THE FREE TABULATOR COMPLETION OF A DOUBLE CATEGORY

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MOTIVATION & OVERVIEW

A double category $\mathbb{D}$ consists of:

- objects $A, B, C, D, \ldots$
- tight morphisms $A \to B$

$\Big\}\ \mathbb{D}_0$

- loose morphisms $A \nrightarrow B$
- cells

$\Big\}\ \mathbb{D}_1$

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "\shortmid"{marking}] & D \end{tikzcd}$$

Same as a pseudocategory in CAT:

$$\begin{tikzcd} \mathbb{D}_0 \arrow[r, "\mathrm{id}" description] & \mathbb{D}_1 \arrow[l, shift right=2ex, "\mathrm{dom}"'] \arrow[l, shift left=2ex, "\mathrm{cod}"] & \mathbb{D}_2 = \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[l, "\odot" description] \arrow[l, shift right=2ex, "\pi_1"'] \arrow[l, shift left=2ex, "\pi_2"] \end{tikzcd}$$

- How can we reduce limits indexed by double categories

$$\mathbb{J} \xrightarrow{F} \mathbb{D}$$

to limits indexed by categories?

$$\mathrm{I} \xrightarrow{\hat{F}} \mathbb{D}_0$$

- Can we understand the span construction as a right adjoint?

$$\mathbb{D} \xrightarrow{\text{lax}} \mathbb{S}\mathrm{pan}$$

GOAL: Answer these questions via the free tabulator completion!

TWO-DIMENSIONAL CATEGORICAL STRUCTURES

Double categories $\mathbb{D}$

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "\shortmid"{marking}] & D \end{tikzcd}$$

2-categories $\mathbb{T}\mathrm{i}(\mathcal{K})$

$$\begin{tikzcd} A \arrow[r, "id_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "g"] \\ B \arrow[r, "id_B"', "\shortmid"{marking}] & B \end{tikzcd}$$

Bicategories $\mathbb{L}\mathrm{o}(\mathcal{B})$

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "1_A"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "1_B"] \\ A \arrow[r, "q"', "\shortmid"{marking}] & B \end{tikzcd}$$

Monoidal categories $\mathbb{L}\mathrm{o}(\mathcal{V}, \otimes, I)$

$$\begin{tikzcd} * \arrow[r, "A", "\shortmid"{marking}] \arrow[d, "1_*"'] \arrow[dr, phantom, "f"] & * \arrow[d, "1_*"] \\ * \arrow[r, "B"', "\shortmid"{marking}] & * \end{tikzcd}$$

EXAMPLES

$\mathbb{S}\mathrm{pan}(\mathcal{C})$, $\mathcal{C}$ – has chosen pullbacks

$$\begin{tikzcd}
A \arrow[d, "f"'] & X \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\alpha"] & B \arrow[d, "g"] \\
C & Y \arrow[l, "q_1"] \arrow[r, "q_2"'] & D
\end{tikzcd}$$

$\mathbb{D}\mathrm{ist}$

$$\begin{tikzcd}
A \times B^{op} \arrow[drr, "P"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\Downarrow\alpha"] & & \\
C \times D^{op} \arrow[rr, "Q"'] & & \mathrm{Set}
\end{tikzcd}$$

$\mathbb{R}\mathrm{el}$

$$\begin{tikzcd}
A \arrow[r, "R \subseteq A \times B", "\shortmid" marking] \arrow[d, "f"'] & B \arrow[d, "g"] \\
C \arrow[r, "S \subseteq C \times D"', "\shortmid" marking] & D
\end{tikzcd}$$

$$R(a,b) \Downarrow S(fa,gb)$$

$\mathbb{R}(\mathcal{C})$, $\mathcal{C}$ – restriction category

$$\begin{tikzcd}
A \arrow[r, "p", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\geq"] & B \arrow[d, "g"] \\
C \arrow[r, "q"', "\shortmid" marking] & D
\end{tikzcd}$$

$gp = qf\bar{p}$

f, g total maps

TABULATORS

tabulator of a loose morphism $p : A \nrightarrow B$

$$\begin{tikzcd} T_p \arrow[r, "id", "\mid" marking] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & T_p \arrow[d, "\pi_B"] \\ A \arrow[r, "p"', "\mid" marking] & B \end{tikzcd}$$

universal property

$$\begin{tikzcd} X \arrow[r, "id_X", "\mid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & X \arrow[d, "g"] \\ A \arrow[r, "p"', "\mid" marking] & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} X \arrow[d, "\langle f,g,\alpha\rangle"] \\ T_p \end{tikzcd}$$

- Tabulators are limits of $2_\ell \longrightarrow \mathbb{D}$
- Cotabulators are dual via tight opposite $\mathbb{D}^{opt}$
- Natural with respect to cells:

$$\begin{tikzcd} A \arrow[r, "p", "\mid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "\mid" marking] & D \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} T_p \arrow[d, "T\alpha"] \\ T_q \end{tikzcd}$$

- $\mathbb{D}$ admits all tabulators if and only if

$$\begin{tikzcd} \mathbb{D}_0 \arrow[r, shift left, "id"] \arrow[r, phantom, "\perp"] & \mathbb{D}_1 \arrow[l, shift left, "T_1"] \end{tikzcd}$$

EXAMPLES OF TABULATORS

2-categories. powers by 2

$$\begin{tikzcd} A^2 \arrow[r, "id", "\shortmid"{marking}] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\pi_A"] & A^2 \arrow[d, "\pi_2"] \\ A \arrow[r, "id_A"', "\shortmid"{marking}] & A \end{tikzcd}$$

universal property

$$\begin{tikzcd} X \arrow[r, "id_X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & X \arrow[d, "g"] \\ A \arrow[r, "id_A"', "\shortmid"{marking}] & A \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X \arrow[d, "\langle f, g, \alpha \rangle"] \\ A^2 \end{tikzcd}$$

bicategories: unique 2-cell from identity

$$\begin{tikzcd} A \arrow[r, "id_A", "\shortmid"{marking}] \arrow[d, "1_A"'] \arrow[dr, phantom, "\exists!"] & A \arrow[d, "1_A"] \\ A \arrow[r, "P"', "\shortmid"{marking}] & A \end{tikzcd}$$

monoidal categories: unique morphism from unit

$$\begin{tikzcd} * \arrow[r, "I", "\shortmid"{marking}] \arrow[d, "1_*"'] \arrow[dr, phantom, "\exists!"] & * \arrow[d, "1_*"] \\ * \arrow[r, "A"', "\shortmid"{marking}] & * \end{tikzcd}$$

MORE EXAMPLES OF TABULATORS

$\mathbb{S}pan(\mathcal{C})$

$$\begin{tikzcd} X \arrow[d, "p_1"'] & X \arrow[l, "1_X"'] \arrow[r, "1_X"] \arrow[d, "1_X"] & X \arrow[d, "p_2"] \\ A & X \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

$\mathbb{R}el$

$$\begin{tikzcd} R \arrow[r, "R \subseteq R \times R", "\shortmid"{marking}] \arrow[d, "\pi_A"'] & R \arrow[d, "\pi_B"] \\ A \arrow[r, "R \subseteq A \times B"', "\shortmid"{marking}] & B \end{tikzcd}$$

$$\begin{array}{c} (R(a,b), R(a,b)) \\ \Downarrow \\ R(a,b) \end{array}$$

$\mathbb{R}(\mathcal{C})$, $\mathcal{C}$ - restriction category

$$\begin{tikzcd} X \arrow[r, "1_X", "\shortmid"{marking}] \arrow[d, "\pi"'] & X \arrow[d, "p\pi"] \\ A \arrow[r, "p"', "\shortmid"{marking}] & B \end{tikzcd}$$

π, $p\pi$ total maps

If the restriction idempotent $\bar{p}$ splits (i.e. $\pi u = \bar{p}$ and $u\pi = 1_X$), then the tabulator of p exists + satisfies 2D universal property.

(CO) TABULATORS IN $\mathbb{D}\mathrm{ist}$

Tabulator of a distributor $P : A^{op} \times B \to \mathrm{Set}$

$$\begin{tikzcd} T_P \arrow[r, "id", "\shortmid"{marking}] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_P"] & T_P \arrow[d, "\pi_B"] \\ A \arrow[r, "P"', "\shortmid"{marking}] & B \end{tikzcd}$$

- objects $(a \in A,\ b \in B,\ x \in P(a,b))$
- morphisms $(f,g) : (a,b,x) \longrightarrow (a',b',x')$

$$\begin{tikzcd} a \arrow[r, squiggly, "x"] \arrow[d, "f"'] & b \arrow[d, "g"] \\ a' \arrow[r, squiggly, "x'"'] & b' \end{tikzcd}$$

$$P(f, 1_{b'})(x') = P(1_a, g)(x)$$

Cotabulator of a distributor $P : A^{op} \times B \to \mathrm{Set}$

$$\begin{tikzcd} A \arrow[r, "P", "\shortmid"{marking}] \arrow[d, "\amalg_A"'] \arrow[dr, phantom, "\amalg_P"] & B \arrow[d, "\amalg_B"] \\ \perp P \arrow[r, "id"', "\shortmid"{marking}] & \perp P \end{tikzcd}$$

- set of objects $ob(A) + ob(B)$
- Hom-sets

$$\perp P(x,y) = \begin{cases} A(x,y) & \text{if } x,y \in A \\ B(x,y) & \text{if } x,y \in B \\ P(x,y) & \text{if } x \in A \text{ and } y \in B \\ \emptyset & \text{otherwise} \end{cases}$$

BINARY TABULATORS

binary tabulator of $A \overset{p}{\nrightarrow} B \overset{q}{\nrightarrow} C$

$$\begin{tikzcd}
T(p,q) \arrow[r, "id", "\shortmid"{marking}] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & T(p,q) \arrow[r, "id", "\shortmid"{marking}] \arrow[d, "\pi_B"] \arrow[dr, phantom, "\pi_q"] & T(p,q) \arrow[d, "\pi_C"] \\
A \arrow[r, "p"', "\shortmid"{marking}] & B \arrow[r, "q"', "\shortmid"{marking}] & C
\end{tikzcd}$$

Lemma: Suppose $\mathbb{D}$ has tabulators. Then binary tabulators $\Longleftrightarrow$ pullbacks of left projections along right projections.

$$\begin{tikzcd}
& & T(p,q) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner"{rotate=-45}] & & \\
& Tp \arrow[dl] \arrow[dr] \arrow[d, phantom, "\pi_p"] & {} & Tq \arrow[dl] \arrow[dr] \arrow[d, phantom, "\pi_q"] & \\
A \arrow[rr, "p"', "\shortmid"{marking}] & {} & B \arrow[rr, "q"', "\shortmid"{marking}] & {} & C
\end{tikzcd}$$

The following are equivalent:

- $\mathbb{D}$ admits all binary tabulators
- limits of normal lax $3_\ell \to \mathbb{D}$ exist

$$\begin{tikzcd}
A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "1_A"'] & B \arrow[r, "q", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & C \arrow[d, "1_C"] \\
A \arrow[rr, "r"', "\shortmid"{marking}] & {} & C
\end{tikzcd}$$

- There exists right adjoints

$$\begin{tikzcd}
\mathbb{D}_2 \arrow[rr, hook, shift right] \arrow[dr, dashed, bend right, "T_2'"'] & & {[3_\ell, \mathbb{D}]_{n\ell}} \arrow[ll, shift right, "T"'] \arrow[dl, dashed, bend left, "T_2"] \\
& \mathbb{D}_0 \arrow[ul, "\dashv"{sloped}] \arrow[ur, "\dashv"{sloped}'] &
\end{tikzcd}$$

THE SPAN CONSTRUCTION

adequate triple $(\mathcal{C}, \mathcal{L}, \mathcal{R})$

- $\mathcal{C}$ – category
- $\mathcal{L}, \mathcal{R}$ – wide subcategories of $\mathcal{C}$
- Pullbacks of $\mathcal{L}$-morphisms along $\mathcal{R}$-morphisms exist in $\mathcal{C}$ and the classes are stable

AdTrip: 2-category of adequate triples

DblTab: 2-category of double categories with tabulators and binary tabulators

$$\mathrm{DblTab}_\ell \xrightarrow{(-)_0} \mathrm{AdTrip}$$

$$\mathbb{D} \longmapsto (\mathbb{D}_0, \mathcal{L}P, \mathcal{R}P)$$

left projections right projections

$$\mathrm{AdTrip} \xrightarrow{\mathbb{S}\mathrm{pan}} \mathrm{DblTab}_{ps}$$

$$(\mathcal{C}, \mathcal{L}, \mathcal{R}) \longmapsto \mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{L}, \mathcal{R})$$

- objects and tight morphisms from $\mathcal{C}$
- loose morphisms $A \xleftarrow{\ell \in \mathcal{L}} X \xrightarrow{r \in \mathcal{R}} B$

$$\begin{tikzcd} \mathrm{AdTrip} \arrow[r, bend right, "\mathbb{S}\mathrm{pan}"'] \arrow[r, phantom, "\perp"] & \mathrm{DblTab}_\ell \arrow[l, bend right, "(-)_0"'] \end{tikzcd}$$

unit $\mathbb{D} \xrightarrow{\eta_{\mathbb{D}}} \mathbb{S}\mathrm{pan}(\mathbb{D}_0, \mathcal{L}P, \mathcal{R}P)$

$$A \overset{p}{\longrightarrow\!\!\!\!\!\!\!+}\; B \longmapsto A \xleftarrow{\pi_A} Tp \xrightarrow{\pi_B} B$$

THE FREE COMPLETION UNDER TABULATORS

$$\begin{tikzcd} \mathbb{D}\mathrm{blTab}_\ell \arrow[r, shift right=2, "U"'] \arrow[r, phantom, "\perp"] & \mathbb{D}\mathrm{bl}_\ell \arrow[l, shift right=2, "\mathbb{F}"'] \end{tikzcd}$$

Given $\mathbb{D}$, we may freely add tabulators and binary tabulators:

$$\begin{tikzcd} \hat{p} \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "s_p"'] \arrow[dr, phantom, "\tau_p"] & \hat{p} \arrow[d, "t_p"] \\ A \arrow[r, "p"', "\shortmid"{marking}] & B \end{tikzcd}
\qquad
\begin{tikzcd} & \widehat{(p,q)} \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] \arrow[dd, phantom, "\circlearrowright"] & \\ \hat{p} \arrow[dr, "t_p"'] & & \hat{q} \arrow[dl, "s_q"] \\ & B & \end{tikzcd}$$

$$\begin{tikzcd} & A \arrow[dl, bend right, "1"'] \arrow[d, "i_A"] \arrow[dr, bend left, "1"] & \\ A & \widehat{\mathrm{id}}_A \arrow[l, "s_A"] \arrow[r, "t_A"'] & A \end{tikzcd}
\qquad
\begin{tikzcd} \hat{p} \arrow[d, "s_p"'] & \widehat{(p,q)} \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] \arrow[d, "c_{p,q}"] & \hat{q} \arrow[d, "t_q"] \\ A & \widehat{p \odot q} \arrow[l, "s_{p\odot q}"] \arrow[r, "t_{p\odot q}"'] & C \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "\shortmid"{marking}] & D \end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd} A \arrow[d, "f"'] & \hat{p} \arrow[l, "s_p"'] \arrow[r, "t_p"] \arrow[d, "\hat{\alpha}"] & B \arrow[d, "g"] \\ C & \hat{q} \arrow[l, "s_q"] \arrow[r, "t_q"'] & D \end{tikzcd}$$

... modulo several equations.

Write $\mathcal{LS}(\mathbb{D}) = \mathbb{F}(\mathbb{D})_0$, the <u>loose subdivision</u> of a double category.

$$\frac{\mathbb{D} \xrightarrow{\text{lax}} \mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{L}, \mathcal{R})}{(\mathcal{LS}(\mathbb{D}), \mathcal{LP}, \mathcal{RP}) \longrightarrow (\mathcal{C}, \mathcal{L}, \mathcal{R})}$$

REDUCTION OF LIMITS IN DOUBLE CATEGORIES

$$\begin{tikzcd} \mathbb{J} \arrow[r, "!"] \arrow[d, "F"'] & \mathbb{1} \arrow[dl, "\lim F"] \arrow[dl, phantom, "\xLeftarrow{\pi}", near start, swap] \\ \mathbb{D} & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \lim F \arrow[r, "id", "\shortmid"{marking}] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & \lim F \arrow[d, "\pi_B"] \\ A \arrow[r, "p"', "\shortmid"{marking}] & B \end{tikzcd}$$

Theorem (Grandis & Paré, 1999): A double category admits limits indexed by any small double category $\mathbb{J}$ if and only if it admits small products, equalisers, & tabulators.

Proof strategy: Given a diagram $F: \mathbb{J} \to \mathbb{D}$, construct new diagram $\mathcal{I} \xrightarrow{G} \mathbb{D}_0$ into the underlying category of $\mathbb{D}$ using tabulators and binary tabulators.

Then show that $\mathrm{Cone}(F) \cong \mathrm{Cone}(G)$. By the classical result, the limit of any $G: \mathcal{I} \to \mathbb{D}_0$ exists iff $\mathbb{D}_0$ has products and equalisers. $\square$

Original proof is 5 pages long and very explicit; maybe a more abstract proof is possible?

$$\mathbb{J} \xrightarrow{F} \mathbb{D} \quad \rightsquigarrow \quad \mathbb{F}(\mathbb{J}) \xrightarrow{F'} \mathbb{D}$$

Show that $\mathbb{T}_i(\mathcal{LS}(\mathbb{J})) \longrightarrow \mathbb{F}(\mathbb{J})$ is an initial functor of double cats.

THE NERVE OF A DOUBLE CATEGORY

$$\Delta \xrightarrow{\mathbb{L}_0} \mathrm{Dbl}_{n\ell}, \qquad \mathbb{L}_0(k) = \underline{k}_\ell$$

$$\{0 \mathrel{\mapstochar\rightarrow} 1 \mathrel{\mapstochar\rightarrow} \cdots \mathrel{\mapstochar\rightarrow} k-1\}$$

$$\mathrm{Dbl}_{n\ell} \xrightarrow{N} [\Delta^{op}, \mathrm{CAT}]$$

$$N(\mathbb{D})_k = [\underline{k}_\ell, \mathbb{D}]_{n\ell}$$

For example:

- $N(\mathbb{D})_0 \cong \mathbb{D}_0$ objects & tight morphisms
- $N(\mathbb{D})_1 \cong \mathbb{D}_1$ loose morphisms & cells
- $N(\mathbb{D})_2 \rightsquigarrow$ objects are

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "1"'] & B \arrow[r, "q", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & C \arrow[d, "1"] \\ A \arrow[rr, "r"', "\shortmid"{marking}] & & C \end{tikzcd}$$

Theorem: The nerve is 2-fully faithful.

Proof: Follows Lack & Paoli (2008).

$$D := \left\{ \begin{tikzcd} 0 \arrow[r, shift left=2] \arrow[r, shift right=2] & 1 \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & 2 \end{tikzcd} \right\} \rightarrowtail \Delta$$

Lemma: $\mathrm{Dbl}_{n\ell} \xrightarrow{N} [D^{op}, \mathrm{CAT}]$ is locally fully faithful.

$$\begin{tikzcd}[column sep=large] \mathbb{J}_0 \arrow[r, "\mathrm{id}" description] \arrow[d, bend right, "F_0"'] \arrow[d, bend left, "G_0"] \arrow[d, phantom, "\overset{\alpha_0}{\Rightarrow}"] & \mathbb{J}_1 \arrow[l, shift right=3, "\mathrm{dom}"'] \arrow[l, shift left=3, "\mathrm{cod}"] \arrow[d, bend right, "F_1"'] \arrow[d, bend left, "G_1"] \arrow[d, phantom, "\overset{\alpha_1}{\Rightarrow}"] & {[\underline{2}_\ell, \mathbb{J}]} \arrow[l, shift right=3] \arrow[l] \arrow[l, shift left=3] \arrow[d, bend right, "F_2"'] \arrow[d, bend left, "G_2"] \arrow[d, phantom, "\overset{\alpha_2}{\Rightarrow}"] \\ \mathbb{D}_0 \arrow[r, "\mathrm{id}" description] & \mathbb{D}_1 \arrow[l, shift right=3, "\mathrm{dom}"'] \arrow[l, shift left=3, "\mathrm{cod}"] & {[\underline{2}_\ell, \mathbb{D}]} \arrow[l, shift right=3] \arrow[l] \arrow[l, shift left=3] \end{tikzcd}$$

LOOSE SUBDIVISION AS A COLAX COLIMIT

The loose subdivision $\mathcal{LS}(\mathbb{J})$ is the colax colimit of the "truncated nerve":

$$
\begin{tikzcd}
\mathbb{J}_0 \arrow[r, "\mathrm{id}" description] \arrow[dr, bend right] & \mathbb{J}_1 \arrow[l, shift right=3, "\mathrm{dom}"'] \arrow[l, shift left=3, "\mathrm{cod}"] \arrow[d] & \mathbb{J}_2 \arrow[l, shift right=3, "\pi_1"'] \arrow[l, "\odot" description] \arrow[l, shift left=3, "\pi_2"] \arrow[dl, bend left] \\
& \mathcal{LS}(\mathbb{J}) &
\end{tikzcd}
$$

can replace with $[\mathbb{2}_\ell, \mathbb{J}]_{n\ell}$

If $\mathbb{D}$ has tabulators and binary tabulators:

$$
\begin{tikzcd}
\mathbb{D}_1 \arrow[rr, "\mathrm{dom}"] \arrow[dr, "T_1"'] & {} \arrow[d, phantom, "\Rightarrow" near start] & \mathbb{D}_0 \arrow[dl, "1"] \\
& \mathbb{D}_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}_1 \arrow[rr, "\mathrm{cod}"] \arrow[dr, "T_1"'] & {} \arrow[d, phantom, "\Rightarrow" near start] & \mathbb{D}_0 \arrow[dl, "1"] \\
& \mathbb{D}_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}_0 \arrow[rr, "\mathrm{id}"] \arrow[dr, "1"'] & {} \arrow[d, phantom, "\Rightarrow" near start] & \mathbb{D}_1 \arrow[dl, "T_1"] \\
& \mathbb{D}_0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{D}_2 \arrow[rr, "\odot"] \arrow[dr, "T_2"'] & {} \arrow[d, phantom, "\Rightarrow" near start] & \mathbb{D}_1 \arrow[dl, "T_1"] \\
& \mathbb{D}_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}_2 \arrow[rr, "\pi_1"] \arrow[dr, "T_2"'] & {} \arrow[d, phantom, "\Rightarrow" near start] & \mathbb{D}_1 \arrow[dl, "T_1"] \\
& \mathbb{D}_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}_2 \arrow[rr, "\pi_2"] \arrow[dr, "T_2"'] & {} \arrow[d, phantom, "\Rightarrow" near start] & \mathbb{D}_1 \arrow[dl, "T_1"] \\
& \mathbb{D}_0 &
\end{tikzcd}
$$

By universal property of colax colimits, each normal lax functor $F : \mathbb{J} \longrightarrow \mathbb{D}$ induces a unique $\hat{F} : \mathcal{LS}(\mathbb{J}) \longrightarrow \mathbb{D}_0$.

$$
\begin{tikzcd}
\mathbb{J}_0 \arrow[r, "\mathrm{id}" description] \arrow[d, "F_1"'] & \mathbb{J}_1 \arrow[l, shift right=3, "\mathrm{dom}"'] \arrow[l, shift left=3, "\mathrm{cod}"] \arrow[d, "F_2"] & \mathbb{J}_2 \arrow[l, shift right=3, "\pi_1"'] \arrow[l, "\odot" description] \arrow[l, shift left=3, "\pi_2"] \arrow[d, "F_2"] \\
\mathbb{D}_0 \arrow[r, "\mathrm{id}" description] \arrow[dr, bend right, "1"'] & \mathbb{D}_1 \arrow[l, shift right=3, "\mathrm{dom}"'] \arrow[l, shift left=3, "\mathrm{cod}"] \arrow[d, "T_1"] & \mathbb{D}_2 \arrow[l, shift right=3, "\pi_1"'] \arrow[l, "\odot" description] \arrow[l, shift left=3, "\pi_2"] \arrow[dl, bend left, "T_2"] \\
& \mathbb{D}_0 &
\end{tikzcd}
$$

By local fully faithfulness of truncated nerve, we may show that $\mathrm{Cone}(F) \cong \mathrm{Cone}(\hat{F})$.

EXAMPLE: COMPUTING THE EM-CATEGORY OF A COMONAD

Consider $\mathbb{M}\mathrm{nd}^{\mathrm{opt}}$ freely generated by cells

$$
\begin{tikzcd}
* \arrow[r, "m", "\shortmid"{marking}] \arrow[d, "1"'] \arrow[dr, phantom, "\varepsilon"] & * \arrow[d, "1"] \\
* \arrow[r, "\mathrm{id}_*"', "\shortmid"{marking}] & *
\end{tikzcd}
\qquad
\begin{tikzcd}
* \arrow[rr, "m", "\shortmid"{marking}] \arrow[d, "1"'] \arrow[drr, phantom, "\delta"] & & * \arrow[d, "1"] \\
* \arrow[r, "m"', "\shortmid"{marking}] & * \arrow[r, "m"', "\shortmid"{marking}] & *
\end{tikzcd}
$$

modulo the equations for a comonad.

Let $\mathbb{M}\mathrm{nd}^{\mathrm{opt}} \xrightarrow{F} \mathbb{D}\mathrm{ist}$ send $* \xrightarrow{m} *$ to the distributor $A(-, S-) : A^{\mathrm{op}} \times A \longrightarrow \mathrm{Set}$ for an endofunctor $S : A \to A$.

This diagram determines a comonad (S, ε, δ) on A, and its limit is the EM-category.

The functor $\mathcal{LS}(\mathbb{M}\mathrm{nd}^{\mathrm{opt}}) \xrightarrow{\hat{F}} \mathrm{Cat}$ is determined by the diagram

$$
\begin{tikzcd}
A \arrow[d, "i"] & & (1_A \downarrow S) \times_A (1_A \downarrow S) \arrow[dl, shift right, "\pi_1"'] \arrow[dl, shift left, "\pi_2"] \arrow[d, "c"] \\
1_A \downarrow 1_A & 1_A \downarrow S \arrow[l, "\hat{\varepsilon}"] \arrow[r, "\hat{\delta}"'] & 1_A \downarrow S^2
\end{tikzcd}
$$

$i(x) = (x, x, 1_x : x \to x)$

$\hat{\varepsilon}(x, y, x \xrightarrow{f} Sy) = (x, y, x \xrightarrow{\varepsilon_y \circ f} y)$

$\hat{\delta}(x, y, x \xrightarrow{f} Sy) = (x, y, x \xrightarrow{\mu_y \circ f} S^2 y)$

$c(x, y, z, x \xrightarrow{f} Sy, y \xrightarrow{g} Sz) = (x, z, x \xrightarrow{Sg \circ f} Sz)$

The objects of $\lim(\hat{F})$ are $(x, y, x \xrightarrow{f} Sy)$ such that $x = y$, $\varepsilon_x \cdot f = 1_x$, and $\mu_f \cdot f = Sf \cdot f$.

LOOSE MORPHISMS BETWEEN TABULATORS

Tight morphisms between tabulators come from naturality of limits:

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "\shortmid"{marking}] & D \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} Tp \arrow[d, "T\alpha"] \\ Tq \end{tikzcd}$$

Loose morphisms between tabulators are limits indexed by a loose distributor:

parallel tabulators

$$\begin{tikzcd} 2_\ell \arrow[r, "\mathbb{H}\mathrm{om}", "\shortmid\shortmid"{marking}] \arrow[d, "p"'] \arrow[dr, phantom, "\Phi"] & 2_\ell \arrow[d, "q"] \\ \mathbb{D} \arrow[r, "\mathbb{H}\mathrm{om}"', "\shortmid\shortmid"{marking}] & \mathbb{D} \end{tikzcd}$$

Gives 2D universal property of tabulators.

$\lim \Phi : Tp \nrightarrow Tq$ determined by data

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "1"'] & B \arrow[r, "r_1", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & Y \arrow[d, "1"] \\ A \arrow[rr, "r_3"', "\shortmid"{marking}] & & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "r_2", "\shortmid"{marking}] \arrow[d, "1"'] & X \arrow[r, "q", "\shortmid"{marking}] \arrow[d, phantom, "\beta"] & Y \arrow[d, "1"] \\ A \arrow[rr, "r_3"', "\shortmid"{marking}] & & Y \end{tikzcd}$$

If $\mathbb{D}$ admits parallel tabulators and parallel binary tabulators, the limits indexed by $\mathbb{J} : \mathbb{S} \nrightarrow \mathbb{T}$ reduce to:

$$\begin{tikzcd} \mathcal{L}\mathbb{S}(\mathbb{S}) \arrow[d] & \mathcal{L}\mathbb{S}(\mathbb{J}) \arrow[l] \arrow[r] \arrow[d] & \mathcal{L}\mathbb{S}(\mathbb{T}) \arrow[d] \\ \mathbb{D}_0 & \mathbb{D}_1 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & \mathbb{D}_0 \end{tikzcd}$$

SUMMARY & RELATED WORK

- We introduced the free completion $\mathbb{F}(\mathbb{D})$ of $\mathbb{D}$ under tabulators and binary tabulators.

- Showed that limits of double functors

$$\mathbb{J} \xrightarrow{F} \mathbb{D}$$

may be reduced to limits of functors

$$\mathbb{F}(\mathbb{D})_0 = \mathcal{LS}(\mathbb{D}) \xrightarrow{\hat{F}} \mathbb{D}_0$$

if $\mathbb{D}$ has tabulators & binary tabulators.

- Characterised $\mathbb{S}\mathrm{pan}$ as a right adjoint.

$$\mathsf{AdTrip} \underset{\mathbb{S}\mathrm{pan}}{\overset{(-)_0}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathsf{DblTab}_\ell \underset{\longrightarrow}{\overset{\mathbb{F}}{\overset{\longleftarrow}{\perp}}} \mathsf{Dbl}_\ell$$

- Dawson, Paré, & Pronk show that $\mathbb{S}\mathrm{pan}$ is a left adjoint (TAC, 2010) and a right adjoint (unpublished) using NcoVDCs.

- Niefield (2012) and Grandis & Paré (2017) study $\mathbb{D} \longrightarrow \mathbb{S}\mathrm{pan}(\mathbb{D}_0)$ when $\mathbb{D}$ has tabulators.

- Mac Lane (CW, Ch. IX, S.5) and Loregian (2021) use the subdivision category $\mathcal{C}^\S$ to show that ends of $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{V}$ corespond to limits $\mathcal{C}^\S \to \mathcal{V}$.

- Carlson & Patterson (2026) construct the flattening of $\mathbb{D}$, closely related to $\mathcal{LS}(\mathbb{D})$.