RESTRICTION LIMITS VS DOUBLE-CATEGORICAL LIMITS

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MOTIVATION & OUTLINE

- Both functions and partial functions are important in mathematics $\rightsquigarrow$ what structures in category theory should be used to study them?
- The category Par of sets & partial functions is the Kleisli category for the monad $(-)+1 : \mathrm{Set} \to \mathrm{Set}$.

$$A \nrightarrow B \quad \rightsquigarrow \quad A \longrightarrow B+1 \quad \rightsquigarrow \quad A \leftarrowtail X \to B$$

- Par is less nice than Set as a category; instead we may use restriction categories & double categories.

1. Background on restriction categories and double categories
2. Restriction limits
3. Limits indexed by double categories
4. Limits indexed by loose distributors

GOAL: Show that restriction limits are certain double-categorical limits, using partial functions as a case study.

BACKGROUND ON RESTRICTION CATEGORIES & DOUBLE CATEGORIES

RESTRICTION CATEGORIES

A restriction category is a category $\mathcal{C}$ equipped with a restriction operator

$$f : A \longrightarrow B \quad \rightsquigarrow \quad \bar{f} : A \longrightarrow A$$

satisfying the following axioms:

(R1) $f \circ \bar{f} = f$ $\quad \forall\, f : A \to B$

(R2) $\bar{f} \circ \bar{g} = \bar{g} \circ \bar{f}$ $\quad \forall\, f : A \to B,\ g : A \to C$

(R3) $\overline{g \circ \bar{f}} = \bar{g} \circ \bar{f}$ $\quad \forall\, f : A \to B,\ g : A \to C$

(R4) $\bar{g} \circ f = f \circ \overline{g \circ f}$ $\quad \forall\, f : A \to B,\ g : B \to C$

Morphism $f : A \longrightarrow B$ is total if $\bar{f} = 1_A$.

Let $\mathrm{Total}(\mathcal{C})$ be the wide subcategory of total maps.

Each $\mathcal{C}(A,B)$ is a poset where $f \leq g$ if $f = g \circ \bar{f}$.

The category Par of sets & partial functions is a restriction category where

$$A \overset{i}{\hookleftarrow} X \overset{f}{\longrightarrow} B \quad \rightsquigarrow \quad A \overset{i}{\hookleftarrow} X \overset{i}{\hookrightarrow} A$$

A restriction functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ satisfies $F\bar{f} = \overline{Ff}$.

A (lax) transformation $\alpha : F \Rightarrow G$ has components $\alpha_A : FA \longrightarrow GA$ that are total and satisfy

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "\geq"] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd} \quad \rightsquigarrow \quad Ff \circ \alpha_B = Gf \circ \alpha_A \circ \overline{Ff}$$

Let RCat_ℓ be the 2-category of restriction categories.

DOUBLE CATEGORIES

A double category $\mathbb{D}$ consists of:

- objects $A, B, C, D, \ldots$
- tight morphisms $f : A \to B$ (drawn vertical)
- loose morphisms $p : A \nrightarrow B$ (drawn horizontal)
- cells

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "\shortmid"{marking}] & D \end{tikzcd}$$

- tight identities 1_A and composites $g \circ f$
- loose identities id_A and composities $p \odot q$
- left unitor, right unitor, associator isocells

$\mathrm{id}_A \odot p \cong p \qquad p \odot \mathrm{id}_B \cong p \qquad p \odot (q \odot r) \cong (p \odot q) \odot r$

There is a strict double category $\mathbb{P}\mathrm{ar}$ whose:

- objects are sets
- tight morphisms are functions
- loose morphisms are partial functions
- cells are diagrams

$$\begin{tikzcd} A \arrow[d, "f"'] & X \arrow[l, "i"', tail] \arrow[r, "p"] \arrow[d, "\alpha"] & B \arrow[d, "g"] \\ C & Y \arrow[l, "j", tail] \arrow[r, "q"'] & D \end{tikzcd}$$

$\alpha : X \to Y$ is unique if it exists $\rightsquigarrow$ thin double category

The underlying 2-category whose 1-cells are loose morphisms recovers Par as a poset-enriched category.

NORMAL LAX FUNCTORS & TRANSFORMATIONS

A normal lax functor $F : \mathbb{C} \to \mathbb{D}$ is an assignment

$$\begin{tikzcd} A \arrow[r, "p", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "|"{marking}] & D \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} FA \arrow[r, "Fp", "|"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\alpha"] & FB \arrow[d, "Fg"] \\ FC \arrow[r, "Fq"', "|"{marking}] & FD \end{tikzcd}$$

preserving tight identities & composites and loose identities together with compositor cells:

$$\begin{tikzcd} FA \arrow[r, "Fp \odot Fq", "|"{marking}] \arrow[d, "1"'] \arrow[dr, phantom, "\underline{c}(p,q)"] & FC \arrow[d, "1"] \\ FA \arrow[r, "F(p\odot q)"', "|"{marking}] & FC \end{tikzcd}$$

natural + coherent

Strict functor if laxators are identities.

A transformation $\varphi : F \Rightarrow G$ consists of cells

$$\begin{tikzcd} FA \arrow[r, "Fp", "|"{marking}] \arrow[d, "\varphi_A"'] \arrow[dr, phantom, "\varphi_p"] & FB \arrow[d, "\varphi_B"] \\ GA \arrow[r, "Gp"', "|"{marking}] & GB \end{tikzcd}$$

natural

such that $\varphi_{id_A} = id_{\varphi_A}$ and

$$\begin{array}{|c|} \hline \varphi_p \odot \varphi_q \\ \hline \underline{c}(p,q) \\ \hline \end{array} \quad = \quad \begin{array}{|c|} \hline \underline{c}(p,q) \\ \hline \varphi_{p \odot q} \\ \hline \end{array}$$

Let $\mathbf{Dbl}_{nl}$ be the 2-category of double categories and normal lax functors. Similarly $\mathbf{Dbl}_s$ for strict double functors.

FROM RESTRICTION CATEGORIES TO DOUBLE CATEGORIES

Let $\mathcal{C}$ be a restriction category. There is a strict double category $\mathbb{R}(\mathcal{C})$ whose:

- objects are objects of $\mathcal{C}$
- loose morphisms are morphisms of $\mathcal{C}$
- tight morphisms are the total morphisms
- cells are given by

$$\begin{tikzcd} A \arrow[r, "p", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\geq"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "|"{marking}] & D \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{array}{c} g \circ p \leq q \circ f \\ \text{i.e.} \\ g \circ p = q \circ f \circ \bar{p} \end{array}$$

Determines a 2-functor $\mathbb{R} : \mathrm{RCat}_\ell \longrightarrow \mathrm{Dbl}_s$ which is locally fully faithful.

<u>Example</u>: $\mathbb{R}(\mathrm{Par}) \cong \mathbb{P}\mathrm{ar}$.

RESTRICTION LIMITS

RESTRICTION LIMITS

finite category

The restriction limit of $F: J \to \mathcal{C}$ is a cone with total components

$$\begin{tikzcd} & \lim F \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] \arrow[d, phantom, "\circlearrowleft"] & \\ FA \arrow[rr, "Ff"'] & {} & FB \end{tikzcd}$$

such that for any lax cone

$$\begin{tikzcd} & X \arrow[dl, "\varphi_A"'] \arrow[dr, "\varphi_B"] \arrow[d, phantom, "\geq"] & \\ FA \arrow[rr, "Ff"'] & {} & FB \end{tikzcd} \qquad \varphi_B = Ff \circ \varphi_A \circ \overline{\varphi_B}$$

there exists a unique morphism $u: X \to \lim F$ such that $\pi_A \circ u = \varphi_A \circ e$ where $e = \prod_{A \in J} \overline{\varphi_A}$ is the composite of the restriction idempotents.

For a (finite) category J and restriction category $\mathcal{C}$, there is a restriction category $\mathcal{C}^J$ whose:

- objects are functors $F: J \to \mathcal{C}$
- morphisms $\alpha: F \Rightarrow G$ consist of a morphism $\alpha_A: FA \to GA$ for each $A \in J$ such that

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "\geq"] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd} \qquad \alpha_B \circ Ff = Gf \circ \alpha_A \circ \overline{\alpha_B \circ Ff}$$

for each $f: A \to B$ in J.

Prop (Cockett & Lack): There is a right adjoint to $\Delta: \mathcal{C} \to \mathcal{C}^J$ in RCatℓ if and only if each $F: J \to \mathcal{C}$ admits a restriction limit.

EXAMPLES: RESTRICTION TERMINAL OBJECTS & PRODUCTS

$\boxed{\text{Case } J=\emptyset:}$ A restriction terminal object is a right adjoint to $\mathcal{C}\longrightarrow 1$ in RCat_ℓ. It is an object T together with a total map $t_A: A\longrightarrow T$ for each object A, such that $t_T = 1_T$ and $t_B\circ f = t_A\overline{f}$ for each morphism $f:A\longrightarrow B$.

$\boxed{\text{Case } J=\{0,1\}:}$ The restriction product of A and B is an object $A\times B$ together with total morphisms $A\xleftarrow{\pi_A}A\times B\xrightarrow{\pi_B}B$ such that for any span $A\xleftarrow{f}X\xrightarrow{g}B$ there is a unique morphism $\langle f,g\rangle: X\longrightarrow A\times B$ such that $\pi_A\circ\langle f,g\rangle = f\circ\overline{g}$ and $\pi_B\circ\langle f,g\rangle = g\circ\overline{f}$.

$$
\begin{tikzcd}
 & X \arrow[dl, bend right, "f"'] \arrow[dr, bend left, "g"] \arrow[d, "\langle f,g\rangle"] \arrow[dl, phantom, near start, "\geq"] \arrow[dr, phantom, near start, "\leq"] & \\
A & A\times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}
$$

<u>Example</u>: In Par, the restriction terminal is the singleton set and the restriction product is the cartesian product.

$$
\begin{tikzcd}
 & Y \arrow[r, "i"] \arrow[ddl, "f"'] & X & Z \arrow[l, "j"'] \arrow[ddr, "g"] & \\
 & & Y\times_X Z \arrow[u] \arrow[ur] \arrow[d, "f\times g"] & & \\
A & & A\times B \arrow[ll, "\pi_A"] \arrow[rr, "\pi_B"'] & & B
\end{tikzcd}
$$

pullback of i and j

SPLITTING OF RESTRICTION IDEMPOTENTS

$\boxed{\text{Case } J = \{0 \to 1\}:}$ The limit of a morphism $f: A \longrightarrow B$ consists of a cone

$$\begin{tikzcd} & P \arrow[dl, "\pi"'] \arrow[dr, "f\pi"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

π, $f\pi$ are total

such that

$$\begin{tikzcd} & Q \arrow[dl, "q_1"'] \arrow[dr, "q_2"] \arrow[d, phantom, "\geqslant"] & \\ A \arrow[rr, "f"'] & {} & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} & Q \arrow[d, "\exists! u"] \arrow[ddl, bend right, "q_1"'] \arrow[ddr, bend left, "q_2"] \arrow[dl, phantom, "\geqslant"] \arrow[dr, phantom, "\leqslant"] & \\ {} & P \arrow[dl, "\pi"'] \arrow[dr, "f\pi"] & {} \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

where $\pi \circ u = q_1 \circ \overline{q_2}$.

Apply to the cone:

$$\begin{tikzcd} & A \arrow[dl, "1_A"'] \arrow[dr, "f"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} & A \arrow[d, "u"] \arrow[ddl, bend right, "1_A"'] \arrow[ddr, bend left, "f"] \arrow[dl, phantom, "\geqslant"] \arrow[dr, phantom, "\leqslant"] & \\ {} & P \arrow[dl, "\pi"'] \arrow[dr, "f\pi"] & {} \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

Unique $u: A \longrightarrow B$ such that $\pi \circ u = \overline{f}$.

Can also show that $u \circ \pi = 1_P$

<u>Example</u>: The limit of a partial function $A \overset{i}{\hookleftarrow} X \overset{f}{\longrightarrow} B$ is X.

<u>Prop (Cockett & Lack)</u>: A restriction category $\mathcal{C}$ admits all limits iff restriction idempotents split and $\mathrm{Total}(\mathcal{C})$ admits finite limits.

LIMITS INDEXED BY DOUBLE CATEGORIES

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A cone (X, φ) over a normal lax functor $F : \mathbb{J} \longrightarrow \mathbb{D}$ is an object $X \in \mathbb{D}$ & a transformation

$$
\begin{tikzcd}
\mathbb{J} \arrow[r, "!"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\varphi}{\Longleftarrow}"] & \mathbb{1} \arrow[dl, "X"] \\
\mathbb{D} &
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{J} \arrow[r, "!"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\psi}{\Longleftarrow}"] & \mathbb{1} \arrow[dl, "Y"'] \arrow[dl, bend left=60, "X"] \arrow[dl, phantom, bend left=30, "\Uparrow f"] \\
\mathbb{D} &
\end{tikzcd}
$$

A morphism of cones $f : (X, \varphi) \longrightarrow (Y, \psi)$ is a tight morphism $f : X \longrightarrow Y$ such that $\psi \circ f = \varphi$.

A limit of $F : \mathbb{J} \longrightarrow \mathbb{D}$ is a terminal object $(\lim F, \pi)$ in the category of cones over F.

$$
\begin{tikzcd}
\lim F \arrow[d, "\pi_A"'] \\
A
\end{tikzcd}
\qquad
\begin{tikzcd}
\lim F \arrow[r, "id", "\shortmid"{marking}] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & \lim F \arrow[d, "\pi_B"] \\
A \arrow[r, "p"', "\shortmid"{marking}] & B
\end{tikzcd}
$$

components of π

Universal property states for any cone (X, φ) over F there exists a unique tight morphism $u : X \longrightarrow \lim F$ such that

$$
\begin{tikzcd}
X \arrow[r, "id", "\shortmid"{marking}] \arrow[d, "u"'] \arrow[dr, phantom, "id_u"] & X \arrow[d, "u"] \\
\lim F \arrow[r, "id"', "\shortmid"{marking}] \arrow[d, "\varphi_A"'] \arrow[dr, phantom, "\pi_p"] & \lim F \arrow[d, "\pi_B"] \\
FA \arrow[r, "Fp"', "\shortmid"{marking}] & FB
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "id", "\shortmid"{marking}] \arrow[d, "\varphi_A"'] \arrow[dr, phantom, "\varphi_p"] & X \arrow[d, "\varphi_B"] \\
FA \arrow[r, "Fp"', "\shortmid"{marking}] & FB
\end{tikzcd}
$$

Theorem (Grandis & Paré): A double category admits limits indexed by any double category $\mathbb{J}$ if and only if it admits tight limits and tabulators.

TIGHT LIMITS & TABULATORS

Category $\mathbb{J} \rightsquigarrow$ double category $\mathbb{T}\mathrm{i}(\mathcal{C})$

- objects and tight morphisms determined by $\mathcal{C}$
- loose morphisms and cells are identities

A tight limit is a limit of $F : \mathbb{T}\mathrm{i}(\mathbb{J}) \to \mathbb{D}$, but this is the same as the limit of the functor $F_0 : \mathbb{J} \to \mathbb{D}_0$ in the underlying category of objects and tight morphisms in $\mathbb{D}$.

Example: $\mathbb{R}(\mathcal{C})$ admits all tight limits if and only if $\mathrm{Total}(\mathcal{C})$ admits all limits.

Tight limits in $\mathbb{P}\mathrm{ar}$ are just limits in Set.

A tabulator is a limit of $\{0 \nrightarrow 1\} \xrightarrow{p} \mathbb{D}$, i.e. a loose morphism $p : A \nrightarrow B$ in $\mathbb{D}$.

$$\begin{tikzcd}
T_p \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & T_p \arrow[d, "\pi_B"] \\
A \arrow[r, "p"', "\shortmid"{marking}] & B
\end{tikzcd}$$

Example: If a restriction category $\mathcal{C}$ admits splittings of restriction idempotents, then $\mathbb{R}(\mathcal{C})$ admits tabulators. In $\mathbb{P}\mathrm{ar}$ we have:

$$\begin{tikzcd}
X \arrow[d, "i"'] & X \arrow[l, "1"'] \arrow[r, "1"] \arrow[d, "1"] & X \arrow[d, "f"] \\
A & X \arrow[l, "i"] \arrow[r, "f"'] & B
\end{tikzcd}$$

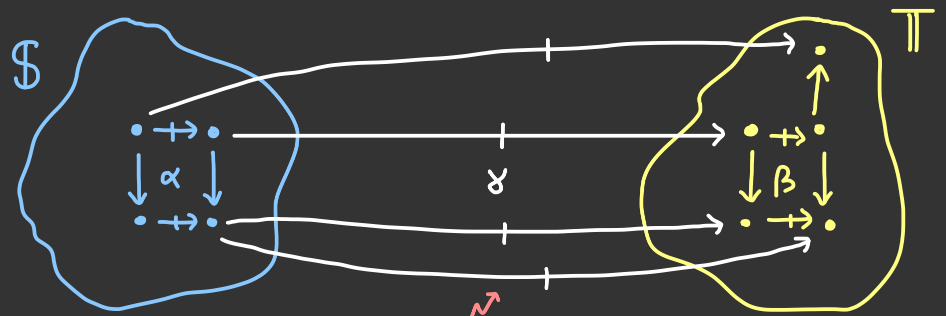
LIMITS INDEXED BY LOOSE DISTRIBUTORS

LOOSE DISTRIBUTORS

A loose distributor $(\mathbb{J},Q) : \mathbb{S} \nrightarrow \mathbb{T}$ is a strict functor $Q : \mathbb{J} \to 2_\ell = \{0 \nrightarrow 1\}$ such that $Q^{-1}\{0\} = \mathbb{S}$ and $Q^{-1}\{1\} = \mathbb{T}$.

$$\begin{tikzcd}
\mathbb{S} \arrow[r, hook] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{J} \arrow[d, "Q"] & \mathbb{T} \arrow[l, hook'] \arrow[d, "!"] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathbb{1} \arrow[r, "s"'] & 2_\ell & \mathbb{1} \arrow[l, "t"]
\end{tikzcd}$$

Picture to have in mind:



loose heteromorphisms & heterocells

Double category $\mathbb{D} \rightsquigarrow$ Hom loose distributor

$\mathrm{Hom} : \mathbb{D} \nrightarrow \mathbb{D}$ determined by $\pi : \mathbb{D} \times 2_\ell \to 2_\ell$.

An alteration Φ with frame

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "{(\mathbb{J},Q)}", "\shortmid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Phi"] & \mathbb{T} \arrow[d, "G"] \\
\mathbb{S}' \arrow[r, "{(\mathbb{J}',Q')}"', "\shortmid" marking] & \mathbb{T}'
\end{tikzcd}$$

F, G normal lax functors

is a normal lax functor $\Phi : \mathbb{J} \to \mathbb{J}'$ s.t.

$$\begin{tikzcd}
\mathbb{S} \arrow[r, hook] \arrow[d, "F"'] & \mathbb{J} \arrow[d, "\Phi"] & \mathbb{T} \arrow[l, hook'] \arrow[d, "G"] \\
\mathbb{S}' \arrow[r, hook] \arrow[d, "!"'] & \mathbb{J}' \arrow[d, "Q'"] & \mathbb{T}' \arrow[l, hook'] \arrow[d, "!"] \\
\mathbb{1} \arrow[r, "s"'] & 2_\ell & \mathbb{1} \arrow[l, "t"]
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{S} \arrow[r, hook] \arrow[d, "!"'] & \mathbb{J} \arrow[d, "Q"] & \mathbb{T} \arrow[l, hook'] \arrow[d, "!"] \\
\mathbb{1} \arrow[r, "s"'] & 2_\ell & \mathbb{1} \arrow[l, "t"]
\end{tikzcd}$$

LIMITS INDEXED BY LOOSE DISTRIBUTORS

Suppose $(\lim F, \varphi)$ and $(\lim G, \psi)$ are limits of $F : \mathbb{S} \to \mathbb{D}$ and $G : \mathbb{T} \to \mathbb{D}$. A limit of an alteration

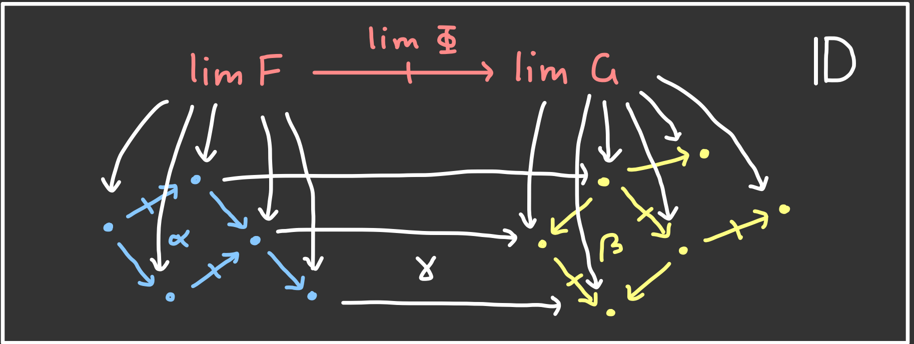
$$\begin{tikzcd} \mathbb{S} \arrow[r, "{(\mathbb{J},Q)}", "\shortmid\shortmid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Phi"] & \mathbb{T} \arrow[d, "G"] \\ \mathbb{D} \arrow[r, "\mathrm{Hom}"', "\shortmid\shortmid" marking] & \mathbb{D} \end{tikzcd}$$

is a loose morphism $\lim \Phi : \lim F \nrightarrow \lim G$ in $\mathbb{D}$ and a terminal cone θ which provides a cell θ_p in $\mathbb{D}$ for each heteromorphism $p : A \nrightarrow B$ in $\mathbb{J}$

+ axioms.

$$\begin{tikzcd} \lim F \arrow[r, "\lim \Phi", "\shortmid" marking] \arrow[d, "\varphi_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[d, "\psi_B"] \\ FA \arrow[r, "\Phi_p"', "\shortmid" marking] & GB \end{tikzcd}$$

Picture to have in mind:



<u>Note</u>: F and G could coincide / overlap.

An alteration $\Phi : \mathbb{J} \to \mathbb{D} \times \mathbb{2}_\ell$ is same as a span

$$\begin{tikzcd} \mathbb{J} \arrow[r, "Q"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\overset{\theta}{\Leftarrow}"] & \mathbb{2}_\ell \arrow[dl, bend left, "\lim : \lim F \nrightarrow \lim G"] \\ \mathbb{D} & \end{tikzcd}$$

and its limit is a pointwise right extension in $\mathbf{Dbl}_{n\ell}$; i.e. pullback along $s,t : \mathbb{1} \to \mathbb{2}_\ell$ yields limit cones.

COMPANIONS & CONJOINTS

A companion is a limit of the alteration:

$$\begin{tikzcd} \mathbf{2}_t \arrow[r, "\shortmid\shortmid" marking] \arrow[d, "f"'] & \mathbb{1} \arrow[d, "B"] \\ \mathbb{D} \arrow[r, "\shortmid\shortmid" marking, "\mathrm{Hom}"'] & \mathbb{D} \end{tikzcd} \rightsquigarrow \boxed{\begin{tikzcd} A \arrow[d, "f"] & \\ B \arrow[r, "\shortmid" marking, "\mathrm{id}"'] & B \end{tikzcd}}$$

The limit is a loose morphism $f_*: A \to\!\!\!\!\shortmid\ B$

and a cell:

$$\begin{tikzcd} A \arrow[r, "\shortmid" marking, "f_*"] \arrow[d, "f"'] \arrow[dr, phantom, "\pi"] & B \arrow[d, "1_B"] \\ B \arrow[r, "\shortmid" marking, "\mathrm{id}"'] & B \end{tikzcd} \quad + \text{U.P.}$$

<u>Example</u>: $\mathbb{R}(\mathcal{C})$ admits all companions since each total morphism is a morphism.

In $\mathbb{P}\mathrm{ar}$, the companion of $f: A \to B$ is

$$A \xleftarrow{1_A} A \xrightarrow{f} B$$

A conjoint is a limit of the alteration:

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\shortmid\shortmid" marking] \arrow[d, "B"'] & \mathbf{2}_t \arrow[d, "f"] \\ \mathbb{D} \arrow[r, "\shortmid\shortmid" marking, "\mathrm{Hom}"'] & \mathbb{D} \end{tikzcd} \rightsquigarrow \boxed{\begin{tikzcd} & A \arrow[d, "f"'] \\ B \arrow[r, "\shortmid" marking, "\mathrm{id}"'] & B \end{tikzcd}}$$

The limit is a loose morphism $f^*: B \to\!\!\!\!\shortmid\ A$

and a cell:

$$\begin{tikzcd} B \arrow[r, "\shortmid" marking, "f^*"] \arrow[d, "1_B"'] \arrow[dr, phantom, "\pi"] & A \arrow[d, "f"] \\ B \arrow[r, "\shortmid" marking, "\mathrm{id}"'] & B \end{tikzcd} \quad + \text{U.P.}$$

<u>Example</u>: In $\mathbb{R}(\mathcal{C})$, the conjoint of a total morphism $f: A \to B$ is a morphism $f^*: B \to A$ such that $f \circ f^* = \overline{f^*}$ and $f^* \circ f = 1_A$.

In $\mathbb{P}\mathrm{ar}$, these are the injective functions.

PARALLEL LIMITS

A parallel product is a limit of the alteration:

$$\begin{tikzcd} \mathbb{1}+\mathbb{1} \arrow[r, "\mathrm{Hom}", "\shortmid"{marking}] \arrow[d, "\{A,C\}"'] & \mathbb{1}+\mathbb{1} \arrow[d, "\{B,D\}"] \\ \mathbb{D} \arrow[r, "\mathrm{Hom}"', "\shortmid"{marking}] & \mathbb{D} \end{tikzcd} \rightsquigarrow \boxed{\begin{tikzcd} A \arrow[r, "p"', "\shortmid"{marking}] & B \\ C \arrow[r, "q", "\shortmid"{marking}] & D \end{tikzcd}}$$

The limit is a loose morphism $p \times q : A \times C \to B \times D$

together with projection cells

$$\begin{tikzcd} A \times C \arrow[r, "p \times q", "\shortmid"{marking}] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & B \times D \arrow[d, "\pi_B"] \\ A \arrow[r, "p"', "\shortmid"{marking}] & B \end{tikzcd} \qquad \begin{tikzcd} A \times C \arrow[r, "p \times q", "\shortmid"{marking}] \arrow[d, "\pi_C"'] \arrow[dr, phantom, "\pi_q"] & B \times D \arrow[d, "\pi_D"] \\ C \arrow[r, "q"', "\shortmid"{marking}] & D \end{tikzcd}$$

satisfying an appropriate universal property.

A parallel terminal object is a limit of the unique alteration from $\emptyset \xrightarrow{\mathrm{Hom}} \emptyset$.

Prop: $\mathbb{R}(\mathcal{C})$ admits finite parallel products and a parallel terminal object if and only if $\mathcal{C}$ is a cartesian restriction category.

In this case, $\mathbb{R}(\mathcal{C})$ is a cartesian monoidal double category.

In general, a tight parallel limit is a limit whose shape is $\mathbb{T}\mathrm{i}(J) \xrightarrow{\mathrm{Hom}} \mathbb{T}\mathrm{i}(J)$.

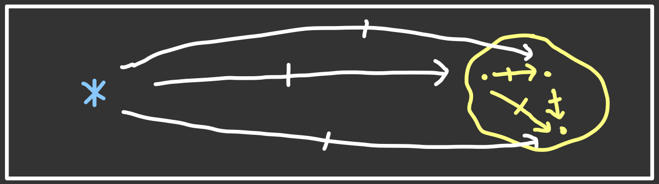
Example: $\mathbb{P}\mathrm{ar}$ admits all tight parallel limits.

RESTRICTION LIMITS AS DOUBLE-CATEGORICAL LIMITS

$\mathcal{C}$ category $\rightsquigarrow$ $\mathbb{L}o(\mathcal{C})$ double category whose:

- objects and loose morphisms determined by $\mathcal{C}$;
- tight morphisms and cells are identities.

There is a loose distributor $\mathbb{1} \xrightarrow{!^*} \mathbb{L}o(J)$



A unique loose heteromorphism for each object in $\mathcal{C}$ + a unique heterocell for each loose morphism in $\mathcal{C}$.

$$\begin{tikzcd} \mathbb{1} \arrow[r, "!^*"] \arrow[d, "X"'] \arrow[dr, phantom, "\Phi"] & \mathbb{L}o(J) \arrow[d, "F"] \\ \mathbb{R}(\mathcal{C}) \arrow[r, "\mathrm{Hom}"'] & \mathbb{R}(\mathcal{C}) \end{tikzcd}$$

F strict,

Φ normal colax

An alteration Φ is precisely a lax cone over a functor $F : J \longrightarrow \mathcal{C}$.

$$\begin{tikzcd} X \arrow[rr, "\Phi_B"] \arrow[d, "1_X"'] & \arrow[d, phantom, "\geqslant"] & FB \arrow[d, "1_{FB}"] \\ X \arrow[r, "\Phi_A"'] & FA \arrow[r, "Ff"'] & FB \end{tikzcd}$$

$\Phi_B = Ff \circ \Phi_A \circ \overline{\Phi_B}$

and its limit is a (loose) morphism $X \xrightarrow{u} \lim F$ together with cells

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "1_X"'] \arrow[dr, phantom, "\geqslant"] & \lim F \arrow[d, "\pi_A"] \\ X \arrow[r, "\Phi_A"'] & FA \end{tikzcd}$$

$\pi_A \circ u = \Phi_A \circ \overline{u}$

total components

satisfying a U.P. $\rightsquigarrow$ a restriction limit!

SUMMARY & FUTURE WORK

- Both restriction categories and double categories are useful structures in which to study sets & partial functions.

- We showed that restriction limits are the same as certain double-categorical limits indexed by loose distributors.

$$\begin{tikzcd} \mathbb{1} \arrow[r, "!^*", "\shortmid"{marking}] \arrow[d, "X"'] \arrow[dr, phantom, "\Phi"] & \mathbb{L}o(\mathcal{C}) \arrow[d, "F"] \\ \mathbb{R}(\mathcal{C}) \arrow[r, "\mathrm{Hom}"', "\shortmid"{marking}] & \mathbb{R}(\mathcal{C}) \end{tikzcd}$$

Many avenues for future work:

- Examine local (co)products and the relationship with join/meet restriction categories (Guo).

- Show that restriction profunctors (DeWolf, Lin) induce loose distributors.

- Characterise which double categories arise from restriction categories.

- Upcoming project in Adjoint School 2026 with Nathanael Arkor.