COMPANIONS & CONJOINTS ARE (CO)LIMITS

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MOTIVATION & OVERVIEW

A double category $\mathbb{D}$ consists of:

- objects $A, B, C, \ldots$
- tight morphisms $A \longrightarrow B$
- loose morphisms $A \nrightarrow B$
- cells/squares

$\left.\begin{matrix}\text{objects} \\ \text{tight morphisms}\end{matrix}\right\} \mathbb{D}_0$

$\left.\begin{matrix}\text{loose morphisms} \\ \text{cells/squares}\end{matrix}\right\} \mathbb{D}_1$

$$\begin{tikzcd} A \arrow[r, "p", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "|" marking] & D \end{tikzcd}$$

A double category is a pseudo category object in CAT:

$$\begin{tikzcd} \mathbb{D}_0 \arrow[r, "\mathrm{id}" description] & \mathbb{D}_1 \arrow[l, bend right, "\mathrm{dom}"'] \arrow[l, bend left, "\mathrm{cod}"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[l, "\odot"'] \end{tikzcd}$$

Companions and conjoints determine loose morphisms from tight morphisms.

$$f : A \longrightarrow B \quad \rightsquigarrow \quad \begin{matrix} f_* : A \nrightarrow B \\ f^* : B \nrightarrow A \end{matrix}$$

Many double categories admit all companions and conjoints; these might be called:

- fibrant double categories
- framed bicategories
- (proarrow) equipments

GOAL: Show companions/conjoints are (co)limits!

ANALOGY: SPLITTING OF IDEMPOTENTS

The splitting of an idempotent $e:A\to A$ is an object B and morphisms $r:A\to B$ and $s:B\to A$ such that $rs=1_B$ and $sr=e$.

Consider an idempotent $e:A\to A$ and a diagram

$$\begin{tikzcd} & B \arrow[dl, "s"'] \arrow[dr, "s"] \arrow[d, phantom, "\circlearrowright"] & \\ A \arrow[rr, "e"'] & & A \end{tikzcd} \qquad (1)$$

such that pasting with (1) gives a bijection:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f"] \arrow[d, phantom, "\circlearrowright"] & \\ A \arrow[rr, "e"'] & & A \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} X \arrow[d, "g"] \\ B \end{tikzcd}$$

Consider an idempotent $e:A\to A$ and a diagram

$$\begin{tikzcd} A \arrow[rr, "e"] \arrow[dr, "r"'] & \arrow[d, phantom, "\circlearrowright"] & A \arrow[dl, "r"] \\ & B & \end{tikzcd} \qquad (2)$$

such that pasting with (2) gives a bijection:

$$\begin{tikzcd} A \arrow[rr, "e"] \arrow[dr, "h"'] & \arrow[d, phantom, "\circlearrowright"] & A \arrow[dl, "h"] \\ & Y & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} B \arrow[d, "k"] \\ Y \end{tikzcd}$$

Exercise: Show that an idempotent $e:A\to A$ splits <u>iff</u> there is a diagram (1) with the above U.P. <u>iff</u> there is a diagram (2) with the above U.P.

COMPANIONS & CONJOINTS VIA STRUCTURE

A companion of a tight morphism $f : A \longrightarrow B$ is a loose morphism $f_* : A \to\!\!\!\!\!\shortmid\; B$ and cells

$$
\begin{tikzcd}
A \arrow[r, "f_*", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\rho"] & B \arrow[d, "1_B"] \\
B \arrow[r, "\mathrm{id}_B"', "\shortmid"{marking}] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A", "\shortmid"{marking}] \arrow[d, "1_A"'] \arrow[dr, phantom, "\sigma"] & A \arrow[d, "f"] \\
A \arrow[r, "f_*"', "\shortmid"{marking}] & B
\end{tikzcd}
$$

such that the following equations hold.

$$
\begin{array}{|c|}\hline \sigma \\ \hline \rho \\ \hline \end{array} = \boxed{\mathrm{id}_f}
\qquad
\begin{array}{|c|c|}\hline \sigma & \rho \\ \hline \end{array} = \boxed{1_{f_*}}
$$

A conjoint of a tight morphism $f : A \longrightarrow B$ is a loose morphism $f^* : B \to\!\!\!\!\!\shortmid\; A$ and cells

$$
\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\varepsilon"] & A \arrow[d, "1_A"] \\
B \arrow[r, "f^*"', "\shortmid"{marking}] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "f^*", "\shortmid"{marking}] \arrow[d, "1_B"'] \arrow[dr, phantom, "\eta"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathrm{id}_B"', "\shortmid"{marking}] & B
\end{tikzcd}
$$

such that the following equations hold.

$$
\begin{array}{|c|}\hline \varepsilon \\ \hline \eta \\ \hline \end{array} = \boxed{\mathrm{id}_f}
\qquad
\begin{array}{|c|c|}\hline \eta & \varepsilon \\ \hline \end{array} = \boxed{1_{f^*}}
$$

EXAMPLES OF CONJOINTS

- In $\mathbb{S}\mathrm{pan}(\mathcal{E})$, the conjoint of $f:A\to B$ is the span $B \xleftarrow{f} A \xrightarrow{1_A} A$ with cells:

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "1_A"'] \arrow[r, "1_A"] \arrow[d, "1_A"] & A \arrow[d, "1_A"] \\
B & A \arrow[l, "f"] \arrow[r, "1_A"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[d, "1_B"'] & A \arrow[l, "f"'] \arrow[r, "1_A"] \arrow[d, "f"] & A \arrow[d, "f"] \\
B & B \arrow[l, "1_B"] \arrow[r, "1_B"'] & B
\end{tikzcd}
$$

- In $\mathbb{R}\mathrm{ing}$, the conjoint of $f:R\to S$ is the (S,R)-bimodule S with left action $s \triangleright s' = s\,s'$ and right action $s \triangleleft r = s\cdot f(r)$ and bimodule maps $1_S : S \to S$ to the trivial (S,S)-bimodule and $f: R \to S$ from the trivial (R,R)-bimodule.

- In $\mathbb{D}\mathrm{ist}$, the conjoint of a functor $f:A\to B$ is $B(-,f-):B^{op}\times A \longrightarrow \mathrm{Set}$ with cells:

$$
\begin{tikzcd}
B^{op}\times A \arrow[dd, "1\times f"'] \arrow[dr, bend left, "{B(-,f-)}"] & \\
\arrow[r, phantom, "\circlearrowright"] & \mathrm{Set} \\
B^{op}\times B \arrow[ur, bend right, "{B(-,-)}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A^{op}\times A \arrow[dd, "f\times 1"'] \arrow[dr, bend left, "{A(-,-)}"] & \\
\arrow[r, phantom, "\Downarrow"] & \mathrm{Set} \\
B^{op}\times A \arrow[ur, bend right, "{B(-,f-)}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A(a,a') \arrow[d, "f_{a,a'}"] \\
B(fa,fa')
\end{tikzcd}
$$

- In $\mathbb{Q}(\mathcal{K})$, the conjoint of $r:A\to B$ is a 1-cell $\ell : B \to A$ and 2-cells:

$$
\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "r"'] & A \arrow[d, "1_A"] \\
B \arrow[r, "\ell"'] \arrow[ur, phantom, "\varepsilon\,\Nearrow"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\ell"] \arrow[d, "1_B"'] & A \arrow[d, "r"] \\
B \arrow[r, "1_B"'] \arrow[ur, phantom, "\eta\,\Nearrow"] & B
\end{tikzcd}
$$

CONJOINTS VIA UNIVERSAL PROPERTY

Consider a tight morphism $f : A \to B$ and a cell

$$\begin{tikzcd} B \arrow[r, "f^*", "|" marking] \arrow[d, "1_B"'] \arrow[dr, phantom, "\eta"] & A \arrow[d, "f"] \\ B \arrow[r, "\mathrm{id}_B"', "|" marking] & B \end{tikzcd}$$

such that composing with η gives a bijection:

$$\begin{tikzcd} X \arrow[r, "p", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, "fh"] \\ B \arrow[r, "\mathrm{id}_B"', "|" marking] & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} X \arrow[r, "p", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\bar{\alpha}"] & Y \arrow[d, "h"] \\ B \arrow[r, "f^*"', "|" marking] & A \end{tikzcd}$$

Exercise: Show that $f : A \to B$ admits a conjoint <u>iff</u> there exists η with universal property above.

Consider a tight morphism $f : A \to B$ and a cell

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\varepsilon"] & A \arrow[d, "1_A"] \\ B \arrow[r, "f^*"', "|" marking] & A \end{tikzcd}$$

such that composing with ε gives a bijection:

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A", "|" marking] \arrow[d, "kf"'] \arrow[dr, phantom, "\beta"] & A \arrow[d, "j"] \\ X \arrow[r, "p"', "|" marking] & Y \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} B \arrow[r, "f^*", "|" marking] \arrow[d, "k"'] \arrow[dr, phantom, "\bar{\beta}"] & A \arrow[d, "j"] \\ X \arrow[r, "p"', "|" marking] & Y \end{tikzcd}$$

Exercise: Show that $f : A \to B$ admits a conjoint <u>iff</u> there exists ε with universal property above.

SHAPES, DIAGRAMS, & CONES

shape ⇝ span $S \xleftarrow{s} J \xrightarrow{t} T$ in Cat

$p \in J \iff p : A \nrightarrow B$ s.t. $s(p)=A$, $t(p)=B$

diagram ⇝ commutative diagram

$$\begin{tikzcd}
S \arrow[d, "F"'] & J \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "H"] & T \arrow[d, "G"] \\
\mathbb{D}_0 & \mathbb{D}_1 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & \mathbb{D}_0
\end{tikzcd} \qquad (*)$$

double category $\mathbb{D}$

This determines a functorial assignment:

$$\begin{tikzcd}
A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\
A' \arrow[r, "p'"', "\shortmid"{marking}] & B'
\end{tikzcd} \in J \longmapsto
\begin{tikzcd}
FA \arrow[r, "Hp", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "H\alpha"] & GB \arrow[d, "Gg"] \\
FA' \arrow[r, "Hp'"', "\shortmid"{marking}] & GB'
\end{tikzcd} \in \mathbb{D}$$

cone over (F,H,G) ⇝ loose morphism $q : X \nrightarrow Y$ and

$$\begin{tikzcd}
 & S \arrow[d, "F"] & & J \arrow[ll, "s"'] \arrow[rr, "t"] \arrow[d, "H"] & & T \arrow[d, "G"] \\
1 \arrow[ur, phantom, "\swarrow\varphi"] \arrow[r, phantom, "\Rightarrow"] \arrow[dr, phantom, "\searrow X"] & \mathbb{D}_0 & 1 \arrow[ur, phantom, "\swarrow\theta"] \arrow[r, phantom, "\Rightarrow"] \arrow[dr, phantom, "\searrow q"] & \mathbb{D}_1 \arrow[l, "\mathrm{dom}"] \arrow[rr, "\mathrm{cod}"'] & 1 \arrow[ur, phantom, "\swarrow\psi"] \arrow[r, phantom, "\Rightarrow"] \arrow[dr, phantom, "\searrow Y"] & \mathbb{D}_0
\end{tikzcd}$$

$$\boxed{\begin{aligned} \varphi \cdot s &= \mathrm{dom} \cdot \theta \\ \psi \cdot t &= \mathrm{cod} \cdot \theta \end{aligned}}$$

- a tight morphism $\varphi_A : X \to FA$ for each $A \in S$;
- a tight morphism $\psi_B : Y \to GB$ for each $B \in T$;
- for each $p : A \nrightarrow B$ in J, a cell θ_p as below;

$$\begin{tikzcd}
X \arrow[r, "q", "\shortmid"{marking}] \arrow[d, "\varphi_A"'] \arrow[dr, phantom, "\theta_p"] & Y \arrow[d, "\psi_B"] \\
FA \arrow[r, "Hp"', "\shortmid"{marking}] & GB
\end{tikzcd}$$

- φ, ψ, θ are natural with respect to morphisms in S, T, J.

LIMITS INDEXED BY SPANS OF FUNCTORS

A **limit** of a diagram

$$\begin{tikzcd}
S \arrow[d, "F"'] & J \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "H"] & T \arrow[d, "G"] \\
\mathbb{D}_0 & \mathbb{D}_1 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & \mathbb{D}_0
\end{tikzcd}$$

in a double category $\mathbb{D}$ is a loose morphism $\lim : \lim F \nrightarrow \lim G$, where $(\lim F, \varphi)$ is a limit of F and $(\lim G, \psi)$ is a limit of G, and a natural family of cells θ_p indexed by $p \in J$, with the following U.P.:

$$\begin{tikzcd}
\lim F \arrow[r, "\lim", "\shortmid"{marking}] \arrow[d, "\varphi_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[d, "\psi_B"] \\
FA \arrow[r, "Hp"', "\shortmid"{marking}] & GB
\end{tikzcd}$$

Composing with θ_p gives a bijection between cones $(\varphi', \theta', \psi')$ w/ apex $q : X \nrightarrow Y$ over (F, H, G) and cells from $q : X \nrightarrow Y$ to $\lim : \lim F \nrightarrow \lim G$.

$$\begin{tikzcd}
X \arrow[r, "q", "\shortmid"{marking}] \arrow[d, "\varphi'_A"'] \arrow[dr, phantom, "\theta'_p"] & Y \arrow[d, "\psi'_B"] \\
FA \arrow[r, "Hp"', "\shortmid"{marking}] & GB
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
X \arrow[r, "q", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, "g"] \\
\lim F \arrow[r, "\lim"', "\shortmid"{marking}] & \lim G
\end{tikzcd}$$

Summary: A limit in $\mathbb{D}$ indexed by $S \xleftarrow{s} J \xrightarrow{t} T$ is a terminal cone over a diagram (F, H, G) in the **2-category $[\{\bullet \leftarrow \bullet \rightarrow \bullet\}, \mathrm{CAT}]$** that is preserved by left, right : $[\{\bullet \leftarrow \bullet \rightarrow \bullet\}, \mathrm{CAT}] \rightrightarrows \mathrm{CAT}$.

CONJOINTS AS LIMITS & COLIMITS

diagram

$$\begin{tikzcd}
1 \arrow[r, equal] \arrow[d, "B"'] & 1 \arrow[r, "cod"] \arrow[d, "id_B"] & 2 \arrow[d, "f"] \\
\mathbb{D}_0 & \mathbb{D}_1 \arrow[l, "dom"] \arrow[r, "cod"'] & \mathbb{D}_0
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
 & A \arrow[d, "f"] \\
B \arrow[r, "id_B"', "\shortmid"{marking}] & B
\end{tikzcd}$$

limit

$$\begin{tikzcd}
B \arrow[r, "\lim(id, f) = f^*", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & A \arrow[d, "f"] \\
B \arrow[r, "id"', "\shortmid"{marking}] & B
\end{tikzcd}$$

$$\boxed{\begin{array}{l} \lim(B : 1 \to \mathbb{D}_0) = B \\ \lim(f : 2 \to \mathbb{D}_0) = A \end{array}}$$

U.P.

$$\begin{tikzcd}
X \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, "fh"] \\
B \arrow[r, "id_B"', "\shortmid"{marking}] & B
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
X \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\bar{\alpha}"] & Y \arrow[d, "h"] \\
B \arrow[r, "f^*"', "\shortmid"{marking}] & A
\end{tikzcd}$$

diagram

$$\begin{tikzcd}
2 \arrow[d, "f"'] & 1 \arrow[l, "dom"'] \arrow[r, equal] \arrow[d, "id_A"] & 1 \arrow[d, "A"] \\
\mathbb{D}_0 & \mathbb{D}_1 \arrow[l, "dom"] \arrow[r, "cod"'] & \mathbb{D}_0
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A \arrow[r, "id_A", "\shortmid"{marking}] \arrow[d, "f"'] & A \\
B &
\end{tikzcd}$$

colimit

$$\begin{tikzcd}
A \arrow[r, "id_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\varepsilon"] & A \arrow[d, equal] \\
B \arrow[r, "colim(f, id) = f^*"', "\shortmid"{marking}] & A
\end{tikzcd}$$

$$\boxed{\begin{array}{l} colim(A : 1 \to \mathbb{D}_0) = A \\ colim(f : 2 \to \mathbb{D}_0) = B \end{array}}$$

U.P.

$$\begin{tikzcd}
A \arrow[r, "id_A", "\shortmid"{marking}] \arrow[d, "fk"'] \arrow[dr, phantom, "\beta"] & A \arrow[d, "j"] \\
X \arrow[r, "p"', "\shortmid"{marking}] & Y
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
B \arrow[r, "f^*", "\shortmid"{marking}] \arrow[d, "k"'] \arrow[dr, phantom, "\bar{\beta}"] & A \arrow[d, "j"] \\
X \arrow[r, "p"', "\shortmid"{marking}] & Y
\end{tikzcd}$$

PARALLEL LIMITS & MAIN THEOREM

$$\begin{tikzcd} J \arrow[r, equal] \arrow[d] & J \arrow[r, equal] \arrow[d, "H"] & J \arrow[d] \\ \mathbb{D}_0 & \mathbb{D}_1 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & \mathbb{D}_0 \end{tikzcd}$$

← shape for a parallel limit

- A parallel (co)limit is precisely a limit in $\mathbb{D}_1$ that is preserved by $\mathrm{dom},\mathrm{cod} : \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$.

- E.g. a parallel product of $p : A \nrightarrow B$ and $q : C \nrightarrow D$ is a loose morphism $p \times q$ and cells

$$\begin{tikzcd} A \times C \arrow[r, "p\times q", "|" marking] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & B \times D \arrow[d, "\pi_B"] \\ A \arrow[r, "p"', "|" marking] & B \end{tikzcd} \qquad \begin{tikzcd} A \times C \arrow[r, "p\times q", "|" marking] \arrow[d, "\pi_C"'] \arrow[dr, phantom, "\pi_q"] & B \times D \arrow[d, "\pi_D"] \\ C \arrow[r, "q"', "|" marking] & D \end{tikzcd}$$

with suitable U.P..

Theorem: A double category $\mathbb{D}$ admits all (co)limits indexed by spans of functors <u>iff</u> $\mathbb{D}$ admits:

- companions
- conjoints

$\Big\}$ (co)restrictions

- parallel (co)products
- parallel (co)equalisers

$\Big\}$ parallel (co)limits

<u>iff</u>

- $\langle \mathrm{dom}, \mathrm{cod} \rangle : \mathbb{D}_1 \longrightarrow \mathbb{D}_0 \times \mathbb{D}_0$ is a bifibration.
- $\mathbb{D}_1$ admits all (co)limits and these are preserved by $\mathrm{dom},\mathrm{cod} : \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$.

SUMMARY & FURTHER WORK

- We introduced (co)limits in double categories indexed by spans of functors, and showed companions and conjoints are (co)limits.
- Companions and conjoints are preserved by any normal (co)lax double functor — they are absolute (co)limits.
- Conjecture: A double category admits all absolute colimits <u>iff</u> it has companions, conjoints, and (parallel) splitting of idempotents.

This talk presented a small part of a larger story on limits in double categories.

spans in $\mathcal{CAT}$	$S \longleftarrow J \longrightarrow T$

generalise shapes

loose distributors between double cats.	$\mathbb{S} \xrightarrow{\mathbb{J}} \mathbb{T}$

span $S_0 \leftarrow J \rightarrow T_0$ with compatible left action of $S_1 \rightrightarrows S_0$ and right action of $T_1 \rightrightarrows T_0$