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow V$ correspond to limits $\mathcal{C}^\S \rightarrow V$.

- Carlson & Patterson (2026) construct the flattening of $\mathbb{I}\mathbb{D}$, closely related to $\mathbb{L}\mathbb{S}(\mathbb{I}\mathbb{D})$.