

LIMITS IN IDOUBLE CATEGORIES, REVISITED

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MOTIVATION & OVERVIEW

01

- Double categories are a 2-dimensional structure with 2 kinds of morphism.
- The prototypical example is the double category \mathbf{IRel} of sets, functions, and relations.
- Limits in double categories were studied in the seminal work of Grandis and Paré in 1999.
- In their work, limits are indexed by double categories, and are *objects* — but many examples are not objects!
- Today, I will introduce limits indexed by *loose distributors* between double categories; these are *loose morphisms*.
- Our running example is the double category $\mathbf{IRel}(\mathcal{C})$.

PART 1: Background on double categories & relations.

$$\begin{array}{ccc}
 A & \xrightarrow{R} & B \\
 f \downarrow & & \downarrow g \\
 C & \xrightarrow{S} & D
 \end{array}
 \quad \rightsquigarrow \quad
 \begin{array}{ccc}
 R & \xrightarrow{\langle l_R, r_R \rangle} & A \times B \\
 \downarrow & \curvearrowright & \downarrow f \times g \\
 S & \xrightarrow{\langle l_S, r_S \rangle} & C \times D
 \end{array}$$

PART 2: Limits indexed by double categories II

$$\begin{array}{ccc}
 & \lim F & \\
 \gamma_A \swarrow & & \searrow \gamma_B \\
 FA & \xrightarrow{Ff} & FB
 \end{array}
 \qquad
 \begin{array}{ccc}
 \lim F & \xrightarrow{id} & \lim F \\
 \gamma_C \downarrow & \gamma_P & \downarrow \gamma_D \\
 FC & \xrightarrow{Fp} & FD
 \end{array}$$

PART 3: Limits indexed by loose distributors $\mathbb{I}_s \dashrightarrow \mathbb{I}_t$

$$\begin{array}{ccc}
 \lim F & \xrightarrow{\lim \Phi} & \lim G \\
 \gamma_A \downarrow & \Theta_q & \downarrow \gamma_X \\
 FA & \xrightarrow{\Phi_q} & GX
 \end{array}$$

BACKGROUND ON DOUBLE CATEGORIES AND RELATIONS

DOUBLE CATEGORIES

02

A **double category** \mathbb{D} consists of:

- objects A, B, C, D, \dots
- **tight morphisms** $A \rightarrow B$ (usually drawn vertically)
- **loose morphisms** $A \rightrightarrows B$ (usually drawn horizontally)
- cells

$$\begin{array}{ccc} A & \xrightarrow{p} & C \\ f \downarrow & \alpha & \downarrow g \\ B & \xrightarrow{q} & D \end{array}$$

- Identities 1_A and composition $g \circ f$ in tight direction
- Identities id_A and composition $p \circ q$ in loose direction

$\begin{array}{c} \bullet \xrightarrow{\text{id}_A \circ p} \bullet \\ \parallel \quad \underline{l}(p) \quad \parallel \\ \bullet \xrightarrow{p} \bullet \end{array}$	$\begin{array}{c} \bullet \xrightarrow{p \circ \text{id}_B} \bullet \\ \parallel \quad \underline{r}(p) \quad \parallel \\ \bullet \xrightarrow{p} \bullet \end{array}$	$\begin{array}{c} \bullet \xrightarrow{p \circ (q \circ r)} \bullet \\ \parallel \quad \underline{a}(p, q, r) \quad \parallel \\ \bullet \xrightarrow{(p \circ q) \circ r} \bullet \end{array}$
left unitor	right unitor	associator

A double category is a **pseudo category object** in the 2-category CAT of locally small categories.

$$\begin{array}{ccccc} & \xleftarrow{\text{dom}} & & \xleftarrow{\pi_1} & \\ D_0 & \xrightleftharpoons[\text{cod}]{\text{id}} & D_1 & \xrightleftharpoons[\pi_2]{\circ} & D_2 = D_1 \times_{D_0} D_1 \end{array}$$

Examples

- Categories, monoidal categories, 2-categories, bicategories
- **Span** - sets, functions, spans, span morphisms
- **IDist** - categories, functors, distributors/profunctors
- **Mat(V)** - sets, functions, matrices in V dist. monoidal
- **Mod(V)** - monoids, homomorphisms, bimodules in nice V
- **Loc** - locales/frames, homomorphisms, left exact functions

DOUBLE FUNCTORS & TRANSFORMATIONS

03

A **lax double functor** $F: \mathbb{C} \rightarrow \mathbb{D}$ is an assignment

$$\begin{array}{ccc} A & \xrightarrow{p} & C \\ f \downarrow & \alpha & \downarrow g \\ B & \xrightarrow{q} & D \end{array} \rightsquigarrow \begin{array}{ccc} FA & \xrightarrow{Fp} & FC \\ Ff \downarrow & F\alpha & \downarrow Fg \\ FB & \xrightarrow{Fq} & FD \end{array}$$

preserving tight identities & composites, together with unitor and laxator cells (satisfying several axioms):

$$\begin{array}{ccc} FA & \xrightarrow{id_{FA}} & FA \\ 1_{FA} \downarrow & \eta_A & \downarrow 1_{FA} \\ FA & \xrightarrow{F(id_A)} & FA \end{array} \quad \begin{array}{ccccc} FA & \xrightarrow{Fp} & FB & \xrightarrow{Fq} & FC \\ 1_{FA} \downarrow & & \mu_{p,q} & & \downarrow 1_{FC} \\ FA & \xrightarrow{F(p \circ q)} & FC \end{array}$$

- Called **normal** if η_A is identity cell, **pseudo** if η_A and $\mu_{p,q}$ are invertible, **strict** if identities.
- For a **colax double functor**, flip η_A and $\mu_{p,q}$.

A **transformation** between (co)lax double functors

$$\begin{array}{ccc} & F & \\ \mathbb{C} & \Downarrow \varphi & \mathbb{D} \\ & G & \end{array}$$

consists of a family of cells

$$\begin{array}{ccc} FA & \xrightarrow{Fp} & FB \\ \varphi_A \downarrow & \varphi_p & \downarrow \varphi_B \\ GA & \xrightarrow{Gp} & GB \end{array}$$

which are natural and satisfy certain coherence axioms.

To study **limits**, we are interested in the 2-category DBL_{nl} of double categories, normal lax functors, and transformations.

For **colimits**, we work with DBL_{nc} instead.

FROM REGULAR CATEGORIES TO RELATIONS

04

• A **regular epimorphism** is a coequaliser of some parallel pair of morphisms.

• A category with finite limits is called **regular** if:

* coequalisers of **kernel pairs** exist;

$$\begin{array}{ccc} X & \xrightarrow{\pi_2} & A \\ \pi_1 \downarrow \lrcorner & & \downarrow f \\ A & \xrightarrow{f} & B \end{array} \Rightarrow X \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\pi_2} \end{array} A \xrightarrow{q} Q$$

* regular epimorphisms are stable under pullback.

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ g \downarrow \lrcorner & & \downarrow f \\ \bullet & \longrightarrow & \bullet \end{array}$$

f reg. epi. \Rightarrow
 g reg. epi.

Given a regular category \mathcal{C} , let $\mathbf{Rel}(\mathcal{C})$ be the double category of relations in \mathcal{C} , whose:

• objects and tight morphisms are the objects and morphisms of \mathcal{C} ;

• loose morphisms $A \xrightarrow{R} B$ are **relations** in \mathcal{C} , that is, monomorphisms $\langle l_R, r_R \rangle: R \rightarrowtail A \times B$.

• cells with boundary

$$\begin{array}{ccc} A & \xrightarrow{R} & B \\ f \downarrow & & \downarrow g \\ C & \xrightarrow{S} & D \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} R & \xrightarrow{\langle l_R, r_R \rangle} & A \times B \\ \downarrow & \curvearrowright & \downarrow f \times g \\ S & \xrightarrow{\langle l_S, r_S \rangle} & C \times D \end{array}$$

THE DOUBLE CATEGORY OF RELATIONS

05

- The **identity relation** $A \xrightarrow{\text{id}_A} A$ is $\langle 1_A, 1_A \rangle: A \rightarrowtail A \times A$.
 \parallel
 Δ_A
- The **composite** $R \circ S$ of relations

$$A \xrightarrow{R} B \xrightarrow{S} C$$

is computed as follows:

$$\begin{array}{ccccc} R \circ S & \xleftarrow{\text{reg. epi.}} & R \times_B S & \longrightarrow & B \\ \text{mono.} \downarrow & (*) & \downarrow \lrcorner & & \downarrow \\ A \times C & \xleftarrow{l_R \times r_S} & R \times S & \xrightarrow{r_R \times l_S} & B \times B \end{array}$$

where $(*)$ is from the **(reg.epi., mono)** factorisation in \mathcal{C} .

$$\begin{array}{ccccc} \text{Ker}(f) & \xrightarrow{\pi_1} & X & \xrightarrow{q} & \text{If} \\ & \xrightarrow{\pi_2} & & & \downarrow i \\ & & & \searrow f & Y \end{array}$$

Exercise: show that cells in $\text{IRel}(\mathcal{C})$ are closed under tight composition (easy) and loose composition (harder).

The double category $\text{IRel}(\mathcal{C})$ determines a (pseudo) category object

$$\mathcal{C} \begin{array}{c} \xleftarrow{\text{dom}} \\ \xleftarrow{\text{id}} \\ \xleftarrow{\text{cod}} \end{array} \text{Rel}(\mathcal{C})_1 \xleftarrow{\odot} \text{Rel}(\mathcal{C})_1 \times_e \text{Rel}(\mathcal{C})_1$$

where $\text{Rel}(\mathcal{C})_1$ is the category of relations and cells.

$$\begin{array}{ccccc} A & & \xrightarrow{R} & B & \\ f \downarrow & \xleftarrow{\text{dom}} & f \downarrow & \downarrow g & \xrightarrow{\text{cod}} & B \\ C & & \xrightarrow{S} & D & & \downarrow g \\ & & & & & D \end{array}$$

LIMITS INDEXED BY DOUBLE CATEGORIES

LIMITS INDEXED BY DOUBLE CATEGORIES

06

• A **(lax) diagram** is a normal lax double functor $F: \mathbb{I} \rightarrow \mathbb{D}$ whose domain, called the **shape** or **index**, is small.

• A **cone** (X, γ) over a diagram $F: \mathbb{I} \rightarrow \mathbb{D}$ is an object X in \mathbb{D} and a transformation γ in DBL_{n1} .

$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{!} & \mathbb{1} \\ F \downarrow & \xleftarrow{\gamma} & \swarrow X \\ \mathbb{D} & & \end{array} = \begin{array}{ccc} \mathbb{I} & \xrightarrow{!} & \mathbb{1} \\ F \downarrow & \xleftarrow{\psi} & \swarrow \gamma \\ \mathbb{D} & & \end{array} \begin{array}{ccc} & \nearrow f & \\ & \nearrow X & \end{array}$$

• A **morphism of cones** $f: (X, \gamma) \rightarrow (Y, \psi)$ is a tight morphism $f: X \rightarrow Y$ such that $\psi \circ f = \gamma$.

• A **limit** of $F: \mathbb{I} \rightarrow \mathbb{D}$ is a terminal object $(\lim F, \pi)$ in the category $\text{Cone}(F)$ of cones over F .

This is an object $\lim F$ and a cone π which provides for each $f: A \rightarrow B$ and $p: C \rightarrow D$

$$\begin{array}{ccc} & \lim F & \\ \pi_A \swarrow & & \searrow \pi_B \\ FA & \xrightarrow{Ff} & FB \end{array} \quad \begin{array}{ccc} \lim F & \xrightarrow{id} & \lim F \\ \pi_C \downarrow & \pi_p & \downarrow \pi_D \\ FC & \xrightarrow{Fp} & FD \end{array}$$

natural w.r.t. cells in \mathbb{I} such that $\pi_{id_A} = id_{\pi_A}$ and

$$\begin{array}{ccccc} \lim F & \xrightarrow{id} & \lim F & \xrightarrow{id} & \lim F \\ \pi_A \downarrow & \pi_p & \downarrow \pi_B & \pi_q & \downarrow \pi_C \\ FA & \xrightarrow{Fp} & FB & \xrightarrow{Fq} & FC \\ \parallel & & \mu_{p,q} & & \parallel \\ FA & \xrightarrow{F(p \circ q)} & & & FC \end{array} = \begin{array}{ccc} \lim F & \xrightarrow{id \circ id} & \lim F \\ \parallel & \cong & \parallel \\ \lim F & \xrightarrow{id} & \lim F \\ \pi_A \downarrow & \pi_{p \circ q} & \downarrow \pi_C \\ FA & \xrightarrow{F(p \circ q)} & FC \end{array}$$

Theorem (Grandis-Paré, 99)

A double category \mathbb{D} admits limits indexed by any double category \mathbb{I} if and only if \mathbb{D} admits **tight limits** and **tabulators**.

TABULATORS

- The **tabulator** of a loose morphism $p: A \rightharpoonup B$ is a cell

$$\begin{array}{ccc} T_p & \xrightarrow{\text{id}} & T_p \\ \pi_A \downarrow & \pi_p & \downarrow \pi_B \\ A & \xrightarrow[p]{} & B \end{array}$$

such that for any cell α as below, there exists a unique tight morphism $u: X \rightarrow T_p$ such that the following equation holds.

$$\begin{array}{ccc} X & \xrightarrow{\text{id}} & X \\ f \downarrow & \alpha & \downarrow g \\ A & \xrightarrow[p]{} & B \end{array} = \begin{array}{ccc} X & \xrightarrow{\text{id}} & X \\ u \downarrow & \text{id}_u & \downarrow u \\ T_p & \xrightarrow{\text{id}} & T_p \\ \pi_A \downarrow & \pi_p & \downarrow \pi_B \\ A & \xrightarrow[p]{} & B \end{array}$$

- A tabulator is a limit whose shape is:

$$\mathcal{D} = \{0 \rightharpoonup 1\}$$

- ID admits all **tabulators** if and only if the functor $\text{id}: \text{ID}_0 \rightarrow \text{ID}_1$ has a **right adjoint** $T: \text{ID}_1 \rightarrow \text{ID}_0$.

$\text{IRel}(\mathcal{C})$ has tabulators.

The tabulator of $R: A \rightharpoonup B$ is the cell:

$$\begin{array}{ccc} R & \xrightarrow{\Delta_R} & R * R \\ 1_R \downarrow & & \downarrow l_R * r_R \\ R & \xrightarrow{\langle l_R, r_R \rangle} & A * B \end{array}$$

TIGHT LIMITS

- For each category \mathcal{C} , there is a double category $\Pi(\mathcal{C})$ whose:
 - objects and tight morphisms are the objects and morphisms of \mathcal{C} ;
 - loose morphisms and cells are identities.
- A **tight limit** is a limit whose shape is $\Pi(\mathcal{C})$ for some category \mathcal{C} .
- Tight limits in a double category \mathbb{D} are precisely limits in the underlying category D_0 of **objects** and **tight morphisms**.

$$\Pi(\mathcal{C}) \xrightarrow{F} \mathbb{D} \quad \rightsquigarrow \quad \mathcal{C} \xrightarrow{F_0} D_0$$

- $\mathbf{I}Rel(\mathcal{C})$ admits all finite tight limits, since \mathcal{C} has all finite limits.

Proposition

The double category $\mathbf{I}Rel(\mathcal{C})$ of relations in a regular category \mathcal{C} admits all limits indexed by finite double categories. If \mathcal{C} admits small limits, then so does $\mathbf{I}Rel(\mathcal{C})$.

Moreover, $\mathbf{I}Rel(\mathcal{C})$ admits all colimits indexed by double categories if and only if \mathcal{C} is cocomplete.

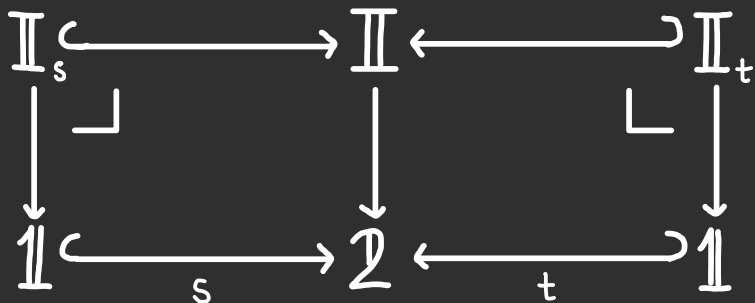
- To capture richer notions of limits in double categories, we need to index by a different type of **shape**!

LIMITS INDEXED BY LOOSE DISTRIBUTORS

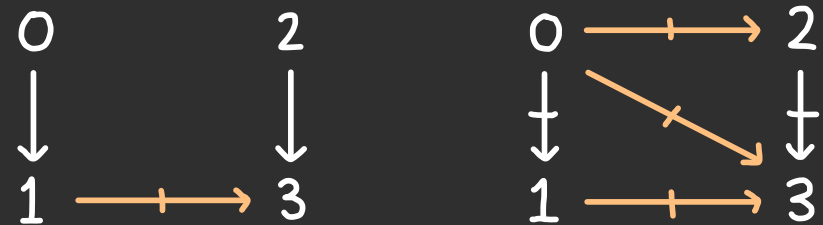
LOOSE DISTRIBUTORS

09

- In ordinary category theory, a **distributor** or **profunctor** $P: \mathcal{C} \rightarrow \mathcal{D}$ between categories is equivalent to a functor into the interval category $\underline{2}$.
- In double category theory, we have two kinds of intervals: **tight** $\{ \cdot \rightarrow \cdot \}$ and **loose** $\{ \cdot \rightrightarrows \cdot \}$.
- A **loose distributor** is a double functor $\mathbb{I} \rightarrow \underline{2}$ into the loose interval.
- A loose distributor is seen as a **morphism** $\mathbb{I}_s \rightrightarrows \mathbb{I}_t$ between double categories as follows.



- We can depict a loose distributor as a double category with some **marked** loose arrows, often called **loose heteromorphisms**.



- For each double category \mathbb{D} , there is a **Hom loose distributor** given by $\mathbb{D} \times \underline{2} \xrightarrow{\pi} \underline{2}$.
- Unlike distributors between categories, loose distributors **do not compose** in general.

ALTERATIONS & MODIFICATIONS

10

- A (normal lax) alteration with frame

$$\begin{array}{ccc} \mathbb{I}_s & \xrightarrow{\mathbb{I}} & \mathbb{I}_t \\ F \downarrow & \Phi & \downarrow G \\ \mathbb{J}_s & \xrightarrow{\mathbb{J}} & \mathbb{J}_t \end{array}$$

F, G unitary lax functors

\mathbb{I}, \mathbb{J} loose distributors

is a normal lax functor $\Phi: \mathbb{I} \rightarrow \mathbb{J}$ over \mathcal{D} such that the following equation holds.

$$\begin{array}{ccccc} \mathbb{I}_s & \xrightarrow{\quad} & \mathbb{I} & \xleftarrow{\quad} & \mathbb{I}_t \\ F \downarrow & & \downarrow \Phi & & \downarrow G \\ \mathbb{J}_s & \xrightarrow{\quad} & \mathbb{J} & \xleftarrow{\quad} & \mathbb{J}_t \\ \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \lrcorner \\ \mathbb{1} & \xrightarrow{s} & \mathcal{D} & \xleftarrow{t} & \mathbb{1} \end{array} = \begin{array}{ccccc} \mathbb{I}_s & \xrightarrow{\quad} & \mathbb{I} & \xleftarrow{\quad} & \mathbb{I}_t \\ \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \lrcorner \\ \mathbb{1} & \xrightarrow{s} & \mathcal{D} & \xleftarrow{t} & \mathbb{1} \end{array}$$

- A modification between alterations

$$\begin{array}{ccc} \mathbb{I}_s & \xrightarrow{\mathbb{I}} & \mathbb{I}_t \\ F \downarrow & \Phi & \downarrow G \\ \mathbb{J}_s & \xrightarrow{\mathbb{J}} & \mathbb{J}_t \end{array} \xRightarrow{m} \begin{array}{ccc} \mathbb{I}_s & \xrightarrow{\mathbb{I}} & \mathbb{I}_t \\ F' \downarrow & \Phi' & \downarrow G' \\ \mathbb{J}_s & \xrightarrow{\mathbb{J}} & \mathbb{J}_t \end{array}$$

is a transformation $m: \Phi \Rightarrow \Phi'$ over \mathcal{D} such that the following diagram "commutes".

$$\begin{array}{ccccc} \mathbb{I}_s & \xrightarrow{\quad} & \mathbb{I} & \xleftarrow{\quad} & \mathbb{I}_t \\ F \left(\Rightarrow \right) F' & \Phi \left(\xRightarrow{m} \right) \Phi' & G \left(\Rightarrow \right) G' \\ \mathbb{J}_s & \xrightarrow{\quad} & \mathbb{J} & \xleftarrow{\quad} & \mathbb{J}_t \end{array}$$

- There is a 2-category $\mathcal{L}Dist_{nl} = DBL_{nl} / \mathcal{D}$ of loose distributors, alterations, and modifications.

LIMITS INDEXED BY LOOSE DISTRIBUTORS

1 1

Suppose $(\lim F, \gamma)$ and $(\lim G, \gamma)$ are limits of $F: \mathbb{I}_s \rightarrow \mathbb{D}$ and $G: \mathbb{I}_t \rightarrow \mathbb{D}$, respectively. The **limit** of an alteration

$$\begin{array}{ccc} \mathbb{I}_s & \xrightarrow[\Phi]{\mathbb{I}} & \mathbb{I}_t \\ F \downarrow & & \downarrow G \\ \mathbb{D} & \xrightarrow[\text{Hom}]{} & \mathbb{D} \end{array}$$

is a loose morphism $\lim \Phi: \lim F \rightarrow \lim G$ in \mathbb{D} and a **terminal cone** Θ which provides for each heteromorphism $p: A \rightarrow X$ in \mathbb{I} a cell Θ_p in \mathbb{D} satisfying several natural axioms.

$$\begin{array}{ccc} \lim F & \xrightarrow{\lim \Phi} & \lim G \\ \gamma_A \downarrow & \Theta_p & \downarrow \gamma_X \\ FA & \xrightarrow[\Phi_p]{} & GX \end{array}$$

Equivalently, an alteration $\Phi: \mathbb{I} \rightarrow \mathbb{D} \times \mathbb{2}$ into the Hom loose distributor determines a span

$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{\text{loose distributor}} & \mathbb{2} \\ \downarrow & \Theta & \uparrow \text{lim} \\ \mathbb{D} & & \end{array}$$

diagram

and its limit is the pointwise right extension in DBL_{ne} , that is, such that the pullback along $s, t: \mathbb{1} \rightarrow \mathbb{2}$ yields limit cones $(\lim F, \gamma)$ and $(\lim G, \gamma)$.

⚠ Limits of alterations can be pathological unless \mathbb{D} is **replete**: $\langle \text{dom}, \text{cod} \rangle: \mathbb{D}_1 \rightarrow \mathbb{D}_0 \times \mathbb{D}_0$ is an isofibration.

$$\begin{array}{ccc} \bullet & \xrightarrow{\lim \Phi} & \bullet \\ \cong \downarrow & & \downarrow \cong \\ \bullet & & \bullet \end{array}$$

COMPANIONS & CONJOINTS

1 2

- A tight morphism $f: A \rightarrow B$ has a **companion** loose morphism $f_*: A \rightharpoonup B$ if there are cells

$$\begin{array}{ccc}
 A & \xrightarrow{f_*} & B \\
 f \downarrow & \sigma & \downarrow 1_B \\
 B & \xrightarrow{id_B} & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{id_A} & A \\
 1_A \downarrow & \tau & \downarrow f \\
 A & \xrightarrow{f_*} & B
 \end{array}$$

such that $\tau \circ \sigma = 1_{f_*}$ and $\sigma \circ \tau = id_f$.

- In $\mathbf{IRel}(\mathcal{C})$, the companion of $f: A \rightarrow B$ is the relation $\langle 1_A, f \rangle: A \rightharpoonup A \times B$ with cells:

$$\begin{array}{ccc}
 A & \xrightarrow{\langle 1_A, f \rangle} & A \times B \\
 f \downarrow & & \downarrow f \times 1_B \\
 B & \xrightarrow{\Delta_B} & B \times B
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{\Delta_A} & A \times A \\
 1_A \downarrow & & \downarrow 1_A \times f \\
 A & \xrightarrow{\langle 1_A, f \rangle} & A \times B
 \end{array}$$

- A tight morphism $f: A \rightarrow B$ has a **conjoint** loose morphism $f^*: B \rightharpoonup A$ if there are cells

$$\begin{array}{ccc}
 A & \xrightarrow{id_A} & A \\
 f \downarrow & \varepsilon & \downarrow 1_A \\
 B & \xrightarrow{f^*} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 B & \xrightarrow{f^*} & A \\
 1_B \downarrow & \eta & \downarrow f \\
 B & \xrightarrow{id_B} & B
 \end{array}$$

such that $\eta \circ \varepsilon = 1_{f^*}$ and $\eta \circ \varepsilon = id_f$.

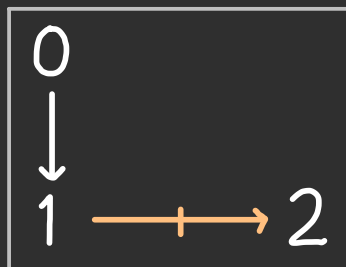
- In $\mathbf{IRel}(\mathcal{C})$, the conjoint of $f: A \rightarrow B$ is the relation $\langle f, 1_A \rangle: A \rightharpoonup B \times A$ with cells:

$$\begin{array}{ccc}
 A & \xrightarrow{\Delta_A} & A \times A \\
 1_A \downarrow & & \downarrow f \times 1_A \\
 A & \xrightarrow{\langle f, 1_A \rangle} & B \times A
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{\langle f, 1_A \rangle} & B \times A \\
 f \downarrow & & \downarrow 1_B \times f \\
 B & \xrightarrow{\Delta_B} & B \times B
 \end{array}$$

COMPANIONS & CONJOINTS AS LIMITS

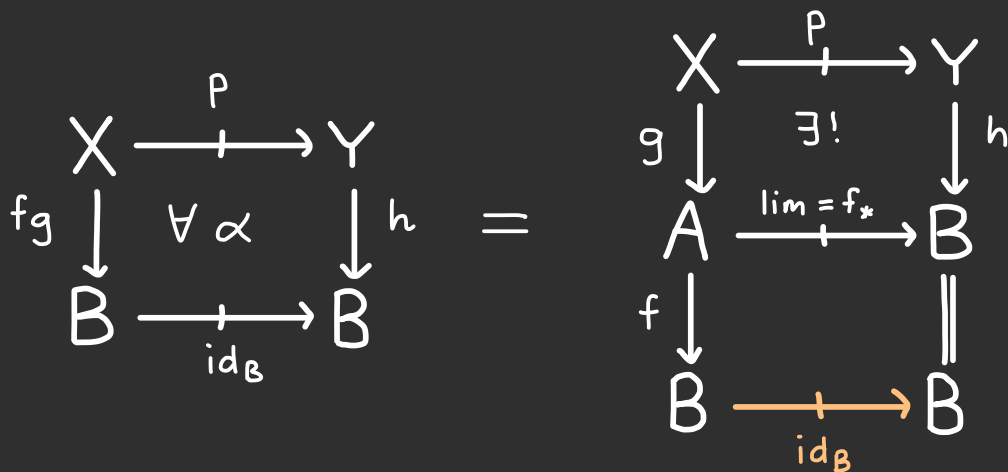
1 3

- A double category admits **companions** if and only if it admits limits of an alteration whose shape is

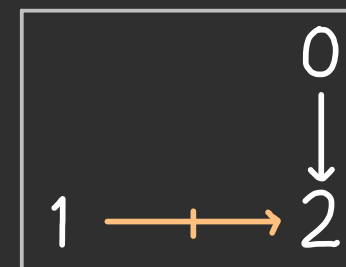


heteromorphism
marked in
orange

where $1 \rightarrow 2$ is sent to an identity loose morphism.



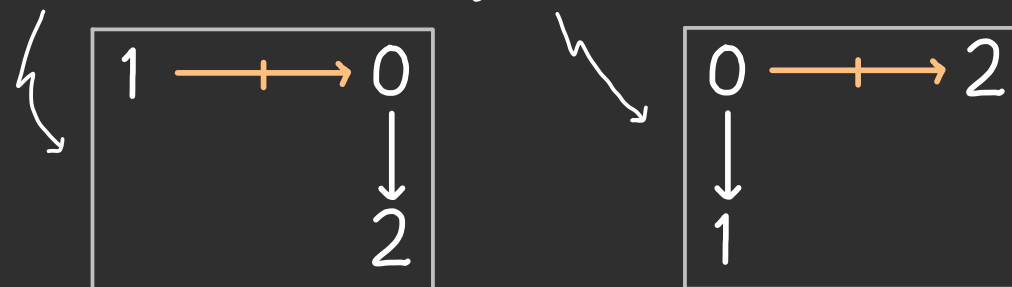
- A double category admits **conjoins** if and only if it admits limits of an alteration whose shape is



heteromorphism
marked in
orange

where $1 \rightarrow 2$ is sent to an identity loose morphism.

- Companions and conjoins also arise as colimits of:



- Companions and conjoins are **absolute limits** – they are preserved by every normal lax double functor.

RESTRICTIONS & EXTENSIONS

14

- A **restriction** of a niche (f, s, g) is a cell $\text{res}(f, s, g)$ with the following universal property:

$$\begin{array}{ccc}
 X & \xrightarrow{r} & Y \\
 fh \downarrow & \alpha & \downarrow gk \\
 C & \xrightarrow{s} & D
 \end{array}
 =
 \begin{array}{ccc}
 X & \xrightarrow{r} & Y \\
 h \downarrow & \exists! & \downarrow k \\
 A & \xrightarrow{f_* \circ s \circ g^*} & B \\
 f \downarrow & \text{res}(f, s, g) & \downarrow g \\
 C & \xrightarrow{s} & D
 \end{array}$$

- An **extension** of a co-niche (h, r, k) is a cell $\text{ext}(h, r, k)$ with the following universal property:

$$\begin{array}{ccc}
 X & \xrightarrow{r} & Y \\
 fh \downarrow & \alpha & \downarrow gk \\
 C & \xrightarrow{s} & D
 \end{array}
 =
 \begin{array}{ccc}
 X & \xrightarrow{r} & Y \\
 h \downarrow & \text{ext}(h, r, k) & \downarrow k \\
 A & \xrightarrow{h^* \circ r \circ k_*} & B \\
 f \downarrow & \exists! & \downarrow g \\
 C & \xrightarrow{s} & D
 \end{array}$$

- For a double category ID , the following are equivalent:
 - * ID has restrictions
 - * ID has extensions
 - * ID has companions and conjoints
 - * the functor $\langle \text{dom}, \text{cod} \rangle: ID_1 \longrightarrow ID_0 \times ID_0$ is a **bifibration**.

- In $IRel(\mathcal{C})$, restrictions are computed by pullback and extensions are computed by factorisation.

$$\begin{array}{ccc}
 \bullet & \xrightarrow{\quad} & A \times B \\
 \downarrow \lrcorner & & \downarrow f \times g \\
 S & \xrightarrow{\langle l_s, r_s \rangle} & C \times D
 \end{array}
 \qquad
 \begin{array}{ccc}
 R & \xrightarrow{\langle l_R, r_R \rangle} & X \times Y \\
 \text{reg. epi} \downarrow & \text{factorise} & \downarrow h \times k \\
 \bullet & \xrightarrow{\text{mono.}} & A \times B
 \end{array}$$

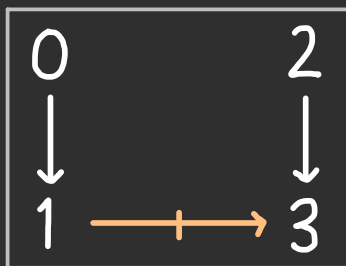
- Consider an identity cell in $IRel(\mathcal{C})$ on $f: A \rightarrow B$.

$$\begin{array}{ccc}
 A & \xrightarrow{id_A} & A \\
 f \downarrow & id_f & \downarrow f \\
 B & \xrightarrow{id_B} & B
 \end{array}
 \qquad
 \begin{array}{l}
 * f \text{ is mono} \iff id_f \cong \text{res}(f, id_B, f) \\
 * f \text{ is reg. epi} \iff id_f \cong \text{ext}(f, id_A, f)
 \end{array}$$

RESTRICTIONS & EXTENSIONS AS (CO)LIMITS

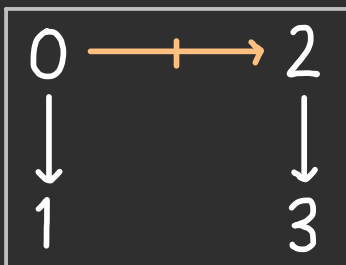
15

- A **restriction** is a limit whose shape is:



heteromorphism
marked in
orange

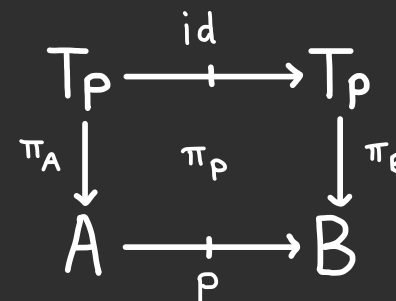
- An **extension** is a colimit whose shape is:



heteromorphism
marked in
orange

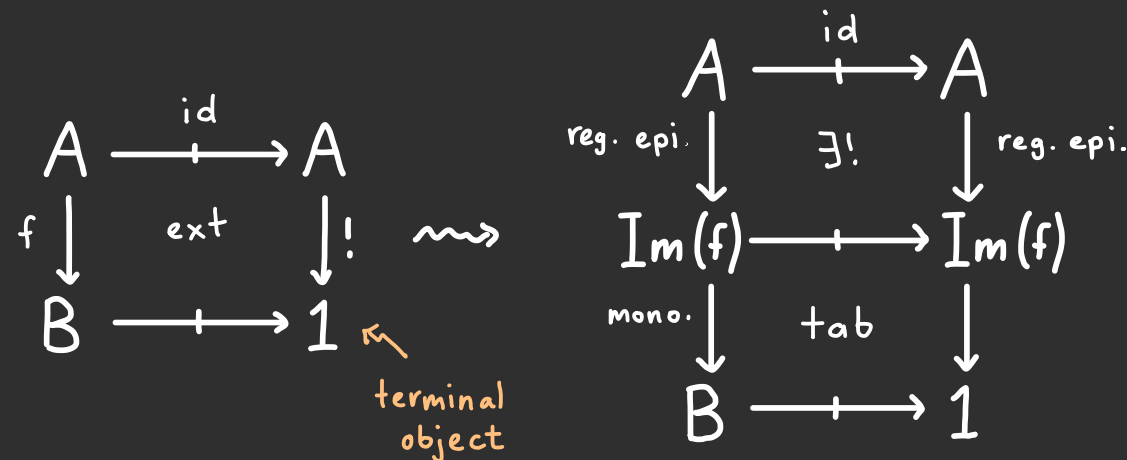
- Restrictions and extensions play an important role throughout double category theory, and often interact well with other kinds of (co)limits.

- A tabulator is called **effective** if the corresponding cell is an extension – a kind of exactness property.



- $\mathbf{IRel}(\mathcal{C})$ has effective tabulators.

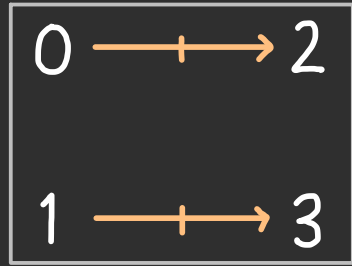
- The image factorisation can be computed as:



PARALLEL PRODUCTS & PARALLEL LIMITS

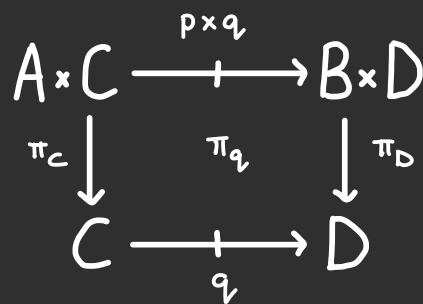
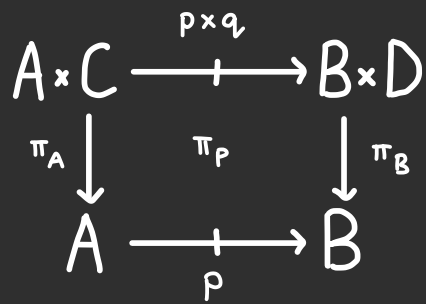
16

- A **parallel product** is a limit whose shape is:



heteromorphisms
marked in
orange

- The parallel product of loose morphisms $p: A \rightrightarrows B$ and $q: C \rightrightarrows D$ is a loose morphism $p \times q: A \times C \rightrightarrows B \times D$ together with projection cells



satisfying the appropriate universal property.

- A double category with all products and parallel products is a **cartesian double category**.
- $\mathbf{IRel}(\mathcal{C})$ has parallel products: given $R: A \rightrightarrows B$ and $S: C \rightrightarrows D$ we have $R \times S: A \times C \rightrightarrows B \times D$.

$$\begin{matrix} \langle l_R, l_S \rangle \times \langle r_R, r_S \rangle \\ R \times S \longrightarrow (A \times C) \times (B \times D) \end{matrix}$$

- More generally, a **tight parallel limit** is a limit whose shape is the Hom loose distributor

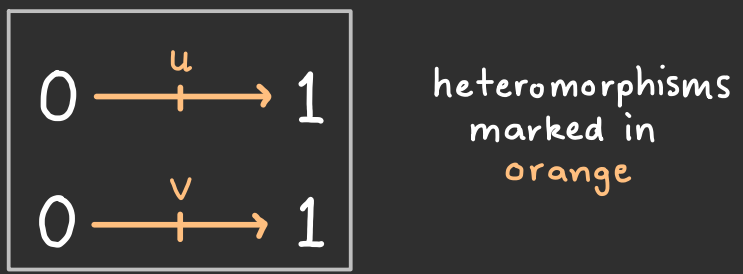
$$\prod_i(\mathcal{C}) \xrightarrow{\text{Hom}} \prod_i(\mathcal{C})$$

for some category \mathcal{C} .

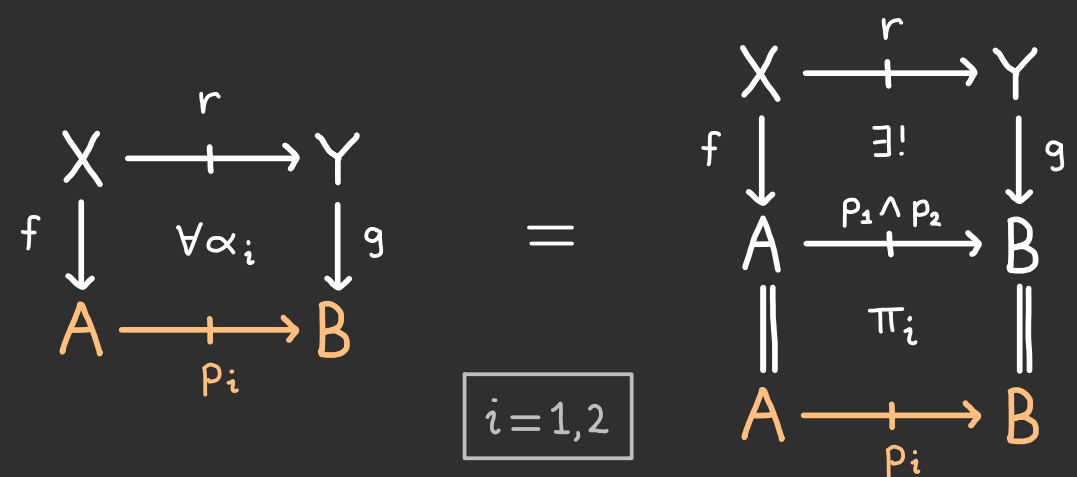
- $\mathbf{IRel}(\mathcal{C})$ admits all finite tight parallel limits; colimits are more subtle.

LOCAL PRODUCTS

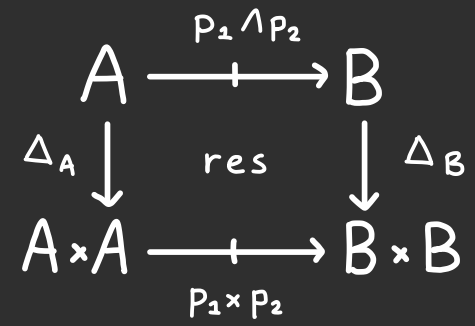
• A **local product** is a limit whose shape is.



• The local product of $p_1, p_2: A \rightrightarrows B$ is a loose morphism $p_1 \wedge p_2: A \rightrightarrows B$ and projection cells with the universal property:

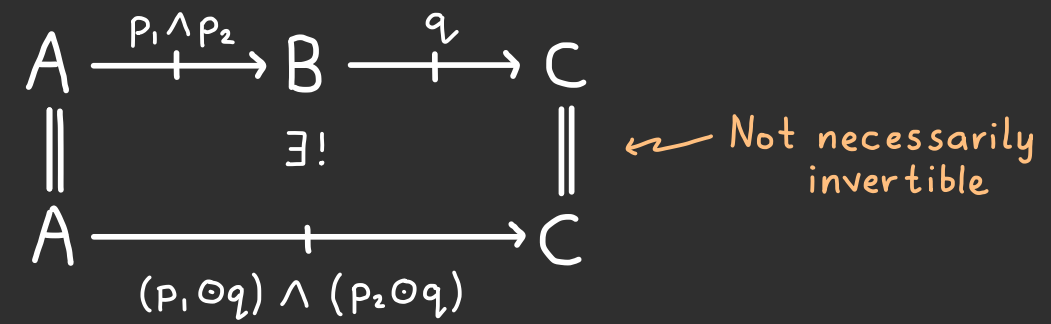


• A cartesian double category with restrictions has local products.



• $\text{Rel}(\mathcal{C})$ has local products (and all local limits) given by "intersection" of relations.

• Local limits are not necessarily preserved by composition with loose morphisms.



PARALLEL TABULATORS

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- A **parallel tabulator** is a limit whose shape is

$$\mathcal{D} \xrightarrow{\text{Hom}} \mathcal{D}$$

- An alteration with this shape determines cells in ID

$$\begin{array}{ccc} A \xrightarrow{p} B \xrightarrow{r_2} D & & A \xrightarrow{r_1} C \xrightarrow{q} D \\ \parallel & \alpha & \parallel \\ A \xrightarrow{r_3} D & & A \xrightarrow{r_3} D \\ & \beta & \end{array}$$

whose parallel tabulator is a loose morphism $T_p \dashrightarrow T_q$ between tabulators and a cone given by cells

$$\begin{array}{ccc} T_p \xrightarrow{\quad} T_q & & T_p \xrightarrow{\quad} T_q & & T_p \xrightarrow{\quad} T_q \\ \pi_A \downarrow & \pi_{r_1} & \downarrow \pi_C & & \pi_B \downarrow & \pi_{r_2} & \downarrow \pi_D & & \pi_A \downarrow & \pi_{r_3} & \downarrow \pi_D \\ A \xrightarrow{r_1} C & & B \xrightarrow{r_2} D & & A \xrightarrow{r_3} D \end{array}$$

which are suitably compatible with α and β .

- $\text{Rel}(\mathcal{C})$ has parallel tabulators, although they are a bit complex to compute.

- A **parallel limit** is a limit whose shape is

$$\mathcal{I} \xrightarrow{\text{Hom}} \mathcal{I}$$

Theorem (Grandis-Paré, 99)

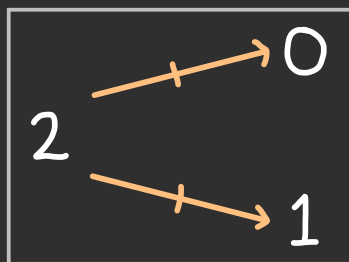
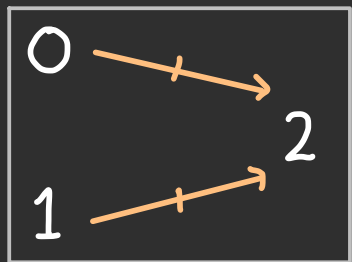
A double category admits parallel limits if and only if it admits **parallel tabulators** and **tight parallel limits**.

Theorem: A double category ID admits limits indexed by loose distributors if and only if ID admits **parallel limits** and **restrictions**.

BIPRODUCTS IN $\mathbf{IRel}(\mathbf{Set})$ AS COLIMITS

19

- Consider the colimits in $\mathbf{IRel}(\mathbf{Set})$ with shapes:



heteromorphisms
marked in
orange

- Given $R: A \rightarrowtail C$ and $S: B \rightarrowtail C$, we have a relation $A + B \rightarrowtail C$ given by:

$$R + S \xrightarrow{\langle \ell_R, r_R \rangle + \langle \ell_S, r_S \rangle} (A \times C) + (B \times C) \cong (A + B) \times C$$

- Dually, given $R': C \rightarrowtail A$ and $S': C \rightarrowtail B$, we have a relation $C \rightarrowtail A + B$ given by:

$$R' + S' \xrightarrow{\langle \ell_{R'}, r_{R'} \rangle + \langle \ell_{S'}, r_{S'} \rangle} (C \times A) + (C \times B) \cong C \times (A + B)$$

- The coprojection(s) are cell(s):

$$\begin{array}{ccc} A & \xrightarrow{R} & C \\ \downarrow \perp_A & \theta_R & \parallel \\ A + B & \xrightarrow{\text{colim}} & C \end{array} \rightsquigarrow \begin{array}{ccc} R & \xrightarrow{\quad} & A \times C \\ \downarrow \perp_R & \lrcorner & \downarrow \perp_A \times 1_C \\ R + S & \xrightarrow{\quad} & (A + B) \times C \end{array}$$

- Since \mathbf{Set} is extensive, these are restriction cells, hence

$$A \xrightarrow{(\perp_A)^*} A + B \xrightarrow{\text{colim}} C = A \xrightarrow{R} C$$

$$B \xrightarrow{(\perp_B)^*} A + B \xrightarrow{\text{colim}} C = B \xrightarrow{R} C$$

$$C \xrightarrow{\text{colim}} A + B \xrightarrow{(\perp_A)^*} A = C \xrightarrow{R'} A$$

$$C \xrightarrow{\text{colim}} A + B \xrightarrow{(\perp_B)^*} B = C \xrightarrow{R} B$$

- Since colimits are unique up to isomorphism, we recover $A + B$ as a "biproduct in the category \mathbf{Rel} ", but demonstrate a far richer universal property in \mathbf{IRel} .

SUMMARY & FURTHER WORK

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- Introduced limits indexed by loose distributors.

Theorem: The double category $\mathbf{IRel}(\mathcal{C})$ of relations in a regular category \mathcal{C} admits all finite limits indexed by double categories, and all finite limits indexed by loose distributors.

- The double category $\mathbf{IRel}(\mathcal{C})$ is far richer than the category of relations, and many natural constructions arise as limits in this setting, including companions, conjoiners, restrictions, extensions, parallel products, local products, and "biproducts".

There are many avenues for further research:

- Exploring the relationship between $\mathbf{IRel}(\mathcal{C})$ and $\mathbf{Span}(\mathcal{C})$ – a reflective adjunction.
- Investigate (co)limits in other double categories of interest to categorical algebra.
- Developing theory of double limit sketches (see Lambert-Patterson, Cartesian double theories).
- Demonstrating how exactness properties in $\mathbf{IRel}(\mathcal{C})$ relate to properties of a regular category (e.g. Barr-exact, Maltsev, Goursat).
- Characterising $\mathbf{IRel}(\mathcal{C})$ as a free (co)completion under certain limits (see Lambert'22, Hoshino-Nasu'25).