LIMITS IN DOUBLE CATEGORIES, REVISITED

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MOTIVATION & OVERVIEW

- Double categories are a 2-dimensional structure with 2 kinds of morphism.
- The prototypical example is the double category $\mathbb{R}\mathrm{el}$ of sets, functions, and relations.
- Limits in double categories were studied in the seminal work of Grandis and Paré in 1999.
- In their work, limits are indexed by double categories, and are objects — but many examples are not objects!
- Today, I will introduce limits indexed by loose distributors between double categories; these are loose morphisms.
- Our running example is the double category $\mathbb{R}\mathrm{el}(\mathcal{C})$.

PART 1: Background on double categories & relations.

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "S"', "\shortmid"{marking}] & D \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} R \arrow[r, tail, "\langle \ell_R, r_R \rangle"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & A \times B \arrow[d, "f \times g"] \\ S \arrow[r, tail, "\langle \ell_S, r_S \rangle"'] & C \times D \end{tikzcd}$$

PART 2: Limits indexed by double categories $\mathbb{I}$

$$\begin{tikzcd} & \lim F \arrow[dl, "\gamma_A"'] \arrow[dr, "\gamma_B"] & \\ FA \arrow[rr, "Ff"'] & & FB \end{tikzcd} \qquad \begin{tikzcd} \lim F \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\gamma_C"'] \arrow[dr, phantom, "\gamma_p"] & \lim F \arrow[d, "\gamma_D"] \\ FC \arrow[r, "Fp"', "\shortmid"{marking}] & FD \end{tikzcd}$$

PART 3: Limits indexed by loose distributors $\mathbb{I}_s \xrightarrow{\shortmid\shortmid} \mathbb{I}_t$

$$\begin{tikzcd} \lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\Theta_q"] & \lim G \arrow[d, "\psi_x"] \\ FA \arrow[r, "\Phi q"', "\shortmid"{marking}] & GX \end{tikzcd}$$

BACKGROUND ON DOUBLE CATEGORIES AND RELATIONS

DOUBLE CATEGORIES

A double category $\mathbb{D}$ consists of:

- objects $A, B, C, D, \ldots$
- tight morphisms $A \longrightarrow B$ (usually drawn vertically)
- loose morphisms $A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ (usually drawn horizontally)
- cells

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "q"', "\shortmid"{marking}] & D \end{tikzcd}$$

- Identities 1_A and composition $g \circ f$ in tight direction
- Identities id_A and composition $p \odot q$ in loose direction

$$\begin{tikzcd} \cdot \arrow[r, "\mathrm{id}_A \odot p", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\ell(p)"] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, "p"', "\shortmid"{marking}] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "p \odot \mathrm{id}_B", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "r(p)"] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, "p"', "\shortmid"{marking}] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "p \odot (q \odot r)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "a(p,q,r)"] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, "(p \odot q) \odot r"', "\shortmid"{marking}] & \cdot \end{tikzcd}$$

left unitor — right unitor — associator

A double category is a pseudo category object in the 2-category CAT of locally small categories.

$$\begin{tikzcd} D_0 \arrow[r, "\mathrm{id}" description] & D_1 \arrow[l, shift right=2, "\mathrm{dom}"'] \arrow[l, shift left=2, "\mathrm{cod}"] & D_2 = D_1 \times_{D_0} D_1 \arrow[l, "\odot" description] \arrow[l, shift right=2, "\pi_1"'] \arrow[l, shift left=2, "\pi_2"] \end{tikzcd}$$

Examples

- Categories, monoidal categories, 2-categories, bicategories
- $\mathbb{S}\mathrm{pan}$ - sets, functions, spans, span morphisms
- $\mathbb{D}\mathrm{ist}$ - categories, functors, distributors/profunctors
- $\mathbb{M}\mathrm{at}(\mathcal{V})$ - sets, functions, matrices in $\mathcal{V}$ dist. monoidal
- $\mathbb{M}\mathrm{od}(\mathcal{V})$ - monoids, homomorphisms, bimodules in nice $\mathcal{V}$
- $\mathbb{L}\mathrm{oc}$ - locales/frames, homomorphisms, left exact functions

DOUBLE FUNCTORS & TRANSFORMATIONS

A lax double functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ is an assignment

$$\begin{tikzcd} A \arrow[r, "p", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "q"', "|"{marking}] & D \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} FA \arrow[r, "Fp", "|"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\alpha"] & FC \arrow[d, "Fg"] \\ FB \arrow[r, "Fq"', "|"{marking}] & FD \end{tikzcd}$$

preserving tight identities & composites, together with unitor and laxator cells (satisfying several axioms):

$$\begin{tikzcd} FA \arrow[r, "\mathrm{id}_{FA}", "|"{marking}] \arrow[d, "1_{FA}"'] \arrow[dr, phantom, "\eta_A"] & FA \arrow[d, "1_{FA}"] \\ FA \arrow[r, "F(\mathrm{id}_A)"', "|"{marking}] & FA \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "Fp", "|"{marking}] \arrow[d, "1_{FA}"'] \arrow[drr, phantom, "\mu_{p,q}"] & FB \arrow[r, "Fq", "|"{marking}] & FC \arrow[d, "1_{FC}"] \\ FA \arrow[rr, "F(p \odot q)"', "|"{marking}] & & FC \end{tikzcd}$$

- Called normal if η_A is identity cell, pseudo if η_A and $\mu_{p,q}$ are invertible, strict if identities.
- For a colax double functor, flip η_A and $\mu_{p,q}$.

A transformation between (co)lax double functors

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \varphi"] & \mathbb{D} \end{tikzcd}$$

consists of a family of cells

$$\begin{tikzcd} FA \arrow[r, "Fp", "|"{marking}] \arrow[d, "\varphi_A"'] \arrow[dr, phantom, "\varphi_p"] & FB \arrow[d, "\varphi_B"] \\ GA \arrow[r, "Gp"', "|"{marking}] & GB \end{tikzcd}$$

which are natural and satisfy certain coherence axioms.

To study limits, we are interested in the 2-category $\mathbb{DBL}_{nl}$ of double categories, normal lax functors, and transformations.

For colimits, we work with $\mathbb{DBL}_{nc}$ instead.

FROM REGULAR CATEGORIES TO RELATIONS

- A regular epimorphism is a coequaliser of some parallel pair of morphisms.

- A category with finite limits is called regular if:

  * coequalisers of kernel pairs exist;

$$\begin{tikzcd} X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} X \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & A \arrow[r, two heads, "q"] & Q \end{tikzcd}$$

  * regular epimorphisms are stable under pullback.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad f \text{ reg. epi.} \Rightarrow g \text{ reg. epi.}$$

Given a regular category $\mathcal{C}$, let $\mathbb{R}\mathrm{el}(\mathcal{C})$ be the double category of relations in $\mathcal{C}$, whose:

- objects and tight morphisms are the objects and morphisms of $\mathcal{C}$;

- loose morphisms $A \overset{R}{\nrightarrow} B$ are relations in $\mathcal{C}$, that is, monomorphisms $\langle \ell_R, r_R \rangle : R \rightarrowtail A \times B$.

- cells with boundary

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "S"', "\shortmid"{marking}] & D \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} R \arrow[r, tail, "\langle \ell_R, r_R \rangle"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & A \times B \arrow[d, "f \times g"] \\ S \arrow[r, tail, "\langle \ell_S, r_S \rangle"'] & C \times D \end{tikzcd}$$

THE DOUBLE CATEGORY OF RELATIONS

- The identity relation $A \overset{\mathrm{id}_A}{\longrightarrow} A$ is $\underset{\Delta_A}{\underset{\parallel}{\langle 1_A, 1_A \rangle}} : A \rightarrowtail A \times A$.

- The composite $R \circ S$ of relations

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid" marking] & B \arrow[r, "S", "\shortmid" marking] & C \end{tikzcd}$$

is computed as follows:

$$\begin{tikzcd}
R \circ S \arrow[d, tail, "\text{mono.}"'] & R \times_B S \arrow[l, two heads, "\text{reg. epi.}"'] \arrow[r] \arrow[d, tail] \arrow[dl, phantom, "(*)"] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, tail] \\
A \times C & R \times S \arrow[l, "\ell_R \times r_S"] \arrow[r, "r_R \times \ell_S"'] & B \times B
\end{tikzcd}$$

where $(*)$ is from the (reg. epi., mono) factorisation in $\mathbb{C}$.

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \arrow[r, two heads, "q"] \arrow[dr, "f"'] & If \arrow[d, tail, "i"] \\
 & & Y
\end{tikzcd}$$

Exercise: show that cells in $\mathbb{R}\mathrm{el}(\mathbb{C})$ are closed under tight composition (easy) and loose composition (harder).

The double category $\mathbb{R}\mathrm{el}(\mathbb{C})$ determines a (pseudo) category object

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{id}" description] & \mathrm{Rel}(\mathbb{C})_1 \arrow[l, shift right=2ex, "\mathrm{dom}"'] \arrow[l, shift left=2ex, "\mathrm{cod}"] & \mathrm{Rel}(\mathbb{C})_1 \times_{\mathbb{C}} \mathrm{Rel}(\mathbb{C})_1 \arrow[l, "\odot"']
\end{tikzcd}$$

where $\mathrm{Rel}(\mathbb{C})_1$ is the category of relations and cells.

$$\begin{tikzcd}
A \arrow[d, "f"'] & & A \arrow[r, "R", "\shortmid" marking] \arrow[d, "f"'] & B \arrow[d, "g"] & & B \arrow[d, "g"] \\
C & \overset{\mathrm{dom}}{\longmapsfrom} & C \arrow[r, "S"', "\shortmid" marking] & D & \overset{\mathrm{cod}}{\longmapsto} & D
\end{tikzcd}$$

LIMITS INDEXED BY DOUBLE CATEGORIES

LIMITS INDEXED BY DOUBLE CATEGORIES

- A (lax) diagram is a normal lax double functor $F: \mathbb{I} \to \mathbb{D}$ whose domain, called the shape or index, is small.

- A cone (X, γ) over a diagram $F: \mathbb{I} \to \mathbb{D}$ is an object X in $\mathbb{D}$ and a transformation γ in $\mathbf{DBL}_{nl}$.

$$\begin{tikzcd} \mathbb{I} \arrow[r, "!"] \arrow[d, "F"'] & \mathbb{1} \arrow[dl, "X"] \\ \mathbb{D} \arrow[ur, phantom, "\Leftarrow\gamma", near start] & \end{tikzcd} = \begin{tikzcd} \mathbb{I} \arrow[r, "!"] \arrow[d, "F"'] & \mathbb{1} \arrow[dl, "Y"'] \arrow[dl, bend left=50, "X"] \\ \mathbb{D} \arrow[ur, phantom, "\Leftarrow\psi", near start] & \end{tikzcd}$$

- A morphism of cones $f: (X, \gamma) \to (Y, \psi)$ is a tight morphism $f: X \to Y$ such that $\psi \circ f = \gamma$.

- A limit of $F: \mathbb{I} \to \mathbb{D}$ is a terminal object $(\lim F, \pi)$ in the category $\mathrm{Cone}(F)$ of cones over F.

This is an object $\lim F$ and a cone π which provides for each $f: A \to B$ and $p: C \nrightarrow D$

$$\begin{tikzcd} & \lim F \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ FA \arrow[rr, "Ff"'] & & FB \end{tikzcd} \qquad \begin{tikzcd} \lim F \arrow[r, "\mathrm{id}", "|" marking] \arrow[d, "\pi_C"'] \arrow[dr, phantom, "\pi_p"] & \lim F \arrow[d, "\pi_D"] \\ FC \arrow[r, "Fp"', "|" marking] & FD \end{tikzcd}$$

natural w.r.t. cells in $\mathbb{I}$ such that $\pi_{\mathrm{id}_A} = \mathrm{id}_{\pi_A}$ and

$$\begin{tikzcd} \lim F \arrow[r, "\mathrm{id}", "|" marking] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & \lim F \arrow[r, "\mathrm{id}", "|" marking] \arrow[d, "\pi_B"] \arrow[dr, phantom, "\pi_q"] & \lim F \arrow[d, "\pi_C"] \\ FA \arrow[r, "Fp", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\mu_{p,q}"] & FB \arrow[r, "Fq", "|" marking] & FC \arrow[d, equal] \\ FA \arrow[rr, "F(p \odot q)"', "|" marking] & & FC \end{tikzcd} = \begin{tikzcd} \lim F \arrow[r, "\mathrm{id} \odot \mathrm{id}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \lim F \arrow[d, equal] \\ \lim F \arrow[r, "\mathrm{id}", "|" marking] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_{p \odot q}"] & \lim F \arrow[d, "\pi_C"] \\ FA \arrow[r, "F(p \odot q)"', "|" marking] & FC \end{tikzcd}$$

Theorem (Grandis-Paré, 99)

A double category $\mathbb{D}$ admits limits indexed by any double category $\mathbb{I}$ <u>if and only if</u> $\mathbb{D}$ admits tight limits and tabulators.

TABULATORS

- The tabulator of a loose morphism $p : A \looparrowright B$ is a cell

$$\begin{tikzcd} T_p \arrow[r, "id", "\shortmid"{marking}] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & T_p \arrow[d, "\pi_B"] \\ A \arrow[r, "p"', "\shortmid"{marking}] & B \end{tikzcd}$$

such that for any cell α as below, there exists a unique tight morphism $u : X \longrightarrow T_p$ such that the following equation holds.

$$\begin{tikzcd} X \arrow[r, "id", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & X \arrow[d, "g"] \\ A \arrow[r, "p"', "\shortmid"{marking}] & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "id", "\shortmid"{marking}] \arrow[d, "u"'] \arrow[dr, phantom, "id_u"] & X \arrow[d, "u"] \\ T_p \arrow[r, "id", "\shortmid"{marking}] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & T_p \arrow[d, "\pi_B"] \\ A \arrow[r, "p"', "\shortmid"{marking}] & B \end{tikzcd}$$

- A tabulator is a limit whose shape is:

$$\mathbb{2} = \{ 0 \looparrowright 1 \}$$

- $\mathbb{D}$ admits all tabulators if and only if the functor $id : \mathbb{D}_0 \longrightarrow \mathbb{D}_1$ has a right adjoint $T : \mathbb{D}_1 \longrightarrow \mathbb{D}_0$.

$\mathbb{R}el(\mathcal{C})$ has tabulators.

The tabulator of $R : A \looparrowright B$ is the cell:

$$\begin{tikzcd} R \arrow[r, tail, "\Delta_R"] \arrow[d, "1_R"'] & R \times R \arrow[d, "\ell_R \times r_R"] \\ R \arrow[r, tail, "\langle \ell_R, r_R \rangle"'] & A \times B \end{tikzcd}$$

TIGHT LIMITS

- For each category $\mathcal{C}$, there is a double category $\mathbb{T}\mathrm{i}(\mathcal{C})$ whose:
  - objects and tight morphisms are the objects and morphisms of $\mathcal{C}$;
  - loose morphisms and cells are identities.
- A tight limit is a limit whose shape is $\mathbb{T}\mathrm{i}(\mathcal{C})$ for some category $\mathcal{C}$.
- Tight limits in a double category $\mathbb{D}$ are precisely limits in the underlying category $\mathbb{D}_0$ of objects and tight morphisms.

$$\mathbb{T}\mathrm{i}(\mathcal{C}) \xrightarrow{F} \mathbb{D} \qquad \rightsquigarrow \qquad \mathcal{C} \xrightarrow{F_0} \mathbb{D}_0$$

- $\mathbb{R}\mathrm{el}(\mathcal{C})$ admits all finite tight limits, since $\mathcal{C}$ has all finite limits.

Proposition

The double category $\mathbb{R}\mathrm{el}(\mathcal{C})$ of relations in a regular category $\mathcal{C}$ admits all limits indexed by finite double categories. If $\mathcal{C}$ admits small limits, then so does $\mathbb{R}\mathrm{el}(\mathcal{C})$.

Moreover, $\mathbb{R}\mathrm{el}(\mathcal{C})$ admits all colimits indexed by double categories if and only if $\mathcal{C}$ is cocomplete.

- To capture richer notions of limits in double categories, we need to index by a different type of shape!

LIMITS INDEXED BY LOOSE DISTRIBUTORS

LOOSE DISTRIBUTORS

- In ordinary category theory, a distributor or profunctor $P : \mathcal{C} \nrightarrow \mathcal{D}$ between categories is equivalent to a functor into the interval category $\underline{2}$.

- In double category theory, we have two kinds of intervals: tight $\{\cdot \to \cdot\}$ and loose $\{\cdot \nrightarrow \cdot\}$.

- A loose distributor is a double functor $\mathbb{I} \longrightarrow \underline{2}$ into the loose interval.

- A loose distributor is seen as a morphism $\mathbb{I}_s \nrightarrow \mathbb{I}_t$ between double categories as follows.

$$
\begin{tikzcd}
\mathbb{I}_s \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{I} \arrow[d] & \mathbb{I}_t \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathbb{1} \arrow[r, hook, "s"'] & \mathbb{2} & \mathbb{1} \arrow[l, hook', "t"]
\end{tikzcd}
$$

- We can depict a loose distributor as a double category with some marked loose arrows, often called loose heteromorphisms.

$$
\begin{tikzcd}
0 \arrow[d] & 2 \arrow[d] \\
1 \arrow[r, "\shortmid"{marking}] & 3
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
0 \arrow[r, "\shortmid"{marking}] \arrow[d, "\shortmid"{marking}] \arrow[dr, "\shortmid"{marking}] & 2 \arrow[d, "\shortmid"{marking}] \\
1 \arrow[r, "\shortmid"{marking}] & 3
\end{tikzcd}
$$

- For each double category $\mathbb{D}$, there is a Hom loose distributor given by $\mathbb{D} \times \mathbb{2} \xrightarrow{\pi} \mathbb{2}$.

- Unlike distributors between categories, loose distributors do not compose in general.

ALTERATIONS & MODIFICATIONS

· A (normal lax) alteration with frame

$$\begin{tikzcd} \mathbb{I}_s \arrow[r, "\mathbb{I}", "\shortmid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Phi"] & \mathbb{I}_t \arrow[d, "G"] \\ \mathbb{J}_s \arrow[r, "\mathbb{J}"', "\shortmid" marking] & \mathbb{J}_t \end{tikzcd}$$

F, G unitary lax functors

$\mathbb{I}, \mathbb{J}$ loose distributors

is a normal lax functor $\Phi : \mathbb{I} \longrightarrow \mathbb{J}$ over $\mathbb{2}$ such that the following equation holds.

$$\begin{tikzcd} \mathbb{I}_s \arrow[r, hook] \arrow[d, "F"'] & \mathbb{I} \arrow[d, "\Phi"] & \mathbb{I}_t \arrow[l, hook'] \arrow[d, "G"] \\ \mathbb{J}_s \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{J} \arrow[d] & \mathbb{J}_t \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ \mathbb{1} \arrow[r, hook, "s"'] & \mathbb{2} & \mathbb{1} \arrow[l, hook', "t"] \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbb{I}_s \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{I} \arrow[d] & \mathbb{I}_t \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ \mathbb{1} \arrow[r, hook, "s"'] & \mathbb{2} & \mathbb{1} \arrow[l, hook', "t"] \end{tikzcd}$$

· A modification between alterations

$$\begin{tikzcd} \mathbb{I}_s \arrow[r, "\mathbb{I}", "\shortmid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Phi"] & \mathbb{I}_t \arrow[d, "G"] \\ \mathbb{J}_s \arrow[r, "\mathbb{J}"', "\shortmid" marking] & \mathbb{J}_t \end{tikzcd} \quad \overset{m}{\Longrightarrow} \quad \begin{tikzcd} \mathbb{I}_s \arrow[r, "\mathbb{I}", "\shortmid" marking] \arrow[d, "F'"'] \arrow[dr, phantom, "\Phi'"] & \mathbb{I}_t \arrow[d, "G'"] \\ \mathbb{J}_s \arrow[r, "\mathbb{J}"', "\shortmid" marking] & \mathbb{J}_t \end{tikzcd}$$

is a transformation $m : \Phi \Rightarrow \Phi'$ over $\mathbb{2}$ such that the following diagram "commutes".

$$\begin{tikzcd} \mathbb{I}_s \arrow[r, hook] \arrow[d, bend right, "F"'] \arrow[d, bend left, "F'"] \arrow[d, phantom, "\Rightarrow"] & \mathbb{I} \arrow[d, bend right, "\Phi"'] \arrow[d, bend left, "\Phi'"] \arrow[d, phantom, "\overset{m}{\Rightarrow}"] & \mathbb{I}_t \arrow[l, hook'] \arrow[d, bend right, "G"'] \arrow[d, bend left, "G'"] \arrow[d, phantom, "\Rightarrow"] \\ \mathbb{J}_s \arrow[r, hook] & \mathbb{J} & \mathbb{J}_t \arrow[l, hook'] \end{tikzcd}$$

· There is a 2-category $\mathbb{L}\mathrm{Dist}_{nl} = \mathrm{DBL}_{nl}/\mathbb{2}$ of loose distributors, alterations, and modifications.

LIMITS INDEXED BY LOOSE DISTRIBUTORS

Suppose $(\lim F, \gamma)$ and $(\lim G, \psi)$ are limits of $F : \mathbb{I}_s \to \mathbb{D}$ and $G : \mathbb{I}_t \to \mathbb{D}$, respectively. The limit of an alteration

$$\begin{tikzcd}
\mathbb{I}_s \arrow[r, "\mathbb{I}", "+" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Phi"] & \mathbb{I}_t \arrow[d, "G"] \\
\mathbb{D} \arrow[r, "\mathrm{Hom}"', "+" marking] & \mathbb{D}
\end{tikzcd}$$

is a loose morphism $\lim \Phi : \lim F \nrightarrow \lim G$ in $\mathbb{D}$ and a terminal cone θ which provides for each heteromorphism $p : A \nrightarrow X$ in $\mathbb{I}$ a cell θ_p in $\mathbb{D}$ satisfying several natural axioms.

$$\begin{tikzcd}
\lim F \arrow[r, "\lim \Phi", "|" marking] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[d, "\psi_X"] \\
FA \arrow[r, "\Phi p"', "|" marking] & GX
\end{tikzcd}$$

· Equivalently, an alteration $\Phi : \mathbb{I} \to \mathbb{D} \times \mathbb{2}$ into the Hom loose distributor determines a span

loose distributor

diagram

$$\begin{tikzcd}
\mathbb{I} \arrow[r] \arrow[d] & \mathbb{2} \arrow[dl, bend left, "\lim"] \arrow[dl, phantom, "\overset{\theta}{\Leftarrow}"] \\
\mathbb{D} &
\end{tikzcd}$$

and its limit is the pointwise right extension in $\mathbf{DBL}_{n\ell}$, that is, such that the pullback along $s, t : \mathbb{1} \to \mathbb{2}$ yields limit cones $(\lim F, \gamma)$ and $(\lim G, \psi)$.

⚠ Limits of alterations can be pathological unless $\mathbb{D}$ is replete : $\langle \mathrm{dom}, \mathrm{cod} \rangle : D_1 \to D_0 \times D_0$ is an isofibration.

$$\begin{tikzcd}
\bullet \arrow[r, "\lim \Phi", "|" marking] \arrow[d, "\cong"'] & \bullet \arrow[d, "\cong"] \\
\bullet & \bullet
\end{tikzcd}$$

COMPANIONS & CONJOINTS

- A tight morphism $f : A \to B$ has a **companion** loose morphism $f_* : A \nrightarrow B$ if there are cells

$$\begin{tikzcd} A \arrow[r, "f_*", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\sigma"] & B \arrow[d, "1_B"] \\ B \arrow[r, "id_B"', "\shortmid"{marking}] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "id_A", "\shortmid"{marking}] \arrow[d, "1_A"'] \arrow[dr, phantom, "\tau"] & A \arrow[d, "f"] \\ A \arrow[r, "f_*"', "\shortmid"{marking}] & B \end{tikzcd}$$

such that $\tau \odot \sigma = 1_{f_*}$ and $\sigma \circ \tau = id_f$.

- In $\mathbb{R}el(\mathcal{C})$, the companion of $f : A \to B$ is the relation $\langle 1_A, f \rangle : A \rightarrowtail A \times B$ with cells:

$$\begin{tikzcd} A \arrow[r, tail, "\langle 1_A, f \rangle"] \arrow[d, "f"'] & A \times B \arrow[d, "f \times 1_B"] \\ B \arrow[r, tail, "\Delta_B"'] & B \times B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, tail, "\Delta_A"] \arrow[d, "1_A"'] & A \times A \arrow[d, "1_A \times f"] \\ A \arrow[r, tail, "\langle 1_A, f \rangle"'] & A \times B \end{tikzcd}$$

- A tight morphism $f : A \to B$ has a **conjoint** loose morphism $f^* : B \nrightarrow A$ if there are cells

$$\begin{tikzcd} A \arrow[r, "id_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\varepsilon"] & A \arrow[d, "1_A"] \\ B \arrow[r, "f^*"', "\shortmid"{marking}] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f^*", "\shortmid"{marking}] \arrow[d, "1_B"'] \arrow[dr, phantom, "\eta"] & A \arrow[d, "f"] \\ B \arrow[r, "id_B"', "\shortmid"{marking}] & B \end{tikzcd}$$

such that $\eta \odot \varepsilon = 1_{f^*}$ and $\eta \circ \varepsilon = id_f$.

- In $\mathbb{R}el(\mathcal{C})$, the conjoint of $f : A \to B$ is the relation $\langle f, 1_A \rangle : A \rightarrowtail B \times A$ with cells:

$$\begin{tikzcd} A \arrow[r, tail, "\Delta_A"] \arrow[d, "1_A"'] & A \times A \arrow[d, "f \times 1_A"] \\ A \arrow[r, tail, "\langle f, 1_A \rangle"'] & B \times A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, tail, "\langle f, 1_A \rangle"] \arrow[d, "f"'] & B \times A \arrow[d, "1_B \times f"] \\ B \arrow[r, tail, "\Delta_B"'] & B \times B \end{tikzcd}$$

COMPANIONS & CONJOINTS AS LIMITS

- A double category admits **companions** if and only if it admits limits of an alteration whose shape is

$$\begin{tikzcd} 0 \arrow[d] & \\ 1 \arrow[r, "\shortmid"{marking}] & 2 \end{tikzcd}$$

heteromorphism marked in orange

where $1 \rightarrow\!\!\!\!\shortmid\, 2$ is sent to an identity loose morphism.

$$\begin{tikzcd} X \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "fg"'] \arrow[dr, phantom, "\forall \alpha"] & Y \arrow[d, "h"] \\ B \arrow[r, "\mathrm{id}_B"', "\shortmid"{marking}] & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\exists !"] & Y \arrow[d, "h"] \\ A \arrow[r, "\lim = f_*", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[d, equal] \\ B \arrow[r, "\mathrm{id}_B"', "\shortmid"{marking}] & B \end{tikzcd}$$

- A double category admits **conjoints** if and only if it admits limits of an alteration whose shape is

$$\begin{tikzcd} & 0 \arrow[d] \\ 1 \arrow[r, "\shortmid"{marking}] & 2 \end{tikzcd}$$

heteromorphism marked in orange

where $1 \rightarrow\!\!\!\!\shortmid\, 2$ is sent to an identity loose morphism.

- Companions and conjoints also arise as colimits of:

$$\begin{tikzcd} 1 \arrow[r, "\shortmid"{marking}] & 0 \arrow[d] \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, "\shortmid"{marking}] \arrow[d] & 2 \\ 1 & \end{tikzcd}$$

- Companions and conjoints are **absolute limits** – they are preserved by every normal lax double functor.

RESTRICTIONS & EXTENSIONS

· A restriction of a niche (f,s,g) is a cell $\mathrm{res}(f,s,g)$ with the following universal property:

$$
\begin{tikzcd}
X \arrow[r, "r", "\shortmid"{marking}] \arrow[d, "fh"'] & Y \arrow[d, "gk"] \\
C \arrow[r, "s"', "\shortmid"{marking}] \arrow[ur, phantom, "\alpha"] & D
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "r", "\shortmid"{marking}] \arrow[d, "h"'] & Y \arrow[d, "k"] \\
A \arrow[r, "f_* \odot s \odot g^*", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[ur, phantom, "\exists!"] & B \arrow[d, "g"] \\
C \arrow[r, "s"', "\shortmid"{marking}] \arrow[ur, phantom, "\mathrm{res}(f,s,g)"] & D
\end{tikzcd}
$$

· An extension of a co-niche (h,r,k) is a cell $\mathrm{ext}(h,r,k)$ with the following universal property:

$$
\begin{tikzcd}
X \arrow[r, "r", "\shortmid"{marking}] \arrow[d, "fh"'] & Y \arrow[d, "gk"] \\
C \arrow[r, "s"', "\shortmid"{marking}] \arrow[ur, phantom, "\alpha"] & D
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "r", "\shortmid"{marking}] \arrow[d, "h"'] & Y \arrow[d, "k"] \\
A \arrow[r, "h^* \odot r \odot k_*", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[ur, phantom, "\mathrm{ext}(h,r,k)"] & B \arrow[d, "g"] \\
C \arrow[r, "s"', "\shortmid"{marking}] \arrow[ur, phantom, "\exists!"] & D
\end{tikzcd}
$$

· For a double category $\mathbb{D}$, the following are equivalent:

* $\mathbb{D}$ has restrictions
* $\mathbb{D}$ has extensions
* $\mathbb{D}$ has companions and conjoints
* the functor $\langle \mathrm{dom}, \mathrm{cod} \rangle : \mathbb{D}_1 \longrightarrow \mathbb{D}_0 \times \mathbb{D}_0$ is a bifibration.

· In $\mathbb{Rel}(\mathcal{C})$, restrictions are computed by pullback and extensions are computed by factorisation.

$$
\begin{tikzcd}
\bullet \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \times B \arrow[d, "f \times g"] \\
S \arrow[r, tail, "\langle l_S, r_S \rangle"'] & C \times D
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, tail, "\langle l_R, r_R \rangle"] \arrow[d, "\text{reg. epi}"'] \arrow[dr, phantom, "\text{factorise}"] & X \times Y \arrow[d, "h \times k"] \\
\bullet \arrow[r, tail, "\text{mono.}"'] & A \times B
\end{tikzcd}
$$

· Consider an identity cell in $\mathbb{Rel}(\mathcal{C})$ on $f : A \to B$.

$$
\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A", "\shortmid"{marking}] \arrow[d, "f"'] & A \arrow[d, "f"] \\
B \arrow[r, "\mathrm{id}_B"', "\shortmid"{marking}] \arrow[ur, phantom, "\mathrm{id}_f"] & B
\end{tikzcd}
$$

* f is mono $\Leftrightarrow$ $\mathrm{id}_f \cong \mathrm{res}(f, \mathrm{id}_B, f)$
* f is reg. epi $\Leftrightarrow$ $\mathrm{id}_f \cong \mathrm{ext}(f, \mathrm{id}_A, f)$

RESTRICTIONS & EXTENSIONS AS (CO)LIMITS

- A restriction is a limit whose shape is:

$$\begin{tikzcd} 0 \arrow[d] & 2 \arrow[d] \\ 1 \arrow[r, "|" marking] & 3 \end{tikzcd}$$

heteromorphism marked in orange

- An extension is a colimit whose shape is:

$$\begin{tikzcd} 0 \arrow[d] \arrow[r, "|" marking] & 2 \arrow[d] \\ 1 & 3 \end{tikzcd}$$

heteromorphism marked in orange

- Restrictions and extensions play an important role throughout double category theory, and often interact well with other kinds of (co)limits.

- A tabulator is called effective if the corresponding cell is an extension – a kind of exactness property.

$$\begin{tikzcd} Tp \arrow[r, "id", "|" marking] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & Tp \arrow[d, "\pi_B"] \\ A \arrow[r, "p"', "|" marking] & B \end{tikzcd}$$

- $\mathbb{R}el(\mathcal{C})$ has effective tabulators.

- The image factorisation can be computed as:

$$\begin{tikzcd} A \arrow[r, "id", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "ext"] & A \arrow[d, "!"] \\ B \arrow[r, "|" marking] & 1 \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r, "id", "|" marking] \arrow[d, "\text{reg. epi.}"'] \arrow[dr, phantom, "\exists!"] & A \arrow[d, "\text{reg. epi.}"] \\ Im(f) \arrow[r, "|" marking] \arrow[d, "\text{mono.}"'] \arrow[dr, phantom, "tab"] & Im(f) \arrow[d] \\ B \arrow[r, "|" marking] & 1 \end{tikzcd}$$

terminal object

PARALLEL PRODUCTS & PARALLEL LIMITS

- A parallel product is a limit whose shape is:

$$\boxed{\begin{tikzcd} 0 \arrow[r, "|" marking] & 2 \\ 1 \arrow[r, "|" marking] & 3 \end{tikzcd}}$$

heteromorphisms
marked in
orange

- The parallel product of loose morphisms $p : A \nrightarrow B$ and $q : C \nrightarrow D$ is a loose morphism $p \times q : A \times C \nrightarrow B \times D$ together with projection cells

$$\begin{tikzcd} A \times C \arrow[r, "p \times q", "|" marking] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & B \times D \arrow[d, "\pi_B"] \\ A \arrow[r, "|" marking, "p"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \times C \arrow[r, "p \times q", "|" marking] \arrow[d, "\pi_C"'] \arrow[dr, phantom, "\pi_q"] & B \times D \arrow[d, "\pi_D"] \\ C \arrow[r, "|" marking, "q"'] & D \end{tikzcd}$$

satisfying the appropriate universal property.

- A double category with all products and parallel products is a cartesian double category.

- $\mathbb{R}\mathrm{el}(\mathcal{C})$ has parallel products: given $R : A \nrightarrow B$ and $S : C \nrightarrow D$ we have $R \times S : A \times C \nrightarrow B \times D$.

$$R \times S \xrightarrow{\langle \ell_R, \ell_S \rangle \times \langle r_R, r_S \rangle} (A \times C) \times (B \times D)$$

- More generally, a tight parallel limit is a limit whose shape is the Hom loose distributor

$$\mathbb{T}\mathrm{i}(\mathcal{C}) \xrightarrow{\text{Hom}} \mathbb{T}\mathrm{i}(\mathcal{C})$$

for some category $\mathcal{C}$.

- $\mathbb{R}\mathrm{el}(\mathcal{C})$ admits all finite tight parallel limits; colimits are more subtle.

LOCAL PRODUCTS

- A local product is a limit whose shape is:

$$\boxed{\begin{tikzcd} 0 \arrow[r, "u", "\shortmid"{marking}] & 1 \\ 0 \arrow[r, "v", "\shortmid"{marking}] & 1 \end{tikzcd}}$$

heteromorphisms marked in orange

- The local product of $p_1, p_2 : A \nrightarrow B$ is a loose morphism $p_1 \wedge p_2 : A \nrightarrow B$ and projection cells with the universal property:

$$\begin{tikzcd} X \arrow[r, "r", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\forall \alpha_i"] & Y \arrow[d, "g"] \\ A \arrow[r, "p_i"', "\shortmid"{marking}] & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "r", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\exists!"] & Y \arrow[d, "g"] \\ A \arrow[r, "p_1 \wedge p_2", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\pi_i"] & B \arrow[d, equal] \\ A \arrow[r, "p_i"', "\shortmid"{marking}] & B \end{tikzcd}$$

$\boxed{i = 1,2}$

- A cartesian double category with restrictions has local products.

$$\begin{tikzcd} A \arrow[r, "p_1 \wedge p_2", "\shortmid"{marking}] \arrow[d, "\Delta_A"'] \arrow[dr, phantom, "\mathrm{res}"] & B \arrow[d, "\Delta_B"] \\ A \times A \arrow[r, "p_1 \times p_2"', "\shortmid"{marking}] & B \times B \end{tikzcd}$$

- $\mathbb{R}\mathrm{el}(\mathcal{C})$ has local products (and all local limits) given by "intersection" of relations.

- Local limits are not necessarily preserved by composition with loose morphisms.

$$\begin{tikzcd} A \arrow[r, "p_1 \wedge p_2", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "q", "\shortmid"{marking}] \arrow[d, phantom, "\exists!"] & C \arrow[d, equal] \\ A \arrow[rr, "(p_1 \odot q) \wedge (p_2 \odot q)"', "\shortmid"{marking}] & & C \end{tikzcd}$$

Not necessarily invertible

PARALLEL TABULATORS

- A parallel tabulator is a limit whose shape is

$$\mathbb{2} \xrightarrow[\text{Hom}]{+\!\!+} \mathbb{2}$$

- An alteration with this shape determines cells in $\mathbb{D}$

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, equal] & B \arrow[r, "r_2"] \arrow[d, phantom, "\alpha"] & D \arrow[d, equal] \\ A \arrow[rr, "r_3"'] & & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "r_1"] \arrow[d, equal] & C \arrow[r, "q"] \arrow[d, phantom, "\beta"] & D \arrow[d, equal] \\ A \arrow[rr, "r_3"'] & & D \end{tikzcd}$$

whose parallel tabulator is a loose morphism $Tp \rightarrow Tq$ between tabulators and a cone given by cells

$$\begin{tikzcd} Tp \arrow[r] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_{r_1}"] & Tq \arrow[d, "\pi_C"] \\ A \arrow[r, "r_1"'] & C \end{tikzcd} \qquad \begin{tikzcd} Tp \arrow[r] \arrow[d, "\pi_B"'] \arrow[dr, phantom, "\pi_{r_2}"] & Tq \arrow[d, "\pi_D"] \\ B \arrow[r, "r_2"'] & D \end{tikzcd} \qquad \begin{tikzcd} Tp \arrow[r] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_{r_3}"] & Tq \arrow[d, "\pi_D"] \\ A \arrow[r, "r_3"'] & D \end{tikzcd}$$

which are suitably compatible with α and β.

- $\mathbb{R}\mathrm{el}(\mathcal{C})$ has parallel tabulators, although they are a bit complex to compute.

- A parallel limit is a limit whose shape is

$$\mathbb{I} \xrightarrow[\text{Hom}]{+\!\!+} \mathbb{I}$$

Theorem (Grandis-Paré, 99)
A double category admits parallel limits if and only if it admits parallel tabulators and tight parallel limits.

Theorem: A double category $\mathbb{D}$ admits limits indexed by loose distributors if and only if $\mathbb{D}$ admits parallel limits and restrictions.

BIPRODUCTS IN $\mathbb{R}\mathrm{el}(\mathrm{Set})$ AS COLIMITS

- Consider the colimits in $\mathbb{R}\mathrm{el}(\mathrm{Set})$ with shapes:

$$\begin{tikzcd} 0 \arrow[dr, "|" marking] & \\ & 2 \\ 1 \arrow[ur, "|" marking] & \end{tikzcd} \qquad \begin{tikzcd} & 0 \\ 2 \arrow[ur, "|" marking] \arrow[dr, "|" marking] & \\ & 1 \end{tikzcd}$$

heteromorphisms marked in orange

- Given $R : A \to\!\!\!\!\!\!\;+\; C$ and $S : B \to\!\!\!\!\!\!\;+\; C$, we have a relation $A + B \to\!\!\!\!\!\!\;+\; C$ given by:

$$R+S \xrightarrow{\langle \ell_R, r_R\rangle + \langle \ell_S, r_S\rangle} (A\times C) + (B \times C) \cong (A+B)\times C$$

- Dually, given $R' : C \to\!\!\!\!\!\!\;+\; A$ and $S' : C \to\!\!\!\!\!\!\;+\; B$, we have a relation $C \to\!\!\!\!\!\!\;+\; A+B$ given by:

$$R'+S' \xrightarrow{\langle \ell_R', r_R'\rangle + \langle \ell_S', r_S'\rangle} (C\times A) + (C \times B) \cong C \times (A+B)$$

- The coprojection(s) are cell(s):

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "\amalg_A"'] \arrow[dr, phantom, "\theta_R"] & C \arrow[d, equal] \\ A+B \arrow[r, "\mathrm{colim}"', "|" marking] & C \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} R \arrow[r, tail] \arrow[d, "\amalg_R"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \times C \arrow[d, "\amalg_A \times 1_C"] \\ R+S \arrow[r, tail] & (A+B)\times C \end{tikzcd}$$

- Since Set is extensive, these are restriction cells, hence

$$A \xrightarrow{(\amalg_A)_*} A+B \xrightarrow{\mathrm{colim}} C \;=\; A \xrightarrow{R} C$$

$$B \xrightarrow{(\amalg_B)_*} A+B \xrightarrow{\mathrm{colim}} C \;=\; B \xrightarrow{R} C$$

$$C \xrightarrow{\mathrm{colim}} A+B \xrightarrow{(\amalg_A)^*} A \;=\; C \xrightarrow{R'} A$$

$$C \xrightarrow{\mathrm{colim}} A+B \xrightarrow{(\amalg_B)^*} B \;=\; C \xrightarrow{R} B$$

- Since colimits are unique up to isomorphism, we recover $A+B$ as a "biproduct in the category Rel", but demonstrate a far richer universal property in $\mathbb{R}\mathrm{el}$.

SUMMARY & FURTHER WORK

- Introduced limits indexed by loose distributors.

> **Theorem:** The double category $\mathbb{R}\mathrm{el}(\mathcal{C})$ of relations in a regular category $\mathcal{C}$ admits all finite limits indexed by double categories, and all finite limits indexed by loose distributors.

- The double category $\mathbb{R}\mathrm{el}(\mathcal{C})$ is far richer than the category of relations, and many natural constructions arise as limits in this setting, including companions, conjoints, restrictions, extensions, parallel products, local products, and "biproducts".

There are many avenues for further research:

- Exploring the relationship between $\mathbb{R}\mathrm{el}(\mathcal{C})$ and $\mathbb{S}\mathrm{pan}(\mathcal{C})$ – a reflective adjunction.
- Investigate (co)limits in other double categories of interest to categorical algebra.
- Developing theory of double limit sketches (see Lambert-Patterson, Cartesian double theories).
- Demonstrating how exactness properties in $\mathbb{R}\mathrm{el}(\mathcal{C})$ relate to properties of a regular category (e.g. Barr-exact, Maltsev, Goursat).
- Characterising $\mathbb{R}\mathrm{el}(\mathcal{C})$ as a free (co)completion under certain limits (see Lambert '22, Hoshino-Nasu '25).