A SECOND LOOK AT LIMITS IN DOUBLE CATEGORIES

BRYCE CLARKE

Tallinn University of Technology, Estonia

bryceclarke.github.io

Joint work with Nathanael Arkor (arkor.co)

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MOTIVATION & OVERVIEW

- Double categories are a 2-dimensional structure consisting of objects, tight & loose morphisms, and cells.
- The prototypical example is $\mathbb{R}\mathrm{el}$, the double category of sets, functions, and relations.
- Monoidal categories, bicategories and 2-categories also examples.
- Marco Grandis & Bob Paré introduced limits in double categories indexed by double categories — these are objects.
- However, these were not expressive enough! They also needed limits of "vertical transformations" – these are loose morphisms.
- However, there are many examples of loose morphisms with universal properties which are not captured!

For example:

- Given relations $R \rightarrowtail A \times B$ and $S \rightarrowtail A \times B$, we may take their intersection to get $R \cap S \rightarrowtail A \times B$
- Given a function $f : A \to B$, we may construct relations $\langle 1_A, f \rangle : A \rightarrowtail A \times B$ and $\langle f, 1_A \rangle : A \rightarrowtail A \times B$
- The coproduct in Set is a biproduct in Rel – why?

OUTLINE: Cover four kinds of "shape" for limits in double categories, with examples.

limit type → / dimension ↓	object	loose morphism
1	category $\mathcal{I}$	span of functors $S \leftarrow \mathcal{I} \rightarrow T$
2	double category $\mathbb{I}$	loose distributor $\mathbb{I}_s \overset{\mathbb{I}}{\,\mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}}\,} \mathbb{I}_t$

DOUBLE CATEGORIES UNPACKED

A double category $\mathbb{D}$ consists of:

- objects $A, B, C, \ldots$
- tight morphisms $A \longrightarrow B$

(objects and tight morphisms: category D_0)

- loose morphisms $A \mathrel{\mapstochar\!\!\longrightarrow} B$
- cells

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "\shortmid"{marking}] & D \end{tikzcd}$$

(loose morphisms and cells: category D_1)

- identity morphisms $1_A : A \longrightarrow A$ and $\mathrm{id}_A : A \mathrel{\mapstochar\!\!\longrightarrow} A$
- composite morphisms $g \circ f : A \longrightarrow C$ and $p \odot q : A \mathrel{\mapstochar\!\!\longrightarrow} C$
- identity cells

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "1_A"'] \arrow[dr, phantom, "1_p"] & B \arrow[d, "1_B"] \\ A \arrow[r, "p"', "\shortmid"{marking}] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{id}_f"] & A \arrow[d, "f"] \\ B \arrow[r, "\mathrm{id}_A"', "\shortmid"{marking}] & B \end{tikzcd}$$

- composite cells

$$\begin{tikzcd} \cdot \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\beta"] & \cdot \arrow[d, "k"] \\ \cdot \arrow[r, "p''"', "\shortmid"{marking}] & \cdot \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \cdot \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "g \circ f"'] \arrow[dr, phantom, "\beta \circ \alpha"] & \cdot \arrow[d, "k \circ h"] \\ \cdot \arrow[r, "p''"', "\shortmid"{marking}] & \cdot \end{tikzcd}$$

$$\begin{tikzcd} \cdot \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[r, "q", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\beta"] & \cdot \arrow[d, "f''"] \\ \cdot \arrow[r, "r"', "\shortmid"{marking}] & \cdot \arrow[r, "s"', "\shortmid"{marking}] & \cdot \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \cdot \arrow[r, "p \odot q", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha \odot \beta"] & \cdot \arrow[d, "f''"] \\ \cdot \arrow[r, "r \odot s"', "\shortmid"{marking}] & \cdot \end{tikzcd}$$

- left unitor, right unitor, & associator cells

$$\begin{tikzcd} \cdot \arrow[r, "\mathrm{id} \odot p", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\underline{\ell}(p)"] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, "p"', "\shortmid"{marking}] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "p \odot \mathrm{id}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\underline{r}(p)"] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, "p"', "\shortmid"{marking}] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "p \odot (q \odot r)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\underline{a}(p,q,r)"] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, "(p \odot q) \odot r"', "\shortmid"{marking}] & \cdot \end{tikzcd}$$

+ axioms and coherence conditions.

DOUBLE CATEGORIES

A double category is a pseudo category object in the 2-category CAT of locally small categories.

$$\begin{tikzcd}[column sep=large] \mathbb{D} := D_0 \arrow[r, "i" description] & D_1 \arrow[l, shift right=3, "s"'] \arrow[l, shift left=3, "t"] & D_1 \times_{D_0} D_1 = D_2 \arrow[l, "m" description] \arrow[l, shift right=3, "\pi_1"'] \arrow[l, shift left=3, "\pi_2"] \end{tikzcd}$$

Therefore, we have commutative diagrams

$$\begin{tikzcd} & D_0 \arrow[dl, "1"'] \arrow[d, "i"] \arrow[dr, "1"] & \\ D_0 & D_1 \arrow[l, "s"] \arrow[r, "t"'] & D_0 \end{tikzcd} \qquad \begin{tikzcd} D_1 \arrow[d, "s"'] & D_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "m"] & D_1 \arrow[d, "t"] \\ D_0 & D_1 \arrow[l, "s"] \arrow[r, "t"'] & D_0 \end{tikzcd}$$

and iterated pullbacks of s and t denoted

$$D_3 = D_1 \times_{D_0} D_1 \times_{D_0} D_1 \qquad D_4 = D_1 \times_{D_0} D_1 \times_{D_0} D_1 \times_{D_0} D_1$$

and invertible natural transformations ℓ, r, and a satisfying the equations $s \cdot \ell = 1 = s \cdot r$, $t \cdot \ell = 1 = t \cdot r$, $t \cdot a = 1$, and $s \cdot a = 1$

$$\begin{tikzcd}[column sep=large] D_1 \arrow[r, "\langle is, 1\rangle"] \arrow[dr, bend right, "1"'] \arrow[dr, phantom, "\ell\Downarrow"] & D_2 \arrow[d, "m"] & D_1 \arrow[l, "\langle 1, it\rangle"'] \arrow[dl, bend left] \arrow[dl, phantom, "\Downarrow r"] \\ & D_1 & \end{tikzcd} \qquad \begin{tikzcd} D_3 \arrow[r, "1\times m"] \arrow[d, "m\times 1"'] \arrow[dr, phantom, "a\Downarrow"] & D_2 \arrow[d, "m"] \\ D_2 \arrow[r, "m"'] & D_1 \end{tikzcd}$$

and satisfying two coherence conditions.

$$\begin{tikzcd} & & D_2 \arrow[dr, "m"] & \\ D_2 \arrow[r] \arrow[urr, bend left, "1"] \arrow[drr, bend right, "1"'] \arrow[ur, phantom, "1\times\ell^{-1}"] \arrow[dr, phantom, "r\times 1"'] & D_3 \arrow[ur] \arrow[dr] \arrow[rr, phantom, "a\Downarrow"] & & D_1 \\ & & D_2 \arrow[ur, "m"'] & \end{tikzcd} = \begin{tikzcd} & D_2 \arrow[dr, "m"] & \\ D_2 \arrow[ur, "1"] \arrow[dr, "1"'] \arrow[rr, phantom, "\circlearrowright"] & & D_1 \\ & D_2 \arrow[ur, "m"'] & \end{tikzcd}$$

$$\begin{tikzcd} & D_3 \arrow[r, "1\times m"] \arrow[d, phantom, "1\times a"] & D_2 \arrow[dr, "m"] & \\ D_4 \arrow[ur, "1\times 1\times m"] \arrow[r] \arrow[dr, "m\times 1\times 1"'] & D_3 \arrow[ur] \arrow[dr] \arrow[d, phantom, "a\times 1"] \arrow[rr, phantom, "a\Downarrow"] & & D_1 \\ & D_3 \arrow[r, "m\times 1"'] & D_2 \arrow[ur, "m"'] & \end{tikzcd} = \begin{tikzcd} & D_3 \arrow[r, "1\times m"] \arrow[dr] & D_2 \arrow[dr, "m"] \arrow[d, phantom, "a"] & \\ D_4 \arrow[ur, "1\times 1\times m"] \arrow[dr, "m\times 1\times 1"'] \arrow[rr, phantom, "\circlearrowright"] & & D_2 \arrow[r] & D_1 \\ & D_3 \arrow[r, "m\times 1"'] \arrow[ur] & D_2 \arrow[ur, "m"'] \arrow[u, phantom, "a"] & \end{tikzcd}$$

SETS, FUNCTIONS, & SPANS

Let $\mathbb{S}\mathrm{pan}$ be the double category whose:

- objects are sets
- tight morphisms are functions
- loose morphisms $(p_1, X, p_2) : A \to\!\!\!\!\!\!+\;\, B$ are spans of functions

$$A \xleftarrow{p_1} X \xrightarrow{p_2} B$$

- cells denoted

$$\begin{tikzcd} A \arrow[r, "{(p_1, X, p_2)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "{(q_1, Y, q_2)}"', "\shortmid"{marking}] & D \end{tikzcd}$$

are commutative diagrams of functions

$$\begin{tikzcd} A \arrow[d, "f"'] & X \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\alpha"] & B \arrow[d, "g"] \\ C & Y \arrow[l, "q_1"] \arrow[r, "q_2"'] & D \end{tikzcd}$$

- the loose identity morphism on A is

$$A \xleftarrow{1_A} A \xrightarrow{1_A} A$$

- the composite of loose morphisms $(p_1, X, p_2) : A \to\!\!\!\!\!\!+\;\, B$ and $(q_1, Y, q_2) : B \to\!\!\!\!\!\!+\;\, C$ is via pullback.

$$\begin{tikzcd} & & X \times_B Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right, "p_1 \pi_X"'] \arrow[ddrr, bend left, "q_2 \pi_Y"] & & \\ & X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & Y \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ A & & B & & C \end{tikzcd}$$

- the unitors and associator are induced by the universal property of the pullback

$$\begin{tikzcd} \mathrm{Set} \arrow[r, "\mathrm{id}"] & \mathrm{Set}^{\{\bullet \leftarrow \bullet \rightarrow \bullet\}} \arrow[l, shift right=2, "\mathrm{dom}"'] \arrow[l, shift left=2, "\mathrm{cod}"] & \mathrm{Set}^{\{\bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet\}} \arrow[l, shift right=2, "\pi_1"'] \arrow[l, "\odot"] \arrow[l, shift left=2, "\pi_2"] \end{tikzcd}$$

SMALL CATEGORIES, FUNCTORS, & DISTRIBUTORS

Let $\mathbb{D}\mathrm{ist}$ be the double category whose:

- objects are small categories
- tight morphisms are functors
- loose morphisms $p : A \nrightarrow B$ are distributors, i.e functors

$$A^{op}\times B \xrightarrow{p} \mathrm{Set}$$

- cells denoted

$$\begin{tikzcd} A \arrow[r, "p", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "|"{marking}] & D \end{tikzcd}$$

are natural transformations

$$\begin{tikzcd} A^{op}\times B \arrow[drr, bend left, "p"] \arrow[d, "f\times g"'] \arrow[rr, phantom, shift right=6, "\Downarrow\alpha"] & & \\ C^{op}\times D \arrow[rr, bend right, "q"'] & & \mathrm{Set} \end{tikzcd}$$

- the loose identity morphism on A is

$$\mathrm{Hom} := A(-,-) : A^{op}\times A \longrightarrow \mathrm{Set}$$

- the composite of loose morphisms $p : A \nrightarrow B$ and $q : B \nrightarrow C$ is via coend:

$$A^{op}\times C \xrightarrow{p\odot q} \mathrm{Set}$$

$$(a,c) \longmapsto \int^{b\in B} p(a,b)\times q(b,c)$$

where the set is defined by the equivalence relation

$$\begin{array}{c} a \xrightarrow{u} b \xrightarrow{v} c \\ \sim \\ a \xrightarrow{u'} b' \xrightarrow{v'} c \end{array} \iff \begin{array}{c} \exists\, w : b \to b' \text{ such that} \\ p(1_a,w)(u) = u' \;\;\&\;\; q(w,1_c)(v') = v \end{array}$$

- the unitors and associator are induced by the coend

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, "\mathrm{id}" description] & \mathrm{Cat}/2 \arrow[l, shift right=3, "\mathrm{dom}"'] \arrow[l, shift left=3, "\mathrm{cod}"] & \mathrm{Cat}/3 \arrow[l, "\odot" description] \arrow[l, shift right=3, "\pi_1"'] \arrow[l, shift left=3, "\pi_2"] \end{tikzcd}$$

LIMITS INDEXED BY CATEGORIES

A limit in a double category $\mathbb{D}$ indexed by a (small) category is a (small) limit in the underlying category D_0 of objects and tight morphisms in $\mathbb{D}$.

- shape ⇝ small category I
- diagram ⇝ functor $F : I \longrightarrow D_0$
- cone ⇝ an object $X \in D_0$ & natural transformation

$$\begin{tikzcd} I \arrow[r, "!"] \arrow[d, "F"'] & 1 \arrow[dl, bend left, "X"] \arrow[dl, phantom, "\overset{\varphi}{\Leftarrow}"] \\ D_0 & \end{tikzcd}$$

- morphism of cones $(X,\gamma) \longrightarrow (Y,\psi)$ ⇝ a tight morphism $f : X \longrightarrow Y$ such that $\psi_A \circ f = \varphi_A$ for all $A \in I$.
- limit ⇝ terminal object $(\lim F, \pi)$ in the category of cones $\mathrm{Cone}(F)$ over F;
- U.P. ⇝ for all cones (X,φ) over F there exists a unique tight morphism $u : X \longrightarrow \lim F$ such that

$$\begin{tikzcd} & X \arrow[dl, "\varphi_A"'] \arrow[dr, "\varphi_B"] & \\ FA \arrow[rr, "Ff"'] & & FB \end{tikzcd} = \begin{tikzcd} & X \arrow[d, "u"] \arrow[ddl, bend right, "\varphi_A"'] \arrow[ddr, bend left, "\varphi_B"] & \\ & \lim F \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ FA \arrow[rr, "Ff"'] & & FB \end{tikzcd}$$

Limits in double categories indexed by categories are not very interesting, as the loose morphisms and cells play no role. Several ways to fix:

1. Index by double categories (Grandis-Paré)
2. Index by spans in Cat
3. Both of the above: index by loose distributors

A 2-CATEGORY OF SPANS OF FUNCTORS

Let $\mathrm{CAT}^{\{\cdot\swarrow\cdot\searrow\cdot\}}$ be the 2-category whose:

- objects are spans of functors in CAT
- 1-cells are morphisms of spans (a triple of functors...)
- 2-cells $(\alpha, \beta, \gamma) : (f_1, g_1, h_1) \longrightarrow (f_2, g_2, h_2)$ are diagrams

$$
\begin{tikzcd}
\bullet \arrow[d, bend right=40, "f_1"'] \arrow[d, bend left=40, "f_2"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & \bullet \arrow[l, "p"'] \arrow[r, "p'"] \arrow[d, bend right=40, "g_1"'] \arrow[d, bend left=40, "g_2"] \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] & \bullet \arrow[d, bend right=40, "h_1"'] \arrow[d, bend left=40, "h_2"] \arrow[d, phantom, "\overset{\gamma}{\Rightarrow}"] \\
\bullet & \bullet \arrow[l, "q"] \arrow[r, "q'"'] & \bullet
\end{tikzcd}
$$

which "commute" in the sense that:

$$q \cdot \beta = \alpha \cdot p \qquad\qquad q' \cdot \beta = \gamma \cdot p'$$

There are 2-functors

$$
\begin{tikzcd}
\mathrm{CAT} \arrow[r, "\mathrm{id}" description] & \mathrm{CAT}^{\{\cdot\swarrow\cdot\searrow\cdot\}} \arrow[l, shift right=3, "\mathrm{dom}"'] \arrow[l, shift left=3, "\mathrm{cod}"]
\end{tikzcd}
$$

where $\mathrm{id}(C) = C \xleftarrow{1} C \xrightarrow{1} C$, $\mathrm{dom}(A \leftarrow X \rightarrow B) = A$ and $\mathrm{cod}(A \leftarrow X \rightarrow B) = B$.

The 2-category $\mathrm{CAT}^{\{\cdot\swarrow\cdot\searrow\cdot\}}$ has a 2-terminal object:

$$1 \longleftarrow 1 \longrightarrow 1$$

Thus, we can consider limits of 1-cells, that is, limits of morphisms of spans in CAT.

$$
\begin{tikzcd}
& S \arrow[dl] \arrow[dd, bend left=40, "f"] & & I \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dd, bend left=40, "g"] & & T \arrow[dl] \arrow[dd, bend left=40, "h"] \\
1 \arrow[dr, "a"'] \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] & {} & 1 \arrow[dr, "x"'] \arrow[r, phantom, "\overset{\beta}{\Rightarrow}"] & {} & 1 \arrow[dr, "b"'] \arrow[r, phantom, "\overset{\gamma}{\Rightarrow}"] & {} \\
& A & & X \arrow[ll, "p"] \arrow[rr, "q"'] & & B
\end{tikzcd}
$$

In particular, want the image of a limit cone under $\mathrm{dom}, \mathrm{cod} : \mathrm{CAT}^{\{\cdot\leftarrow\cdot\rightarrow\cdot\}} \longrightarrow \mathrm{CAT}$ to be a limit cone.

Every double category $\mathbb{D}$ has an underlying span

$$D_0 \xleftarrow{\ s\ } D_1 \xrightarrow{\ t\ } D_0$$

therefore, we can formulate limits in double categories!

LIMITS INDEXED BY SPANS OF FUNCTORS

A limit in a double category $\mathbb{D}$ indexed by a span of functors in Cat is a limit in the underlying span of functors $D_0 \xleftarrow{s} D_1 \xrightarrow{t} D_0$.

- shape ⇝ span $S \longleftarrow I \longrightarrow T$ in Cat (small categories)
- diagram ⇝ morphism of spans in CAT

$$\begin{tikzcd} S \arrow[d, "F"'] & I \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "H"] & T \arrow[d, "G"] \\ D_0 & D_1 \arrow[l, "s"] \arrow[r, "t"'] & D_0 \end{tikzcd} \qquad (*)$$

- cone ⇝

$$\begin{tikzcd} & S \arrow[dl] \arrow[dd, "F"] & & I \arrow[ll, "s"'] \arrow[rr, "t"] \arrow[dl] \arrow[dd, "H"] & & T \arrow[dl] \arrow[dd, "G"] \\ 1 \arrow[dr, "X"'] \arrow[r, phantom, "\overset{\varphi}{\Rightarrow}"] & & 1 \arrow[dr, "P"'] \arrow[r, phantom, "\overset{\theta}{\Rightarrow}"] & & 1 \arrow[dr, "Y"'] \arrow[r, phantom, "\overset{\psi}{\Rightarrow}"] & \\ & D_0 & & D_1 \arrow[ll, "s"] \arrow[rr, "t"'] & & D_0 \end{tikzcd}$$

- limit ⇝ a terminal cone such that images under $\mathrm{dom}, \mathrm{cod} : \mathrm{CAT}^{\{\cdot \leftarrow \cdot \rightarrow \cdot\}} \longrightarrow \mathrm{CAT}$ are terminal cones (limits).

To unpack, we may interpret each span in CAT as a (free double category on a) internal graph in CAT.

Given a span $S \xleftarrow{s} I \xrightarrow{t} T$ we have:

- objects, the objects of $S + T$
- tight morphisms, the morphisms of $S + T$
- loose morphisms $s(x) \overset{x}{\relbar\joinrel\mapstochar\joinrel\rightarrow} t(x)$ for each object x in I; cells are similar.

A limit of $(*)$ is a pair $(\lim F, \varphi)$ and $(\lim G, \psi)$ of limits in D_0, and a universal cone $(\lim(F,G,H), \theta)$ in D_1 ⇝ a loose morphism $\lim : \lim F \relbar\joinrel\mapstochar\joinrel\rightarrow \lim G$ and a cell in $\mathbb{D}$

$$\begin{tikzcd} \lim F \arrow[r, "\lim", "|" marking] \arrow[d, "\varphi_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[d, "\psi_B"] \\ FA \arrow[r, "H_p"', "|" marking] & GB \end{tikzcd}$$

for each $p : A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ in I, natural w.r.t. morphisms (cells) in I.

PARALLEL PRODUCTS / LIMITS

Parallel products have shape $1+1 = 1+1 = 1+1$.

The parallel product of $p_1 : A_1 \to B_1$ and $p_2 : A_2 \to B_2$ is a loose morphism $p_1 \times p_2 : A_1 \times A_2 \to B_1 \times B_2$ and projection cells

$$\begin{tikzcd} A_1 \times A_2 \arrow[r, "p_1 \times p_2", "|" marking] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\pi_i"] & B_1 \times B_2 \arrow[d, "\pi_i"] \\ A_i \arrow[r, "p_i"', "|" marking] & B_i \end{tikzcd} \qquad i = 1, 2$$

such that for every pair cells α_1 and α_2 there exists a unique cell $\langle \alpha_1, \alpha_2 \rangle$ such that $\pi_i \circ \langle \alpha_1, \alpha_2 \rangle = \alpha_i$

$$\begin{tikzcd} X \arrow[r, "q", "|" marking] \arrow[d, "f_i"'] \arrow[dr, phantom, "\alpha_i"] & Y \arrow[d, "g_i"] \\ A_i \arrow[r, "p_i"', "|" marking] & B_i \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "q", "|" marking] \arrow[d, "\langle f_1, f_2 \rangle"'] \arrow[dr, phantom, "\langle \alpha_1, \alpha_2 \rangle"] & Y \arrow[d, "\langle g_1, g_2 \rangle"] \\ A_1 \times A_2 \arrow[r, "p_1 \times p_2"', "|" marking] & B_1 \times B_2 \end{tikzcd}$$

In $\mathbb{S}\mathrm{pan}$, the parallel product of $A \leftarrow X \rightarrow B$ and $C \leftarrow Y \rightarrow D$ is $A \times C \longleftarrow X \times Y \longrightarrow B \times D$.

In $\mathbb{D}\mathrm{ist}$, the parallel product of $p : A^{op} \times B \to \mathrm{Set}$ and $q : C^{op} \times D \to \mathrm{Set}$ is given by:

$$(A \times C)^{op} \times (B \times D) \cong (A^{op} \times B) \times (C^{op} \times D) \xrightarrow{p \times q} \mathrm{Set} \times \mathrm{Set} \xrightarrow{\times} \mathrm{Set}$$

More generally, a tight parallel limit has shape $I = I = I$.

A double category admits all tight parallel limits if and only if D_0 and D_1 admit all limits and the functors $s, t : D_1 \longrightarrow D_0$ preserve them.

Parallel limits were previously treated by Grandis & Paré in "Limits in double categories" (1999) as limits of vertical transformations.

LOCAL PRODUCTS / LIMITS

Local products have shape $1 \longleftarrow 1+1 \longrightarrow 1$.

The local product of $p_1 : A \to B$ and $p_2 : A \to B$ is a loose morphism $p_1 \wedge p_2 : A \to B$ and projection cells

$$\begin{tikzcd} A \arrow[r, "p_1 \wedge p_2", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\pi_i"] & B \arrow[d, equal] \\ A \arrow[r, "p_i"', "|" marking] & B \end{tikzcd} \qquad i = 1, 2$$

such that for every pair cells α_1 and α_2 there exists a unique cell $\langle \alpha_1, \alpha_2 \rangle$ such that $\pi_i \circ \langle \alpha_1, \alpha_2 \rangle = \alpha_i$

$$\begin{tikzcd} X \arrow[r, "q", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha_i"] & Y \arrow[d, "g"] \\ A \arrow[r, "p_i"', "|" marking] & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "q", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\langle \alpha_1, \alpha_2 \rangle"] & Y \arrow[d, "g"] \\ A \arrow[r, "p_1 \wedge p_2"', "|" marking] & B \end{tikzcd}$$

In $\mathbb{S}$pan, the local product of $A \longleftarrow X \longrightarrow B$ and $A \longleftarrow Y \longrightarrow B$ is $A \longleftarrow X \times_{A \times B} Y \longrightarrow B$.

In $\mathbb{D}$ist, the local product of $p : A^{op} \times B \longrightarrow \mathrm{Set}$ and $q : A^{op} \times B \longrightarrow \mathrm{Set}$ is $A^{op} \times B \xrightarrow{\langle p, q \rangle} \mathrm{Set} \times \mathrm{Set} \xrightarrow{\times} \mathrm{Set}$.

More generally, a local limit has shape $1 \longleftarrow I \longrightarrow 1$.

Local colimits were previously treated by Paré in "Composition of modules for lax functors" (2013), however we do not require preservation by loose composition.

$$\begin{tikzcd} A \arrow[rr, "(p_1 \odot q) \vee (p_2 \odot q)", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\exists!"] & & C \arrow[d, equal] \\ A \arrow[r, "p_1 \vee p_2"', "|" marking] & B \arrow[r, "q"', "|" marking] & C \end{tikzcd}$$

Not necessarily invertible

DISCRETE LIMITS

Discrete limits are limits with shape $S \xleftarrow{s} I \xrightarrow{t} T$ where S, I, T are discrete categories.

Given families of objects $(A_x)_{x \in S}$ and $(B_y)_{y \in T}$ and a family of loose morphisms $(p_i : A_{si} \to B_{ti})_{i \in I}$, their limit is a loose morphism $\prod(x \in S) A_x \xrightarrow{\prod p_i} \prod(y \in T) B_y$ and projection cells

$$\begin{tikzcd} \prod_{x \in S} A_x \arrow[r, "\prod p_i", "|" marking] \arrow[d, "\pi_{si}"'] \arrow[dr, phantom, "\pi_i"] & \prod_{y \in T} B_y \arrow[d, "\pi_{ti}"] \\ A_{si} \arrow[r, "p_i"', "|" marking] & B_{ti} \end{tikzcd} \qquad i \in I$$

satisfying the appropriate universal property.

Both $\mathbb{S}\mathrm{pan}$ and $\mathbb{D}\mathrm{ist}$ admit discrete limits.

Discrete limits were previously treated by Paré in the talk "Coherent theories as double Lawvere theories" under the name <u>products</u>.

In 2024, Patterson treated these in more detail in the preprint "Products in double categories, revisited".

Patterson constructs a double category $\mathbb{F}\mathrm{am}(\mathbb{D})$ which is the free cocompletion of $\mathbb{D}$ under discrete colimits.

A double category $\mathbb{D}$ has all discrete colimits if the double functor

$$\mathbb{D} \longrightarrow \mathbb{F}\mathrm{am}(\mathbb{D})$$

has a (colax) left adjoint.

"BICATEGORICAL" (CO)LIMITS

Consider colimits of the shape $1 \leftarrow 1+1 = 1+1$.

A diagram is a pair $p_1 : A \looparrowright B_1$ and $p_2 : A \looparrowright B_2$, whose colimit is a loose morphism $\mathrm{colim}(p_1,p_2) : A \looparrowright B_1 + B_2$ with cells

$$\begin{tikzcd} A \arrow[r, "p_i", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\perp\!\!\perp_i"] & B_i \arrow[d, "\perp\!\!\perp_i"] \\ A \arrow[r, "\mathrm{colim}(p_1,p_2)"', "|" marking] & B_1 + B_2 \end{tikzcd} \qquad i = 1,2$$

such that for any pair of cells α_1 and α_2, there exists a unique cell $[\alpha_1,\alpha_2]$ such that $[\alpha_1,\alpha_2] \circ \perp\!\!\perp_i = \alpha_i$.

$$\begin{tikzcd} A \arrow[r, "p_i", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha_i"] & B_i \arrow[d, "g_i"] \\ C \arrow[r, "q"', "|" marking] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\mathrm{colim}(p_1,p_2)", "|" marking] \arrow[d] \arrow[dr, phantom, "{[\alpha_1,\alpha_2]}"] & B_1 + B_2 \arrow[d, "{[g_1,g_2]}"] \\ C \arrow[r, "q"', "|" marking] & D \end{tikzcd}$$

In $\mathbb{S}\mathrm{pan}$ and $\mathbb{D}\mathrm{ist}$, the cells $\perp\!\!\perp_i$ are restrictions, meaning that $\mathrm{colim}(p_1,p_2) \odot \pi_i^* \cong p_i$ (i.e. they "commute").

The universal property implies that $B_1 + B_2$ is a bicategorical product in the underlying bicategory of $\mathbb{S}\mathrm{pan}$ or $\mathbb{D}\mathrm{ist}$.

It is remarkable that the universal loose morphism into the product $\mathrm{colim}(p_1,p_2) : A \looparrowright B_1 + B_2$ itself has a universal property (of a colimit) — this is not expressible in a bicategory!

By duality, $B_1 + B_2$ is also a bicategorical coproduct in $\mathbb{S}\mathrm{pan}$ or $\mathbb{D}\mathrm{ist}$, hence products and coproducts coincide in this sense, but are far richer in the ambient double category.

RESTRICTIONS, COMPANIONS, & CONJOINTS

A restriction is a limit with the shape $2 \xleftarrow{cod} 1 \xrightarrow{cod} 2$.

A diagram is a niche as below, and its restriction is a loose morphism $\lim(f,p,g): A \mathrel{\mapstochar\rightarrow} B$ and a cell

$$\begin{tikzcd} A \arrow[r, "\lim(f,p,g)", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "res"] & B \arrow[d, "g"] \\ C \arrow[r, "p"', "|" marking] & D \end{tikzcd}$$

such that for any cell α as below, there exists a unique cell $\bar{\alpha}$ such that $res \circ \bar{\alpha} = \alpha$.

$$\begin{tikzcd} X \arrow[r, "q", "|" marking] \arrow[d, "f \circ h"'] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, "g \circ k"] \\ C \arrow[r, "p"', "|" marking] & D \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "q", "|" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\bar{\alpha}"] & Y \arrow[d, "k"] \\ A \arrow[r, "\lim(f,p,g)"', "|" marking] & B \end{tikzcd}$$

A companion in $\mathbb{D}$ is a limit of a diagram

$$\begin{tikzcd} 2 \arrow[d, "f"'] & 1 \arrow[l, "cod"'] \arrow[r, equal] \arrow[d, "id_A"] & 1 \arrow[d, "B"] \\ D_0 & D_1 \arrow[l, "s"] \arrow[r, "t"'] & D_0 \end{tikzcd} \rightsquigarrow \begin{tikzcd} & A \arrow[d, "f"] \\ & B \arrow[r, "id_B"', "|" marking] & B \end{tikzcd}$$

Conjoints are defined dually (reverse loose direction).

Shulman (2008) showed that all extensions exist if and only if all companions & conjoints exist.

In $\mathbb{S}\mathrm{pan}$, the companion and conjoint of $f: A \to B$ is $A = A \xrightarrow{f} B$ and $A \xleftarrow{f} B = B$, respectively.

In $\mathbb{D}\mathrm{ist}$, a functor $f: A \to B$ has companion $B(f-,-): A^{op} \times B \to \mathrm{Set}$ and conjoint $B(-,f-): B^{op} \times A \to \mathrm{Set}$.

Companions and conjoints are also instances of colimits.

TECHNICALITIES: NORMAL LIMITS & REPLETENESS

For a diagram $F : I \longrightarrow D_0$ admiting a limit $(\lim F, \pi)$, we obtain a span $I = I = I$ and a cone:

$$\begin{tikzcd}
 & I \arrow[dl] \arrow[dd, "F"] \arrow[rr, equal] & & I \arrow[dl] \arrow[dd, "i \circ F"] \arrow[rr, equal] & & I \arrow[dl] \arrow[dd, "F"] \\
1 \arrow[dr, "\lim F"'] \arrow[r, phantom, "\overset{\pi}{\Rightarrow}"] & {} & 1 \arrow[dr, "\mathrm{id}_{\lim F}"'] \arrow[r, phantom, "\overset{i \cdot \pi}{\Rightarrow}"] & {} & 1 \arrow[dr, "\lim F"'] \arrow[r, phantom, "\overset{\pi}{\Rightarrow}"] & {} \\
 & D_0 & & D_1 \arrow[ll, "s"] \arrow[rr, "t"'] & & D_0
\end{tikzcd}$$

If this is a limit cone in $\mathrm{CAT}^{\{\cdot \swarrow \cdot \searrow \cdot\}}$, then we say F is a normal limit.

This means the parallel limit consisting entirely of loose identity morphisms is again a loose identity morphism.

In practice, many examples of (co)limits are normal.

Normal limits $(\lim F, \pi)$ admit a two-dimensional universal property, the sense of Grandis & Paré (1999).

Limits in $\mathbb{D}$ indexed by spans of functors are unique up to isomorphism in D_1.

However, given a diagram

$$\begin{tikzcd}
S \arrow[d, "F"'] & I \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "H"] & T \arrow[d, "G"] \\
D_0 & D_1 \arrow[l, "s"] \arrow[r, "t"'] & D_0
\end{tikzcd}$$

admitting a limit $\lim : X \looparrowright Y$ and choices of limits $\lim F \cong X$ and $\lim G \cong Y$, there may not be a loose morphism $p : \lim F \looparrowright \lim G$ and an iso $p \cong \lim$ in D_1.

A double category such that $\langle s,t \rangle : D_1 \longrightarrow D_0 \times D_0$ is an isofibration is called replete, and here we may always reindex limits along tight isomorphisms. This property was called horizontal invariance by Grandis-Paré.

A CONSTRUCTION THEOREM

Theorem: A double category admits all limits indexed by spans of functors if and only if it admits parallel limits and restrictions.

Proof: Given a diagram

$$\begin{tikzcd} S \arrow[d, "F"'] & I \arrow[l, "U"'] \arrow[r, "V"] \arrow[d, "H"] & T \arrow[d, "G"] \\ D_0 & D_1 \arrow[l, "s"] \arrow[r, "t"'] & D_0 \end{tikzcd}$$

construct the parallel limit of the diagram

$$\begin{tikzcd} I \arrow[d, "FU"'] \arrow[r, equal] & I \arrow[d, "H"] \arrow[r, equal] & I \arrow[d, "GV"] \\ D_0 & D_1 \arrow[l, "s"] \arrow[r, "t"'] & D_0 \end{tikzcd}$$

to obtain a loose morphism $\lim H : \lim FU \to\!\!\!\!\!+\ \lim GV$.

Given limits $(\lim F, \varphi)$ and $(\lim G, \psi)$ we obtain cones

$$\begin{tikzcd} I \arrow[r, "\varphi . U"] \arrow[d, "FU"'] & 1 \arrow[dl, "\lim F"] \arrow[dl, phantom, "\Leftarrow", near start, shift right=3] \\ D_0 & \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "\psi . V"] \arrow[d, "GV"'] & 1 \arrow[dl, "\lim G"] \arrow[dl, phantom, "\Leftarrow", near start, shift right=3] \\ D_0 & \end{tikzcd}$$

inducing tight morphisms $\lim F \to \lim FU$ and $\lim G \to \lim GV$.

Finally, take the restriction of

$$\begin{tikzcd} \lim F \arrow[r, "+" marking] \arrow[d] \arrow[dr, phantom, "\mathrm{res}"] & \lim G \arrow[d] \\ \lim FU \arrow[r, "+" marking, "\lim H"'] & \lim GV \end{tikzcd}$$

to obtain the desired limit $\lim : \lim F \to\!\!\!\!\!+\ \lim G$.

<u>EXERCISE</u>. Check that the universal property holds.

DOUBLE FUNCTORS & TRANSFORMATIONS

A lax double functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ is an assignment

$$\begin{tikzcd} A \arrow[r, "p", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "q"', "|" marking] & D \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} FA \arrow[r, "Fp", "|" marking] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\alpha"] & FC \arrow[d, "Fg"] \\ FB \arrow[r, "Fq"', "|" marking] & FD \end{tikzcd}$$

preserving tight identities & composites, together with unitor and laxator cells (satisfying several axioms):

$$\begin{tikzcd} FA \arrow[r, "\mathrm{id}_{FA}", "|" marking] \arrow[d, "1_{FA}"'] \arrow[dr, phantom, "\eta_A"] & FA \arrow[d, "1_{FA}"] \\ FA \arrow[r, "F(\mathrm{id}_A)"', "|" marking] & FA \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "Fp", "|" marking] \arrow[d, "1_{FA}"'] & FB \arrow[r, "Fq", "|" marking] \arrow[d, phantom, "\mu_{p,q}"] & FC \arrow[d, "1_{FC}"] \\ FA \arrow[rr, "F(p \odot q)"', "|" marking] & & FC \end{tikzcd}$$

- Called normal if η_A is identity cell, pseudo if η_A and $\mu_{p,q}$ are invertible, strict if identities.
- For a colax double functor, flip η_A and $\mu_{p,q}$.

A transformation between lax double functors

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \varphi"] & \mathbb{D} \end{tikzcd}$$

consists of a family of cells

$$\begin{tikzcd} FA \arrow[r, "Fp", "|" marking] \arrow[d, "\varphi_A"'] \arrow[dr, phantom, "\varphi_p"] & FB \arrow[d, "\varphi_B"] \\ GA \arrow[r, "Gp"', "|" marking] & GB \end{tikzcd}$$

which are natural and satisfy certain coherence axioms.

Let $\mathbf{DBL}_{n\ell}$ be the 2-category of double categories, normal lax functors, and transformations.

$$\begin{tikzcd} \mathbf{CAT} \arrow[r, bend right, "\Pi_i(-)"'] \arrow[r, phantom, "\top"] & \mathbf{DBL}_{n\ell} \arrow[l, bend right, "(-)_0"'] \end{tikzcd}$$

PRESERVATION OF LIMITS

Let Gph(CAT) be the 2-category of internal graphs in CAT, and let RGph(CAT) be 2-category of internal reflexive graphs in CAT.

A lax functor $F: \mathbb{C} \longrightarrow \mathbb{D}$ is a right adjoint if its underlying morphism of internal graphs in CAT

$$\begin{tikzcd} C_0 \arrow[d, "F_0"'] & C_1 \arrow[l, "s"'] \arrow[r, "s"] \arrow[d, "F_1"] & C_0 \arrow[d, "F_0"] \\ D_0 & D_1 \arrow[l, "t"] \arrow[r, "t"'] & D_0 \end{tikzcd}$$

is a right adjoint in Gph(CAT); i.e. F_0 and F_1 admit left adjoints strictly compatible with source/target functors.

A normal lax functor $F: \mathbb{C} \longrightarrow \mathbb{D}$ is a uniform right adjoint if its underlying morphism of internal reflexive graphs in CAT is a right adjoint in RGph(CAT).

Proposition: Limits in double categories indexed by spans of functors a preserved by right adjoint (lax) double functors. Normal limits are preserved by uniform right adjoints.

Lemma: Companions and conjoints are preserved by any normal (lax/colax) double functor.

Proof: A companion of $f: A \longrightarrow B$ is a pair of cells

$$\begin{tikzcd} A \arrow[r, "f_*"] \arrow[d, "f"'] \arrow[dr, phantom, "\sigma"] & B \arrow[d, "1_B"] \\ B \arrow[r, "id_B"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "id_A"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\tau"] & A \arrow[d, "f"] \\ A \arrow[r, "f_*"'] & B \end{tikzcd}$$

such that $\tau \odot \sigma = 1_{f_*}$ and $\sigma \circ \tau = id_f$. These equations are preserved by an normal double functor.

LIMITS INDEXED BY DOUBLE CATEGORIES

- shape ⇝ small double category $\mathbb{I}$
- diagram ⇝ normal lax double functor $F : \mathbb{I} \longrightarrow \mathbb{D}$
- cone ⇝ object $X \in \mathbb{D}$ and transformation

$$\begin{tikzcd} \mathbb{I} \arrow[r] \arrow[d, "F"'] & \mathbb{1} \arrow[dl, bend left, "X"] \\ \mathbb{D} \arrow[ur, phantom, "\overset{\varphi}{\Leftarrow}"] & \end{tikzcd}$$

- morphism of cones $(X,\gamma) \longrightarrow (Y,\psi)$ ⇝ a tight morphism $f : X \longrightarrow Y$ such that $\psi \circ f = \varphi$.
- limit ⇝ terminal object $(\lim F, \pi)$ in the category of cones $\mathrm{Cone}(F)$ over F;

$$\begin{tikzcd} \lim F \arrow[r, "\mathrm{id}_{\lim F}", "|" marking] \arrow[d, "\pi_A"'] & \lim F \arrow[d, "\pi_B"] \\ FA \arrow[r, "Fp"', "|" marking] \arrow[ur, phantom, "\pi_p"] & FB \end{tikzcd}$$

A tabulator is a limit whose shape is $\mathbb{2} = \{0 \nrightarrow 1\}$

A double category $\mathbb{D}$ admits all tabulators if and only if the functor $\mathrm{id} : D_0 \longrightarrow D_1$ has a right adjoint.

In $\mathbb{S}\mathrm{pan}$, the tabulator of $A \longleftarrow X \longrightarrow B$ is X.

In $\mathbb{D}\mathrm{ist}$, the category of elements of $P : A^{op} \times B \longrightarrow \mathrm{Set}$.

A tight limit is a limit whose shape is $\mathbb{T}\mathrm{i}(\mathcal{C})$, whose objects and morphisms are from $\mathcal{C}$ and loose morphisms/cells are identities, but these are equivalent to limits of functors $\mathcal{C} \longrightarrow D_0$.

Theorem (Grandis–Paré, 1999): A double category $\mathbb{D}$ admits limits indexed by any double category $\mathbb{I}$ <u>if and only if</u> $\mathbb{D}$ admits tight limits and tabulators.

LIMITS INDEXED BY LOOSE DISTRIBUTORS

- A loose distributor $\mathbb{S} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{T}$ is a span $\mathbb{S}_0 \leftarrow \mathbb{I} \rightarrow \mathbb{T}_0$ with a compatible left action of $\mathbb{S}$ and right action of $\mathbb{T}$.

- A loose distributor is a double functor $\mathbb{I} \longrightarrow 2$ into the loose interval.

- A loose distributor is seen as a morphism $\mathbb{I}_s \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{I}_t$ between double categories as follows.

$$\begin{tikzcd}
\mathbb{I}_s \arrow[r, hook] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbb{I} \arrow[d] & \mathbb{I}_t \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, very near start, "\llcorner"] \\
\mathbb{1} \arrow[r, hook, "s"'] & 2 & \mathbb{1} \arrow[l, hook', "t"]
\end{tikzcd}$$

- We have 2-functors below, where $\mathrm{Hom}(\mathbb{D})$ is $\pi : \mathbb{D} \times 2 \rightarrow 2$.

$$\begin{tikzcd}
\mathrm{DBL}_{nl}/2 \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathrm{DBL}_{nl} \arrow[l, "\mathrm{Hom}" description]
\end{tikzcd}$$

- shape $\rightsquigarrow$ a loose distributor $\mathbb{I} \rightarrow 2$ or $\mathbb{I}_s \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{I}_t$

- diagram $\rightsquigarrow$

$$\begin{tikzcd}
\mathbb{I} \arrow[rr, "\langle F, W \rangle"] \arrow[dr, "W"'] & & \mathbb{D} \times 2 \arrow[dl, "\pi"] \\
& 2 &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{I}_s \arrow[r, "\shortmid" marking, "\mathbb{I}"] \arrow[d, "F_s"'] \arrow[dr, phantom, "F"] & \mathbb{I}_t \arrow[d, "F_t"] \\
\mathbb{D} \arrow[r, "\shortmid" marking, "\mathrm{Hom}"'] & \mathbb{D}
\end{tikzcd}
\quad \leftsquigarrow \text{alteration} \qquad \text{diagram } F \text{ with weight } W \rightsquigarrow \quad
\begin{tikzcd}
\mathbb{I} \arrow[r, "W"] \arrow[d, "F"'] & 2 \arrow[dl, bend left, "P"] \arrow[dl, phantom, "\Leftarrow \varphi"] \\
\mathbb{D} &
\end{tikzcd}$$

- cone $\rightsquigarrow$ a modification φ between alterations OR

$$\begin{tikzcd}
\mathbb{I}_s \arrow[r, "\shortmid" marking, "\mathbb{I}"] \arrow[d, "!"'] \arrow[dr, phantom, "!"] & \mathbb{I}_t \arrow[d, "!"] \\
\mathbb{1} \arrow[r, "\shortmid" marking] \arrow[d, "X"'] \arrow[dr, phantom, "P"] & \mathbb{1} \arrow[d, "Y"] \\
\mathbb{D} \arrow[r, "\shortmid" marking, "\mathrm{Hom}"'] & \mathbb{D}
\end{tikzcd}
\overset{\varphi}{\Longrightarrow}
\begin{tikzcd}
\mathbb{I}_s \arrow[r, "\shortmid" marking, "\mathbb{I}"] \arrow[d, "F_s"'] \arrow[dr, phantom, "F"] & \mathbb{I}_t \arrow[d, "F_t"] \\
\mathbb{D} \arrow[r, "\shortmid" marking, "\mathrm{Hom}"'] & \mathbb{D}
\end{tikzcd}$$

- limit $\rightsquigarrow$ terminal cone preserved by $\mathrm{dom}, \mathrm{cod} : \mathrm{DBL}_{nl}/2 \rightarrow \mathrm{DBL}_{nl}$.

MAIN RESULT & SUMMARY

- A parallel tabulator is a limit whose shape is $2 \xrightarrow[\text{Hom}]{} 2$.
- A "homologous limit" is a limit whose shape is $\mathbb{T}_i(S) \to \mathbb{T}_i(T)$, but these are equivalent to limits indexed by spans $S \leftarrow I \rightarrow T$
- A parallel limit is a limit whose shape is $\mathbb{I} \xrightarrow{\text{Hom}} \mathbb{I}$, and a tight parallel limit has shape $\mathbb{T}_i(\mathcal{C}) \xrightarrow{\text{Hom}} \mathbb{T}_i(\mathcal{C})$.

Theorem: A double category $\mathbb{D}$ admits limits indexed by loose distributors if and only if $\mathbb{D}$ admits:

(1) Parallel tabulators
(2) Tight parallel limits
(3) Restrictions

(1) + (2): Parallel limits

(2) + (3): Homologous limits

- The double categories $\mathbb{S}\text{pan}$ and $\mathbb{D}\text{ist}$ admit all limits indexed by double categories and all limits indexed by loose distributors.
- We have shown that limits indexed by loose distributors (or spans of functors) capture many existing concepts including:
  - local and discrete limits
  - bicategorical limits
  - companions and conjoints
- Many current and future research directions.
  - Sufficient conditions for completeness of $\mathbb{S}\text{pan}(\mathcal{E})$, $\mathbb{R}\text{el}(\mathcal{E})$, $\mathbb{M}\text{at}(\mathbb{D})$, $\mathbb{M}\text{od}(\mathbb{D})$, etc.
  - Constructing (co)completions of double categories