

A SECOND LOOK AT LIMITS IN DOUBLE CATEGORIES

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Atlantic Category Theory Seminar (online)

Dalhousie University

14 October 2025

MOTIVATION & OVERVIEW

01

- **Double categories** are a 2-dimensional structure consisting of objects, tight & loose morphisms, and cells.
- The prototypical example is **Rel**, the double category of sets, functions, and relations.
- Monoidal categories, bicategories and 2-categories also examples.
- Marco Grandis & Bob Paré introduced limits in double categories indexed by double categories — these are **objects**.
- However, these were not expressive enough! They also needed limits of "vertical transformations" — these are **loose morphisms**.
- However, there are many examples of loose morphisms with universal properties which are not captured!

For example:

- Given relations $R \rightrightarrows A \times B$ and $S \rightrightarrows A \times B$, we may take their intersection to get $R \cap S \rightrightarrows A \times B$
- Given a function $f: A \rightarrow B$, we may construct relations $\langle 1_A, f \rangle: A \rightrightarrows A \times B$ and $\langle f, 1_B \rangle: A \rightrightarrows A \times B$
- The coproduct in **Set** is a biproduct in **Rel** — why?

OUTLINE: Cover four kinds of "shape" for limits in double categories, with examples.

limit type $\rightarrow$ dimension $\downarrow$	object	loose morphism
1	category $\mathbb{I}$	span of functors $S \leftarrow \mathbb{I} \rightarrow T$
2	double category $\mathbb{I}$	loose distributor $\mathbb{I}_s \xrightarrow{\mathbb{I}} \mathbb{I}_t$

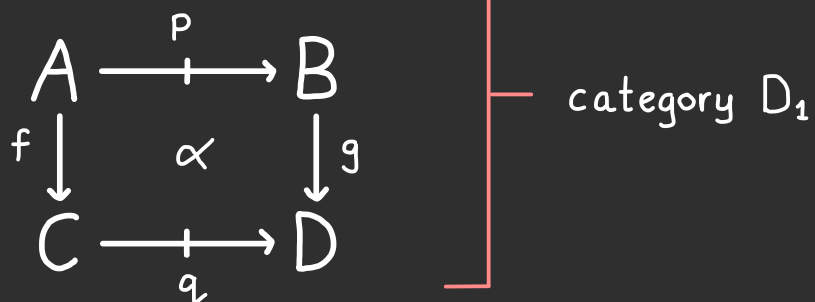
DOUBLE CATEGORIES UNPACKED

A double category ID consists of:

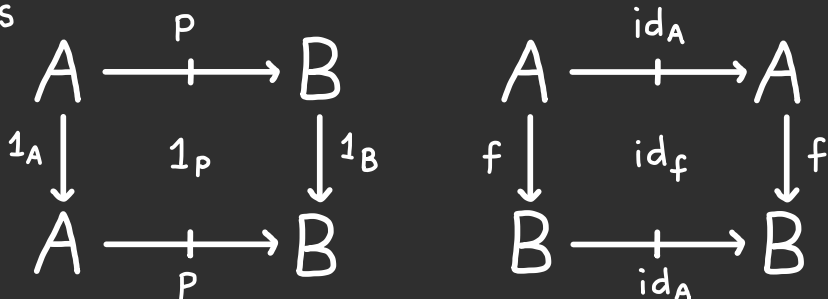
- objects $A, B, C, \dots$
 - tight morphisms $A \longrightarrow B$
- category D_0

- loose morphisms $A \rightrightarrows B$

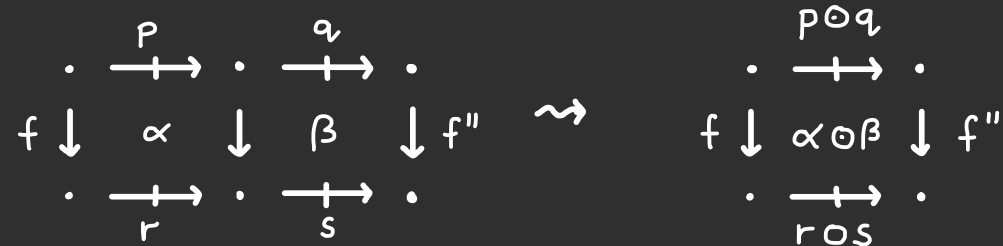
- cells



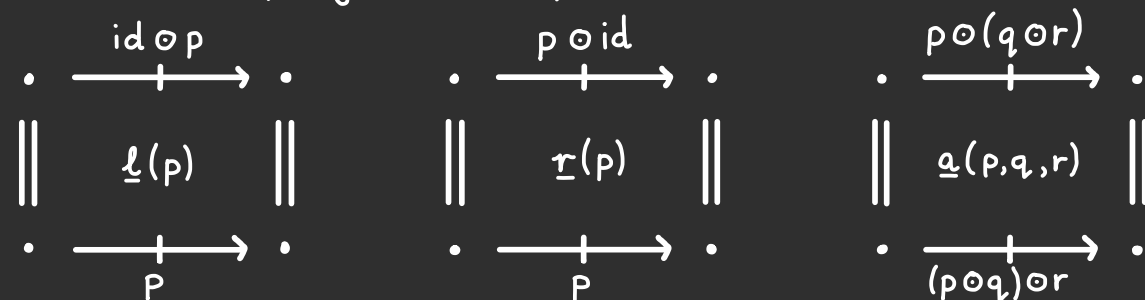
- identity morphisms $1_A: A \longrightarrow A$ and $id_A: A \rightrightarrows A$
- composite morphisms $g \circ f: A \longrightarrow C$ and $p \circ q: A \rightrightarrows C$
- identity cells



- composite cells



- left unitor, right unitor, & associator cells



+ axioms and coherence conditions.

DOUBLE CATEGORIES

A **double category** is a pseudo category object in the 2-category CAT of locally small categories.

$$\text{ID} := D_0 \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{i} \\ \xleftarrow{t} \end{array} D_1 \begin{array}{c} \xleftarrow{\pi_1} \\ \xrightarrow{m} \\ \xleftarrow{\pi_2} \end{array} D_1 \times_{D_0} D_1 = D_2$$

Therefore, we have commutative diagrams

$$\begin{array}{ccc} & D_0 & \\ 1 \swarrow & \downarrow i & \searrow 1 \\ D_0 & \xleftarrow{s} D_1 \xrightarrow{t} & D_0 \end{array} \quad \begin{array}{ccccc} D_1 & \xleftarrow{\pi_1} & D_2 & \xrightarrow{\pi_2} & D_1 \\ s \downarrow & & \downarrow m & & \downarrow t \\ D_0 & \xleftarrow{s} & D_1 & \xrightarrow{t} & D_0 \end{array}$$

and iterated pullbacks of s and t denoted

$$D_3 = D_1 \times_{D_0} D_1 \times_{D_0} D_1 \quad D_4 = D_1 \times_{D_0} D_1 \times_{D_0} D_1 \times_{D_0} D_1$$

and invertible natural transformations $\underline{l}$, $\underline{r}$, and $\underline{a}$ satisfying the equations $s \cdot \underline{l} = 1 = s \cdot \underline{r}$, $t \cdot \underline{l} = 1 = t \cdot \underline{r}$, $t \cdot \underline{a} = 1$, and $s \cdot \underline{a} = 1$

$$\begin{array}{ccc} D_1 & \xrightarrow{\langle is, 1 \rangle} & D_2 \xleftarrow{\langle 1, it \rangle} D_1 \\ & \searrow \underline{l} & \downarrow m \searrow \underline{r} \\ & 1 & D_1 \end{array}$$

$$\begin{array}{ccc} D_3 & \xrightarrow{1 \times m} & D_2 \\ m \times 1 \downarrow & \searrow \underline{a} & \downarrow m \\ D_2 & \xrightarrow{m} & D_1 \end{array}$$

and satisfying two coherence conditions.

$$\begin{array}{ccccc} & & 1 & & \\ & & \searrow & & \nearrow \\ D_2 & \xrightarrow{1} & D_3 & \xrightarrow{1 \times \underline{l}^{-1}} & D_2 \\ & \searrow \underline{r} \times 1 & \downarrow \underline{a} & & \searrow m \\ & & D_2 & & D_1 \end{array}$$

$$= \begin{array}{ccccc} & & D_2 & & \\ 1 \nearrow & & \searrow m & & \\ D_2 & \xrightarrow{1} & D_1 & & \\ & \searrow 1 & \nearrow m & & \\ & & D_2 & & \end{array}$$

$$\begin{array}{ccccc} & & D_3 & \xrightarrow{1 \times m} & D_2 \\ 1 \times 1 \times m \nearrow & & \searrow \underline{1 \times \underline{a}} & & \searrow m \\ D_4 & \xrightarrow{1} & D_3 & \xrightarrow{\underline{a}} & D_1 \\ m \times 1 \times 1 \searrow & & \searrow \underline{a} \times 1 & & \searrow m \\ & & D_3 & \xrightarrow{m \times 1} & D_2 \end{array}$$

$$= \begin{array}{ccccc} & & D_3 & \xrightarrow{1 \times m} & D_2 \\ 1 \times 1 \times m \nearrow & & \searrow \underline{a} & & \searrow m \\ D_4 & \xrightarrow{1} & D_2 & \xrightarrow{1} & D_1 \\ m \times 1 \times 1 \searrow & & \searrow \underline{a} & & \searrow m \\ & & D_3 & \xrightarrow{m \times 1} & D_2 \end{array}$$

SMALL CATEGORIES, FUNCTORS, & DISTRIBUTORS

05

Let **Dist** be the double category whose:

- objects are small categories
- tight morphisms are functors
- loose morphisms $p: A \rightrightarrows B$ are distributors, i.e. functors

$$A^{\text{op}} \times B \xrightarrow{p} \text{Set}$$

- cells denoted

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ f \downarrow & \alpha & \downarrow g \\ C & \xrightarrow{q} & D \end{array}$$

are natural transformations

$$\begin{array}{ccc} A^{\text{op}} \times B & \xrightarrow{p} & \text{Set} \\ f \times g \downarrow & \Downarrow \alpha & \\ C^{\text{op}} \times D & \xrightarrow{q} & \end{array}$$

- the loose identity morphism on A is

$$\text{Hom} := A(-, -) : A^{\text{op}} \times A \longrightarrow \text{Set}$$

- the composite of loose morphisms $p: A \rightrightarrows B$ and $q: B \rightrightarrows C$ is via coend:

$$A^{\text{op}} \times C \xrightarrow{p \circ q} \text{Set}$$

$$(a, c) \longmapsto \int^{b \in B} p(a, b) \times q(b, c)$$

where the set is defined by the equivalence relation

$$\begin{array}{ccc} a \xrightarrow{u} b \xrightarrow{v} c & & \exists w: b \rightarrow b' \text{ such that} \\ a \xrightarrow{u'} b' \xrightarrow{v'} c & \iff & p(1_a, w)(u) = u' \text{ \& } q(w, 1_c)(v') = v \end{array}$$

- the unitors and associator are induced by the coend

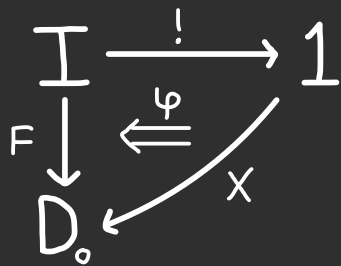
$$\text{Cat} \begin{array}{c} \xleftarrow{\text{dom}} \\ \xleftarrow{\text{id}} \\ \xleftarrow{\text{cod}} \end{array} \text{Cat}/2 \begin{array}{c} \xleftarrow{\pi_1} \\ \xleftarrow{\circ} \\ \xleftarrow{\pi_2} \end{array} \text{Cat}/3$$

LIMITS INDEXED BY CATEGORIES

06

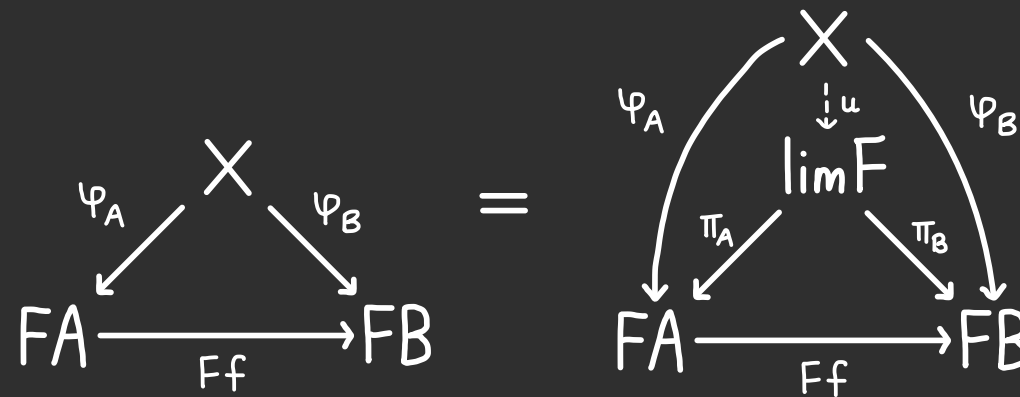
A **limit** in a double category ID **indexed by a (small) category** is a (small) limit in the underlying category D_0 of objects and tight morphisms in ID .

- **shape** $\rightsquigarrow$ small category I
- **diagram** $\rightsquigarrow$ functor $F: I \rightarrow D_0$
- **cone** $\rightsquigarrow$ an object $X \in D_0$ & natural transformation



- **morphism of cones** $(X, \varphi) \rightarrow (Y, \psi) \rightsquigarrow$ a tight morphism $f: X \rightarrow Y$ such that $\psi_A \circ f = \varphi_A$ for all $A \in I$.
- **limit** $\rightsquigarrow$ terminal object $(\lim F, \pi)$ in the category of cones $\text{Cone}(F)$ over F ;

- **U.P.** $\rightsquigarrow$ for all cones (X, φ) over F there exists a unique tight morphism $u: X \rightarrow \lim F$ such that



Limits in double categories indexed by categories are not very interesting, as the **loose morphisms and cells** play no role. Several ways to fix:

1. Index by **double categories** (Grandis-Paré)
2. Index by **spans in Cat**
3. Both of the above: index by **loose distributors**

A 2-CATEGORY OF SPANS OF FUNCTORS

07

Let $CAT^{\{ \cdot \leftarrow \cdot \rightarrow \cdot \}}$ be the 2-category whose:

- objects are spans of functors in CAT
- 1-cells are morphisms of spans (a triple of functors...)
- 2-cells $(\alpha, \beta, \gamma): (f_1, g_1, h_1) \rightarrow (f_2, g_2, h_2)$ are diagrams

$$\begin{array}{ccccc}
 \bullet & \xleftarrow{p} & \bullet & \xrightarrow{p'} & \bullet \\
 f_1 \downarrow \left(\begin{smallmatrix} \alpha \\ \Rightarrow \end{smallmatrix} \right) \downarrow & f_2 & g_1 \downarrow \left(\begin{smallmatrix} \beta \\ \Rightarrow \end{smallmatrix} \right) \downarrow & g_2 & h_1 \downarrow \left(\begin{smallmatrix} \gamma \\ \Rightarrow \end{smallmatrix} \right) \downarrow & h_2 \\
 \bullet & \xleftarrow{q} & \bullet & \xrightarrow{q'} & \bullet
 \end{array}$$

which "commute" in the sense that:

$$q \cdot \beta = \alpha \cdot p \quad q' \cdot \beta = \gamma \cdot p'$$

There are 2-functors

$$CAT \begin{array}{c} \xleftarrow{\text{dom}} \\ \xleftarrow{\text{id}} \\ \xrightarrow{\text{cod}} \end{array} CAT^{\{ \cdot \leftarrow \cdot \rightarrow \cdot \}}$$

where $\text{id}(C) = C \xleftarrow{1} C \xrightarrow{1} C$, $\text{dom}(A \leftarrow X \rightarrow B) = A$ and $\text{cod}(A \leftarrow X \rightarrow B) = B$.

The 2-category $CAT^{\{ \cdot \leftarrow \cdot \rightarrow \cdot \}}$ has a 2-terminal object:

$$1 \longleftarrow 1 \longrightarrow 1$$

Thus, we can consider limits of 1-cells, that is, **limits of morphisms of spans in CAT** .

$$\begin{array}{ccccc}
 S & \xleftarrow{\quad} & I & \xrightarrow{\quad} & T \\
 1_a \swarrow \left(\begin{smallmatrix} \alpha \\ \Rightarrow \end{smallmatrix} \right) \swarrow & f & 1_x \swarrow \left(\begin{smallmatrix} \beta \\ \Rightarrow \end{smallmatrix} \right) \swarrow & g & 1_b \swarrow \left(\begin{smallmatrix} \gamma \\ \Rightarrow \end{smallmatrix} \right) \swarrow & h \\
 A & \xleftarrow{p} & X & \xrightarrow{q} & B
 \end{array}$$

In particular, want the image of a limit cone under $\text{dom}, \text{cod}: CAT^{\{ \cdot \leftarrow \cdot \rightarrow \cdot \}} \rightarrow CAT$ to be a limit cone.

Every double category ID has an underlying span

$$D_0 \xleftarrow{s} D_1 \xrightarrow{t} D_0$$

therefore, we can formulate **limits in double categories!**

LIMITS INDEXED BY SPANS OF FUNCTORS

08

A **limit** in a double category $\mathbb{D}$ indexed by a span of functors in $\mathcal{Cat}$ is a limit in the underlying span of functors $D_0 \xleftarrow{s} D_1 \xrightarrow{t} D_0$.

- **shape** $\rightsquigarrow$ span $S \leftarrow I \rightarrow T$ in $\mathcal{Cat}$ (small categories)
- **diagram** $\rightsquigarrow$ morphism of spans in $\mathcal{CAT}$

$$\begin{array}{ccccc} S & \xleftarrow{s} & I & \xrightarrow{t} & T \\ F \downarrow & & \downarrow H & & \downarrow G \\ D_0 & \xleftarrow{s} & D_1 & \xrightarrow{t} & D_0 \end{array} \quad (*)$$

- **cone** $\rightsquigarrow$

$$\begin{array}{ccccc} S & \xleftarrow{s} & I & \xrightarrow{t} & T \\ \downarrow \varphi_x & & \downarrow \varphi_p & & \downarrow \varphi_y \\ D_0 & \xleftarrow{s} & D_1 & \xrightarrow{t} & D_0 \end{array}$$

- **limit** $\rightsquigarrow$ a terminal cone such that images under $\text{dom}, \text{cod}: \mathcal{CAT}^{\{\leftarrow, \rightarrow\}} \rightarrow \mathcal{CAT}$ are terminal cones (limits).

To unpack, we may interpret each span in $\mathcal{CAT}$ as a (free double category on a) **internal graph** in $\mathcal{CAT}$.

Given a span $S \xleftarrow{s} I \xrightarrow{t} T$ we have:

- objects, the objects of $S + T$
- tight morphisms, the morphisms of $S + T$
- loose morphisms $s(x) \xrightarrow{x} t(x)$ for each object x in I ; cells are similar.

A **limit** of $(*)$ is a pair $(\lim F, \varphi)$ and $(\lim G, \psi)$ of limits in D_0 , and a universal cone $(\lim(F, G, H), \theta)$ in $D_1 \rightsquigarrow$ a loose morphism $\lim: \lim F \dashrightarrow \lim G$ and a cell in $\mathbb{D}$

$$\begin{array}{ccc} \lim F & \xrightarrow{\lim} & \lim G \\ \varphi_A \downarrow & \theta_p & \downarrow \psi_B \\ FA & \xrightarrow{H_p} & GB \end{array}$$

for each $p: A \dashrightarrow B$
in I , natural w.r.t.
morphisms (cells) in I .

PARALLEL PRODUCTS / LIMITS

09

Parallel products have shape $1+1 \Rightarrow 1+1 \Rightarrow 1+1$.

The parallel product of $p_1: A_1 \rightrightarrows B_1$ and $p_2: A_2 \rightrightarrows B_2$ is a loose morphism $p_1 \times p_2: A_1 \times A_2 \rightrightarrows B_1 \times B_2$ and projection cells

$$\begin{array}{ccc} A_1 \times A_2 & \xrightarrow{p_1 \times p_2} & B_1 \times B_2 \\ \pi_i \downarrow & \pi_i & \downarrow \pi_i \\ A_i & \xrightarrow{p_i} & B_i \end{array} \quad i = 1, 2$$

such that for every pair cells α_1 and α_2 there exists a unique cell $\langle \alpha_1, \alpha_2 \rangle$ such that $\pi_i \circ \langle \alpha_1, \alpha_2 \rangle = \alpha_i$

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ f_i \downarrow & \alpha_i & \downarrow g_i \\ A_i & \xrightarrow{p_i} & B_i \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ \langle f_1, f_2 \rangle \downarrow & \langle \alpha_1, \alpha_2 \rangle & \downarrow \langle g_1, g_2 \rangle \\ A_1 \times A_2 & \xrightarrow{p_1 \times p_2} & B_1 \times B_2 \end{array}$$

In $\mathcal{S}pan$, the parallel product of $A \leftarrow X \rightarrow B$ and $C \leftarrow Y \rightarrow D$ is $A \times C \leftarrow X \times Y \rightarrow B \times D$.

In $\mathcal{I}Dist$, the parallel product of $p: A^{\circ p} \times B \rightarrow \mathcal{S}et$ and $q: C^{\circ p} \times D \rightarrow \mathcal{S}et$ is given by:

$$(A \times C)^{\circ p} \times (B \times D) \cong (A^{\circ p} \times B) \times (C^{\circ p} \times D) \xrightarrow{p \times q} \mathcal{S}et \times \mathcal{S}et \xrightarrow{\times} \mathcal{S}et$$

More generally, a ^{tight}parallel limit has shape $I = I = I$.

A double category admits all ^{tight}parallel limits if and only if D_0 and D_1 admit all limits and the functors $s, t: D_1 \rightarrow D_0$ preserve them.

Parallel limits were previously treated by Grandis & Paré in "Limits in double categories" (1999) as limits of vertical transformations.

LOCAL PRODUCTS / LIMITS

10

Local products have shape $1 \longleftarrow 1+1 \longrightarrow 1$.

The local product of $p_1: A \rightrightarrows B$ and $p_2: A \rightrightarrows B$ is a loose morphism $p_1 \wedge p_2: A \rightrightarrows B$ and projection cells

$$\begin{array}{ccc} A & \xrightarrow[p_1 \wedge p_2]{} & B \\ \parallel & \pi_i & \parallel \\ A & \xrightarrow[p_i]{} & B \end{array} \quad i=1,2$$

such that for every pair cells α_1 and α_2 there exists a unique cell $\langle \alpha_1, \alpha_2 \rangle$ such that $\pi_i \circ \langle \alpha_1, \alpha_2 \rangle = \alpha_i$

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ f \downarrow & \alpha_i & \downarrow g \\ A & \xrightarrow[p_i]{} & B \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ f \downarrow & \langle \alpha_1, \alpha_2 \rangle & \downarrow g \\ A & \xrightarrow[p_1 \wedge p_2]{} & B \end{array}$$

In $\mathbb{S}pan$, the local product of $A \longleftarrow X \longrightarrow B$ and $A \longleftarrow Y \longrightarrow B$ is $A \longleftarrow X \times_{A \times B} Y \longrightarrow B$.

In $IDist$, the local product of $p: A^{op} \times B \rightarrow Set$ and $q: A^{op} \times B \rightarrow Set$ is $A^{op} \times B \xrightarrow{\langle p, q \rangle} Set \times Set \xrightarrow{*} Set$.

More generally, a local limit has shape $1 \longleftarrow I \longrightarrow 1$.

Local colimits were previously treated by Paré in "Composition of modules for lax functors" (2013), however we do not require preservation by loose composition.

$$\begin{array}{ccc} A & \xrightarrow[(p_1 \odot q) \vee (p_2 \odot q)]{} & C \\ \parallel & \exists! & \parallel \\ A & \xrightarrow[p_1 \vee p_2]{} B \xrightarrow{q} & C \end{array}$$

← Not necessarily invertible

DISCRETE LIMITS

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Discrete limits are limits with shape $S \xleftarrow{s} I \xrightarrow{t} T$ where S, I, T are discrete categories.

Given families of objects $(A_x)_{x \in S}$ and $(B_y)_{y \in T}$ and a family of loose morphisms $(p_i: A_{s_i} \multimap B_{t_i})_{i \in I}$, their limit is a loose morphism $\prod_{x \in S} A_x \xrightarrow{\prod p_i} \prod_{y \in T} B_y$ and projection cells

$$\begin{array}{ccc}
 \prod_{x \in S} A_x & \xrightarrow{\prod p_i} & \prod_{y \in T} B_y \\
 \downarrow \pi_{s_i} & \pi_i & \downarrow \pi_{t_i} \\
 A_{s_i} & \xrightarrow{p_i} & B_{t_i}
 \end{array} \quad i \in I$$

satisfying the appropriate universal property.

Both $\mathcal{S}pan$ and $\mathcal{I}Dist$ admit discrete limits.

Discrete limits were previously treated by Paré in the talk "Coherent theories as double Lawvere theories" under the name products.

In 2024, Patterson treated these in more detail in the preprint "Products in double categories, revisited".

Patterson constructs a double category $\mathcal{I}Fam(\mathcal{I}D)$ which is the free cocompletion of $\mathcal{I}D$ under discrete colimits.

A double category $\mathcal{I}D$ has all discrete colimits if the double functor

$$\mathcal{I}D \longrightarrow \mathcal{I}Fam(\mathcal{I}D)$$

has a (colax) left adjoint.

"BICATEGORICAL" (CO)LIMITS

12

Consider colimits of the shape $1 \leftarrow 1+1 \rightrightarrows 1+1$.

A diagram is a pair $p_1: A \rightrightarrows B_1$ and $p_2: A \rightrightarrows B_2$, whose colimit is a loose morphism $\text{colim}(p_1, p_2): A \rightrightarrows B_1 + B_2$ with cells

$$\begin{array}{ccc} A & \xrightarrow{p_i} & B_i \\ \parallel & \Downarrow \eta_i & \downarrow \eta_i \\ A & \xrightarrow{\text{colim}(p_1, p_2)} & B_1 + B_2 \end{array} \quad i=1,2$$

such that for any pair of cells α_1 and α_2 , there exists a unique cell $[\alpha_1, \alpha_2]$ such that $[\alpha_1, \alpha_2] \circ \eta_i = \alpha_i$.

$$\begin{array}{ccc} A & \xrightarrow{p_i} & B_i \\ f \downarrow & \alpha_i & \downarrow g_i \\ C & \xrightarrow{q} & D \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\text{colim}(p_1, p_2)} & B_1 + B_2 \\ \downarrow & [\alpha_1, \alpha_2] & \downarrow [g_1, g_2] \\ C & \xrightarrow{q} & D \end{array}$$

In Span and IDist , the cells η_i are restrictions, meaning that $\text{colim}(p_1, p_2) \circ \pi_i^* \cong p_i$ (i.e. they "commute").

The universal property implies that $B_1 + B_2$ is a **bicategorical product** in the underlying bicategory of Span or IDist .

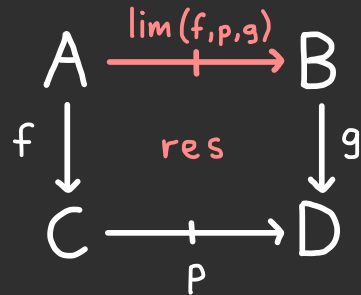
It is remarkable that the universal loose morphism into the product $\text{colim}(p_1, p_2): A \rightrightarrows B_1 + B_2$ itself has a universal property (of a colimit) — this is **not expressible in a bicategory**!

By duality, $B_1 + B_2$ is also a **bicategorical coproduct** in Span or IDist , hence products and coproducts coincide in this sense, but are far richer in the ambient double category.

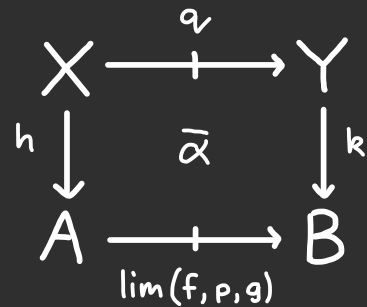
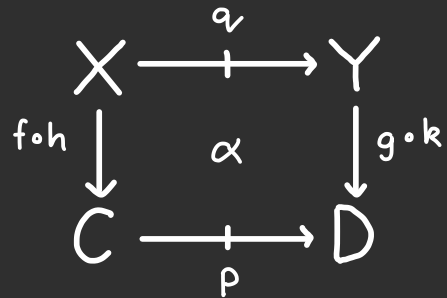
RESTRICTIONS, COMPANIONS, & CONJOINTS

A **restriction** is a limit with the shape $2 \xleftarrow{\text{cod}} 1 \xrightarrow{\text{cod}} 2$.

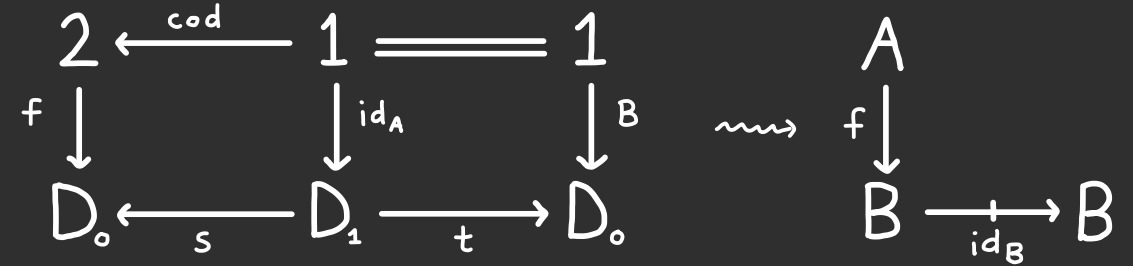
A diagram is a niche as below, and its restriction is a loose morphism $\text{lim}(f, p, g): A \rightarrow B$ and a cell



such that for any cell α as below, there exists a unique cell $\bar{\alpha}$ such that $\text{res} \circ \bar{\alpha} = \alpha$.



A **companion** in $\mathbb{ID}$ is a limit of a diagram



Conjoints are defined dually (reverse loose direction).

Shulman (2008) showed that all extensions exist if and only if all companions & conjoints exist.

In $\mathbb{Span}$, the companion and conjoint of $f: A \rightarrow B$ is

$A = A \xrightarrow{f} B$ and $A \xleftarrow{f} B = B$, respectively.

In $\mathbb{IDist}$, a functor $f: A \rightarrow B$ has companion $B(f, -): A^{\text{op}} \times B \rightarrow \text{Set}$ and conjoint $B(-, f): B^{\text{op}} \times A \rightarrow \text{Set}$.

Companions and conjoints are also instances of colimits.

TECHNICALITIES: NORMAL LIMITS & REPLETENESS

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For a diagram $F: \mathbb{I} \rightarrow D_0$ admitting a limit $(\lim F, \pi)$, we obtain a span $\mathbb{I} = \mathbb{I} = \mathbb{I}$ and a cone:

$$\begin{array}{ccccc}
 \mathbb{I} & \xlongequal{\quad} & \mathbb{I} & \xlongequal{\quad} & \mathbb{I} \\
 \swarrow \pi & \Downarrow F & \swarrow i \cdot \pi & \Downarrow i \circ F & \swarrow \pi \\
 \lim F & \xrightarrow{\quad} & D_0 & \xrightarrow{\quad} & D_0 \\
 \downarrow \text{id}_{\lim F} & & & & \\
 D_0 & \xleftarrow{s} & D_1 & \xrightarrow{t} & D_0
 \end{array}$$

If this is a limit cone in $CAT^{\{x,y,z\}}$, then we say F is a **normal limit**.

This means the parallel limit consisting entirely of loose identity morphisms is again a loose identity morphism.

In practice, many examples of (co)limits are normal.

Normal limits $(\lim F, \pi)$ admit a two-dimensional universal property, the sense of Grandis & Paré (1999).

Limits in D_0 indexed by spans of functors are unique up to isomorphism in D_1 .

However, given a diagram

$$\begin{array}{ccccc}
 S & \xleftarrow{s} & I & \xrightarrow{t} & T \\
 F \downarrow & & \downarrow H & & \downarrow G \\
 D_0 & \xleftarrow{s} & D_1 & \xrightarrow{t} & D_0
 \end{array}$$

admitting a limit $\lim: X \rightarrow Y$ and choices of limits $\lim F \cong X$ and $\lim G \cong Y$, there **may not be** a loose morphism $p: \lim F \rightarrow \lim G$ and an iso $p \cong \lim$ in D_1 .

A double category such that $\langle s, t \rangle: D_1 \rightarrow D_0 \times D_0$ is an isofibration is called **replete**, and here we may always reindex limits along tight isomorphisms. This property was called **horizontal invariance** by Grandis-Paré.

A CONSTRUCTION THEOREM

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Theorem: A double category admits all limits indexed by spans of functors if and only if it admits parallel limits and restrictions.

Proof: Given a diagram

$$\begin{array}{ccccc} S & \xleftarrow{u} & I & \xrightarrow{v} & T \\ F \downarrow & & \downarrow H & & \downarrow G \\ D_0 & \xleftarrow{s} & D_1 & \xrightarrow{t} & D_0 \end{array}$$

construct the parallel limit of the diagram

$$\begin{array}{ccccc} I & \xlongequal{\quad} & I & \xlongequal{\quad} & I \\ F \downarrow & & \downarrow H & & \downarrow G \\ D_0 & \xleftarrow{s} & D_1 & \xrightarrow{t} & D_0 \end{array}$$

to obtain a loose morphism $\lim H : \lim FU \rightarrow \lim GV$.

Given limits $(\lim F, \varphi)$ and $(\lim G, \psi)$ we obtain cones

$$\begin{array}{ccc} I & \xrightarrow{\quad} & 1 \\ F \downarrow & \swarrow \lim F & \\ D_0 & & \end{array}$$

$$\begin{array}{ccc} I & \xrightarrow{\quad} & 1 \\ G \downarrow & \swarrow \lim G & \\ D_0 & & \end{array}$$

inducing tight morphisms $\lim F \rightarrow \lim FU$ and $\lim G \rightarrow \lim GV$.

Finally, take the restriction of

$$\begin{array}{ccc} \lim F & \xrightarrow{\quad} & \lim G \\ \downarrow & \text{res} & \downarrow \\ \lim FU & \xrightarrow{\quad} & \lim GV \\ & \lim H & \end{array}$$

to obtain the desired limit $\lim : \lim F \rightarrow \lim G$.

EXERCISE. Check that the universal property holds.

DOUBLE FUNCTORS & TRANSFORMATIONS

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A **lax double functor** $F: \mathbb{C} \rightarrow \mathbb{D}$ is an assignment

$$\begin{array}{ccc} A & \xrightarrow{p} & C \\ f \downarrow & \alpha & \downarrow g \\ B & \xrightarrow{q} & D \end{array} \rightsquigarrow \begin{array}{ccc} FA & \xrightarrow{Fp} & FC \\ Ff \downarrow & F\alpha & \downarrow Fg \\ FB & \xrightarrow{Fq} & FD \end{array}$$

preserving tight identities & composites, together with unitor and laxator cells (satisfying several axioms):

$$\begin{array}{ccc} FA & \xrightarrow{id_{FA}} & FA \\ 1_{FA} \downarrow & \eta_A & \downarrow 1_{FA} \\ FA & \xrightarrow{F(id_A)} & FA \end{array} \quad \begin{array}{ccccc} FA & \xrightarrow{Fp} & FB & \xrightarrow{Fq} & FC \\ 1_{FA} \downarrow & & \mu_{p,q} & & \downarrow 1_{FC} \\ FA & \xrightarrow{F(p \circ q)} & FC \end{array}$$

- Called **normal** if η_A is identity cell, **pseudo** if η_A and $\mu_{p,q}$ are invertible, **strict** if identities.
- For a **colax double functor**, flip η_A and $\mu_{p,q}$.

A **transformation** between lax double functors

$$\begin{array}{ccc} & F & \\ & \searrow & \\ \mathbb{C} & & \mathbb{D} \\ & \swarrow & \\ & G & \end{array} \quad \begin{array}{c} \Downarrow \varphi \end{array}$$

consists of a family of cells

$$\begin{array}{ccc} FA & \xrightarrow{Fp} & FB \\ \varphi_A \downarrow & \varphi_p & \downarrow \varphi_B \\ GA & \xrightarrow{Gp} & GB \end{array}$$

which are natural and satisfy certain coherence axioms.

Let DBL_{nl} be the 2-category of double categories, normal lax functors, and transformations.

$$\begin{array}{ccc} & (-)_o & \\ CAT & \xleftarrow{\quad} & DBL_{nl} \\ & \xrightarrow{\quad} & \\ & \Pi_i(-) & \end{array}$$

PRESERVATION OF LIMITS

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Let $Gph(CAT)$ be the 2-category of internal graphs in CAT , and let $RGph(CAT)$ be 2-category of internal reflexive graphs in CAT .

A lax functor $F: \mathbb{C} \rightarrow \mathbb{D}$ is a **right adjoint** if its underlying morphism of internal graphs in CAT

$$\begin{array}{ccccc} C_0 & \xleftarrow{s} & C_1 & \xrightarrow{s} & C_0 \\ F_0 \downarrow & & \downarrow F_1 & & \downarrow F_0 \\ D_0 & \xleftarrow{t} & D_1 & \xrightarrow{t} & D_0 \end{array}$$

is a right adjoint in $Gph(CAT)$; i.e. F_0 and F_1 admit left adjoints strictly compatible with source/target functors.

A **normal** lax functor $F: \mathbb{C} \rightarrow \mathbb{D}$ is a **uniform right adjoint** if its underlying morphism of internal reflexive graphs in CAT is a right adjoint in $RGph(CAT)$.

Proposition: Limits in double categories indexed by spans of functors are preserved by right adjoint (lax) double functors. Normal limits are preserved by uniform right adjoints.

Lemma: Companions and conjoints are preserved by any normal (lax/colax) double functor.

Proof: A companion of $f: A \rightarrow B$ is a pair of cells

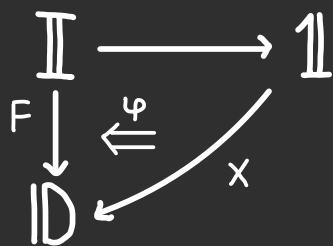
$$\begin{array}{ccc} A & \xrightarrow{f_*} & B \\ f \downarrow & \sigma & \downarrow 1_B \\ B & \xrightarrow{id_B} & B \end{array} \quad \begin{array}{ccc} A & \xrightarrow{id_A} & A \\ 1_A \downarrow & \tau & \downarrow f \\ A & \xrightarrow{f_*} & B \end{array}$$

such that $\tau \odot \sigma = 1_{f_*}$ and $\sigma \circ \tau = id_f$. These equations are preserved by a normal double functor.

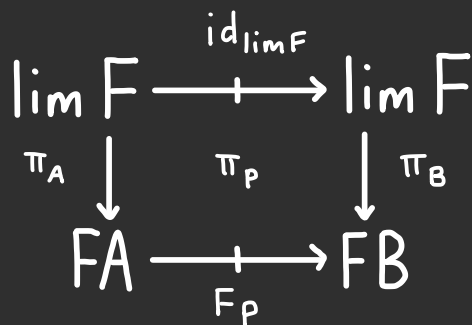
LIMITS INDEXED BY DOUBLE CATEGORIES

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- **shape** $\rightsquigarrow$ small double category $\mathbb{I}$
- **diagram** $\rightsquigarrow$ normal lax double functor $F: \mathbb{I} \longrightarrow \mathbb{D}$
- **cone** $\rightsquigarrow$ object $X \in \mathbb{D}$ and transformation



- **morphism of cones** $(X, \chi) \longrightarrow (Y, \psi) \rightsquigarrow$ a tight morphism $f: X \longrightarrow Y$ such that $\psi \circ f = \chi$.
- **limit** $\rightsquigarrow$ terminal object $(\lim F, \pi)$ in the category of cones $\text{Cone}(F)$ over F ;



A **tabulator** is a limit whose shape is $\mathbb{2} = \{0 \rightrightarrows 1\}$

A double category $\mathbb{D}$ admits all tabulators if and only if the functor $\text{id}: D_0 \longrightarrow D_1$ has a **right adjoint**.

In **Span**, the tabulator of $A \longleftarrow X \longrightarrow B$ is X .

In **IDist**, the category of elements of $P: A^{op} \times B \longrightarrow \text{Set}$.

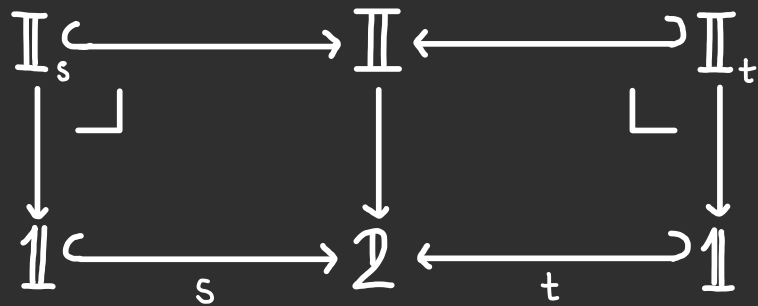
A **tight limit** is a limit whose shape is $\Pi_i(\mathcal{C})$, whose objects and morphisms are from $\mathcal{C}$ and loose morphisms/cells are identities, but these are equivalent to limits of functors $\mathcal{C} \longrightarrow D_0$.

Theorem (Grandis-Paré, 1999): A double category $\mathbb{D}$ admits limits indexed by any double category $\mathbb{I}$ if and only if $\mathbb{D}$ admits **tight limits** and **tabulators**.

LIMITS INDEXED BY LOOSE DISTRIBUTORS

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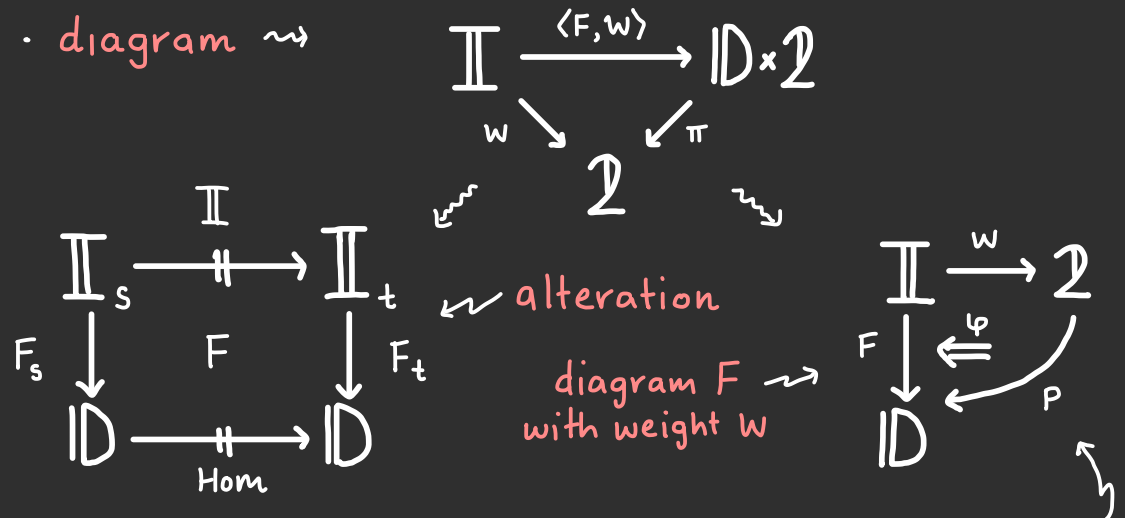
- A loose distributor $\mathbb{S} \dashrightarrow \mathbb{T}$ is a **span** $S_0 \leftarrow \mathbb{I} \rightarrow T_0$ with a **compatible left action** of $\mathbb{S}$ and **right action** of $\mathbb{T}$.
- A **loose distributor** is a double functor $\mathbb{I} \rightarrow \mathbb{2}$ into the loose interval.
- A loose distributor is seen as a **morphism** $\mathbb{I}_s \dashrightarrow \mathbb{I}_t$ between double categories as follows.



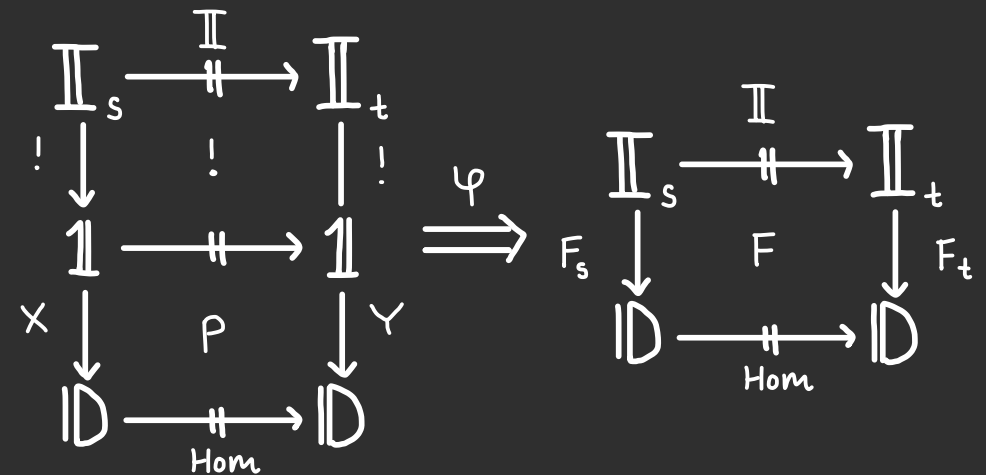
- We have 2-functors below, where $\text{Hom}(\text{ID})$ is $\pi: \text{ID} \times \mathbb{2} \rightarrow \mathbb{2}$.

$$\text{DBL}_{n\ell} / \mathbb{2} \begin{array}{c} \xrightarrow{\text{dom}} \\ \xleftarrow{\text{Hom}} \\ \xrightarrow{\text{cod}} \end{array} \text{DBL}_{n\ell}$$

- **shape** $\leadsto$ a loose distributor $\mathbb{I} \rightarrow \mathbb{2}$ or $\mathbb{I}_s \dashrightarrow \mathbb{I}_t$



- **cone** $\leadsto$ a modification φ between alterations OR



- **limit** $\leadsto$ terminal cone preserved by $\text{dom}, \text{cod}: \text{DBL}_{n\ell} / \mathbb{2} \rightarrow \text{DBL}_{n\ell}$.

MAIN RESULT & SUMMARY

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- A **parallel tabulator** is a limit whose shape is $\mathcal{D} \xrightarrow{\text{Hom}} \mathcal{D}$.
- A "**homologous limit**" is a limit whose shape is $\Pi_i(S) \xrightarrow{\text{Hom}} \Pi_i(T)$, but these are equivalent to limits indexed by spans $S \leftarrow I \rightarrow T$.
- A **parallel limit** is a limit whose shape is $\mathbb{I} \xrightarrow{\text{Hom}} \mathbb{I}$, and a **tight parallel limit** has shape $\Pi_i(\mathcal{C}) \xrightarrow{\text{Hom}} \Pi_i(\mathcal{C})$.

Theorem: A double category ID admits limits indexed by loose distributors if and only if ID admits:

- | | | |
|----------------------------------|---|--------------------------|
| (1) Parallel tabulators | } | Parallel limits |
| (2) Tight parallel limits | | |
| (3) Restrictions | } | Homologous limits |

- The double categories Span and IDist admit all limits indexed by double categories and all limits indexed by loose distributors.
- We have shown that limits indexed by loose distributors (or spans of functors) capture many existing concepts including:
 - * local and discrete limits
 - * bicategorical limits
 - * companions and conjoints
- Many current and future research directions.
 - * Sufficient conditions for completeness of $\text{Span}(\mathcal{E})$, $\text{Rel}(\mathcal{E})$, $\text{Mat}(\text{ID})$, $\text{Mod}(\text{ID})$, etc.
 - * Constructing (co)completions of double categories