A NEW FRAMEWORK FOR LIMITS IN DOUBLE CATEGORIES

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MOTIVATION & OVERVIEW

double categories ⩾ **2-categories** + **bicategories**

- Theory of (co)limits is fundamental, but relatively undeveloped for double categories compared to 2-categories/bicategories.

> GOAL: Develop a sufficiently rich notion of limit in double categories to express all examples of "limit-like" constructions — even those not possible with bicategories.

- Key advantage: double categories have 2 kinds of objects:

$$\begin{tikzcd}
\mathbb{D} = & D_0 \arrow[r, "id" description] & D_1 \arrow[l, shift right=3, "dom"'] \arrow[l, shift left=3, "cod"] & D_1 \times_{D_0} D_1 \arrow[l, "\odot" description] \arrow[l, shift right=3, "\pi_1"'] \arrow[l, shift left=3, "\pi_2"]
\end{tikzcd}$$

objects loose morphisms

- Suggests 2 kinds of limits in double categories!

PART 1: Limits indexed by double categories $\mathbb{I}$

$$\begin{tikzcd}
 & \lim F \arrow[dl, "\gamma_A"'] \arrow[dr, "\gamma_B"] & \\
FA \arrow[rr, "Ff"'] & & FB
\end{tikzcd}
\qquad
\begin{tikzcd}
\lim F \arrow[r, "id", "|" marking] \arrow[d, "\gamma_C"'] \arrow[dr, phantom, "\gamma_p"] & \lim F \arrow[d, "\gamma_D"] \\
FC \arrow[r, "Fp"', "|" marking] & FD
\end{tikzcd}$$

- Introduced by Grandis-Paré in 1999.
- Constructed from limits in D_0 and tabulators.

PART 2: Limits indexed by loose distributors $\mathbb{I} \xrightarrow{P} \mathbb{J}$

$$\begin{tikzcd}
\lim F \arrow[r, "\lim \Phi", "|" marking] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\Theta_q"] & \lim G \arrow[d, "\psi_X"] \\
FA \arrow[r, "\Phi q"', "|" marking] & GX
\end{tikzcd}$$

- Capture parallel limits and many new examples!
- Main theorem: characterising $\mathbb{D}$ which admit all limits.

BACKGROUND ON DOUBLE CATEGORIES

A **double category** $\mathbb{D}$ consists of:

- objects $A, B, C, D, \ldots$
- **tight morphisms** $A \longrightarrow B$ (usually drawn vertically)
- **loose morphisms** $A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ (usually drawn horizontally)
- cells

$$\begin{tikzcd}
A \arrow[r, "p", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\
B \arrow[r, "q"', "|" marking] & D
\end{tikzcd}$$

- Identities 1_A and composition $g \circ f$ in tight direction
- Identities id_A and composition $p \odot q$ in loose direction

$$\begin{tikzcd}
\bullet \arrow[r, "id_A \odot p", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\underline{\ell}(p)"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "p"', "|" marking] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "p \odot id_B", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\underline{r}(p)"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "p"', "|" marking] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "p \odot (q \odot r)", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\underline{a}(p,q,r)"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "(p \odot q) \odot r"', "|" marking] & \bullet
\end{tikzcd}$$

left unitor **right unitor** **associator**

Examples

- **Category** $\mathbb{C}$ $\rightsquigarrow$ double category $\mathbb{T}\mathrm{i}(\mathbb{C})$
  - * objects & tight morphisms come from $\mathbb{C}$
  - * loose morphisms & cells are identities
- 2-categories, monoidal categories, bicategories
- **$\mathbb{R}\mathrm{el}$** – objects are sets, tight morphisms are functions, loose morphisms are relations
- **$\mathbb{S}\mathrm{pan}$** – sets, functions, spans, span morphisms
- **$\mathbb{D}\mathrm{ist}$** – categories, functors, distributors/profunctors

$$\begin{tikzcd}
A \arrow[r, "p", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\
B \arrow[r, "q"', "|" marking] & D
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
A^{op} \times C \arrow[rr, bend left, "p"] \arrow[d, "f^{op} \times g"'] & \Downarrow \alpha & \mathrm{Set} \\
B^{op} \times D \arrow[urr, bend right, "q"'] & &
\end{tikzcd}$$

LIMITS INDEXED BY DOUBLE CATEGORIES

A unitary lax functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ is an assignment

$$\begin{tikzcd}
A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] & & FA \arrow[r, "Fp", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\alpha"] & FC \arrow[d, "Fg"] \\
B \arrow[r, "q"', "\shortmid"{marking}] & D \arrow[r, phantom, "\rightsquigarrow"] & & FB \arrow[r, "Fq"', "\shortmid"{marking}] & FD
\end{tikzcd}$$

preserving tight identities & composites and loose identities, together with laxator cells:

$$\begin{tikzcd}
FA \arrow[r, "Fp", "\shortmid"{marking}] \arrow[d, equal] & FB \arrow[r, "Fq", "\shortmid"{marking}] \arrow[d, phantom, "\underline{c}(p,q)"] & FC \arrow[d, equal] \\
FA \arrow[rr, "F(p \odot q)"', "\shortmid"{marking}] & & FC
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{T}\mathrm{i}(\mathcal{C}) \arrow[r] & \mathbb{D} & \leftrightsquigarrow & \mathcal{C} \arrow[r] & D_0
\end{tikzcd}$$

unitary lax functor   functor

A limit of $F : \mathbb{I} \longrightarrow \mathbb{D}$ is an object $\lim F$ and a terminal cone γ which provides for each $f : A \to B$ and $p : C \xrightarrow{\shortmid} D$

$$\begin{tikzcd}
& \lim F \arrow[dl, "\gamma_A"'] \arrow[dr, "\gamma_B"] & & \lim F \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\gamma_C"'] \arrow[dr, phantom, "\gamma_p"] & \lim F \arrow[d, "\gamma_D"] \\
FA \arrow[rr, "Ff"'] & & FB & FC \arrow[r, "Fp"', "\shortmid"{marking}] & FD
\end{tikzcd}$$

natural w.r.t. cells in $\mathbb{I}$ such that $\gamma_{\mathrm{id}_A} = \mathrm{id}_{\gamma_A}$ and

$$\begin{tikzcd}
\lim F \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\gamma_p"] & \lim F \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\gamma_B"] \arrow[dr, phantom, "\gamma_q"] & \lim F \arrow[d, "\gamma_C"] & & \lim F \arrow[r, "\mathrm{id} \odot \mathrm{id}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \lim F \arrow[d, equal] \\
FA \arrow[r, "Fp", "\shortmid"{marking}] \arrow[d, equal] & FB \arrow[r, "Fq", "\shortmid"{marking}] \arrow[d, phantom, "\underline{c}(p,q)"] & FC \arrow[d, equal] & = & \lim F \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\gamma_{p \odot q}"] & \lim F \arrow[d, "\gamma_C"] \\
FA \arrow[rr, "F(p \odot q)"', "\shortmid"{marking}] & & FC & & FA \arrow[r, "F(p \odot q)"', "\shortmid"{marking}] & FC
\end{tikzcd}$$

TABULATORS & TIGHT LIMITS

- A tabulator is a limit whose shape is

$$\mathbb{2} = \{0 \xrightarrow{\shortmid} 1\}$$

- The tabulator of a loose morphism $p : A \xrightarrow{\shortmid} B$ is a cone

$$\begin{tikzcd}
Tp \arrow[r, "\mathrm{id}", "\shortmid" marking] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & Tp \arrow[d, "\pi_B"] \\
A \arrow[r, "p"', "\shortmid" marking] & B
\end{tikzcd}$$

- A double category $\mathbb{D}$ admits all tabulators if and only if the functor $\mathrm{id} : D_0 \longrightarrow D_1$ has a right adjoint.

- In $\mathbb{S}\mathrm{pan}$, $T(A \leftarrow P \rightarrow B) = P$

- In $\mathbb{R}\mathrm{el}$, $T(R \cdot A \times B \longrightarrow \{\bot \to \top\}) = \{(a,b) \in A \times B \mid R(a,b) = \top\}$

- In $\mathbb{D}\mathrm{ist}$, the category of elements of $P : A^{op} \times B \longrightarrow \mathrm{Set}$.

- A tight limit is a limit whose shape is

$$\mathbb{T}\mathrm{i}(\mathcal{C}) \qquad \mathcal{C} \text{ category}$$

- Tight limits in a double category $\mathbb{D}$ are precisely limits in the underlying category D_0 of objects and tight morphisms.

$$\mathbb{T}\mathrm{i}(\mathcal{C}) \xrightarrow{F} \mathbb{D} \quad \rightsquigarrow \quad \mathcal{C} \xrightarrow{F_0} D_0$$

- Examples: products, equalisers, pullbacks, terminal objects.

Theorem (Grandis-Paré, 99)

A double category $\mathbb{D}$ admits limits indexed by any double category $\mathbb{I}$ <u>if and only if</u> $\mathbb{D}$ admits tight limits and tabulators.

LOOSE DISTRIBUTORS & ALTERATIONS

A loose distributor $P : \mathbb{C} \xrightarrow{\shortmid\shortmid} \mathbb{D}$ is

1) a distributor between pseudo category objects in CAT

$$C_0 \xleftarrow{s} P \xrightarrow{t} D_0 \qquad C_1 \times_{C_0} P \xrightarrow{\triangleright} P \qquad P \times_{D_0} D_1 \xrightarrow{\triangleleft} P$$

2) a functor $\mathbb{P} \longrightarrow \mathbb{2}$ into free double category $\{0 \xrightarrow{\shortmid} 1\}$

3) for each $A \in \mathbb{C}$ and $X \in \mathbb{D}$, a collection $P(A,X)$ of loose heteromorphisms $A \xrightarrow{\shortmid} X$, and set of heterocells

$$\begin{tikzcd}
\mathbb{C} & A \arrow[d, "f"'] & A \arrow[r, "p", "\shortmid" marking] \arrow[d, "f"'] & X \arrow[d, "g"] & X \arrow[d, "g"] & \mathbb{D} \\
 & B & B \arrow[r, "q"', "\shortmid" marking] & Y & Y &
\end{tikzcd}$$

for each frame of tight morphisms and loose heteromorphisms together with compatible left action by $\mathbb{C}$ and right action by $\mathbb{D}$.

collage of $P : \mathbb{C} \xrightarrow{\shortmid\shortmid} \mathbb{D}$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, hook] & \mathbb{P} & \mathbb{D} \arrow[l, hook']
\end{tikzcd}$$

An (unitary lax) alteration with frame

$$\begin{tikzcd}
\mathbb{I} \arrow[r, "P", "\shortmid\shortmid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Phi"] & \mathbb{J} \arrow[d, "G"] \\
\mathbb{D} \arrow[r, "\mathrm{Hom}"', "\shortmid\shortmid" marking] & \mathbb{D}
\end{tikzcd}$$

F, G unitary lax functors

is an assignment on heteromorphisms and heterocells

$$\begin{tikzcd}
A \arrow[r, "p", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & X \arrow[d, "g"] & & FA \arrow[r, "\Phi p", "\shortmid" marking] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Phi\alpha"] & GX \arrow[d, "Gg"] \\
B \arrow[r, "q"', "\shortmid" marking] & Y & \longmapsto & FB \arrow[r, "\Phi q"', "\shortmid" marking] & GY
\end{tikzcd} \quad \text{in } \mathbb{D}$$

preserving identity and composite heterocells together with cells

$$\begin{tikzcd}
FA' \arrow[r, "Fq", "\shortmid" marking] \arrow[d, equal] & FA \arrow[r, "\Phi p", "\shortmid" marking] \arrow[d, phantom, "c^{\triangleright}(q{,}p)"] & GX \arrow[d, equal] & FA \arrow[r, "\Phi p", "\shortmid" marking] \arrow[d, equal] & GX \arrow[r, "Gr", "\shortmid" marking] \arrow[d, phantom, "c^{\triangleleft}(p{,}r)"] & GX' \arrow[d, equal] \\
FA' \arrow[rr, "\Phi(q \triangleright p)"', "\shortmid" marking] & & GX & FA \arrow[rr, "\Phi(p \triangleleft r)"', "\shortmid" marking] & & GX'
\end{tikzcd}$$

LIMITS INDEXED BY LOOSE DISTRIBUTORS

Suppose $(\lim F, \gamma)$ and $(\lim G, \psi)$ are limits of $F : \mathbb{I} \longrightarrow \mathbb{D}$ and $G : \mathbb{J} \longrightarrow \mathbb{D}$, respectively. The limit of an alteration

$$\begin{tikzcd}
\mathbb{I} \arrow[r, "P", "\shortmid\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Phi"] & \mathbb{J} \arrow[d, "G"] \\
\mathbb{D} \arrow[r, "\mathrm{Hom}"', "\shortmid\shortmid"{marking}] & \mathbb{D}
\end{tikzcd}$$

is a loose morphism $\lim \Phi : \lim F \rightarrow\!\!\!\shortmid\, \lim G$ in $\mathbb{D}$ and a terminal cone θ which provides for each $p : A \rightarrow\!\!\!\shortmid\, X$ in $P(A, X)$

$$\begin{tikzcd}
\lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[d, "\psi_X"] \\
FA \arrow[r, "\Phi p"', "\shortmid"{marking}] & GX
\end{tikzcd} \quad \text{in } \mathbb{D}$$

natural w.r.t. heterocells of $P : \mathbb{I} \rightarrow\!\!\!\shortmid\!\shortmid\, \mathbb{J}$ and satisfying

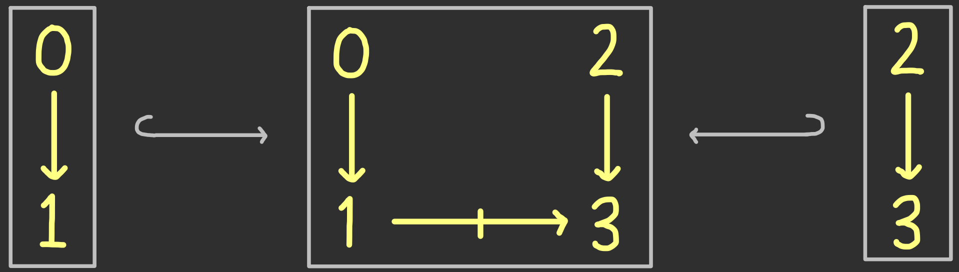
$$\begin{tikzcd}
\lim F \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\gamma_{A'}"'] \arrow[dr, phantom, "\gamma_q"] & \lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[d, "\psi_X"] \\
FA' \arrow[r, "Fq", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\underline{c}^{\triangleright}(q, p)"] & FA \arrow[r, "\Phi p", "\shortmid"{marking}] & GX \arrow[d, equal] \\
FA' \arrow[rr, "\Phi(q \triangleright p)"', "\shortmid"{marking}] & & GX
\end{tikzcd}
=
\begin{tikzcd}
\lim F \arrow[r, "\mathrm{id} \odot \lim \Phi", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \lim G \arrow[d, equal] \\
\lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_{A'}"'] \arrow[dr, phantom, "\theta_{q \triangleright p}"] & \lim G \arrow[d, "\psi_X"] \\
FA' \arrow[r, "\Phi(q \triangleright p)"', "\shortmid"{marking}] & GX
\end{tikzcd}$$

$$\begin{tikzcd}
\lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\psi_X"] \arrow[dr, phantom, "\psi_r"] & \lim G \arrow[d, "\psi_{X'}"] \\
FA \arrow[r, "\Phi p", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\underline{c}^{\triangleleft}(p, r)"] & GX \arrow[r, "Gr", "\shortmid"{marking}] & GX' \arrow[d, equal] \\
FA \arrow[rr, "\Phi(p \triangleleft r)"', "\shortmid"{marking}] & & GX'
\end{tikzcd}
=
\begin{tikzcd}
\lim F \arrow[r, "\lim \Phi \odot \mathrm{id}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \lim G \arrow[d, equal] \\
\lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_{p \triangleleft r}"] & \lim G \arrow[d, "\psi_{X'}"] \\
FA \arrow[r, "\Phi(p \triangleleft r)"', "\shortmid"{marking}] & GX'
\end{tikzcd}$$

⚠ Limits of alterations can be pathological unless $\mathbb{D}$ is replete : $\langle \mathrm{dom}, \mathrm{cod} \rangle : D_1 \longrightarrow D_0 \times D_0$ is an isofibration.

COMPANIONS, CONJOINTS, & RESTRICTIONS ARE LIMITS

- A restriction is a limit whose shape is (the collage)

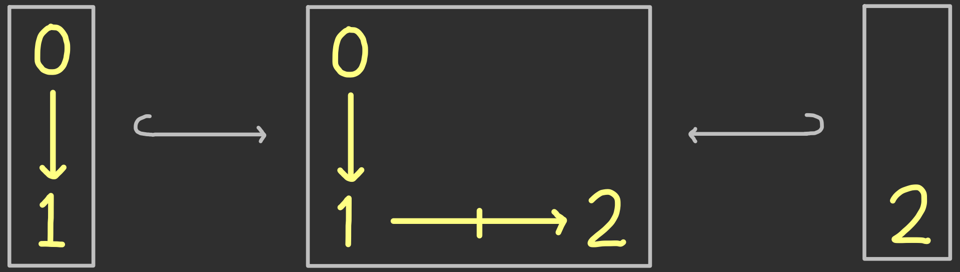


- The universal property states (assuming repleteness)

$$\begin{tikzcd}
X \arrow[r, "q", "\shortmid"{marking}] \arrow[d, "h"'] & Y \arrow[d, "k"] \\
A \arrow[d, "f"'] \arrow[r, phantom, "\forall \alpha"] & C \arrow[d, "g"] \\
B \arrow[r, "h"', "\shortmid"{marking}] & D
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "q", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\exists! \hat{\alpha}"] & Y \arrow[d, "k"] \\
A \arrow[d, "f"'] \arrow[r, "\mathrm{lim}", "\shortmid"{marking}] \arrow[dr, phantom, "\mathrm{res}"] & C \arrow[d, "g"] \\
B \arrow[r, "h"', "\shortmid"{marking}] & D
\end{tikzcd}$$

We choose the limit of a tight morphism to be its domain.

- A double category admits companions if and only if it admits limits whose shape is



- A double category admits restrictions if and only if it admits companions and conjoints.

- $\mathbb{R}\mathrm{el}$, $\mathbb{S}\mathrm{pan}$, and $\mathbb{D}\mathrm{ist}$ admit all restrictions. E.g. in $\mathbb{R}\mathrm{el}$

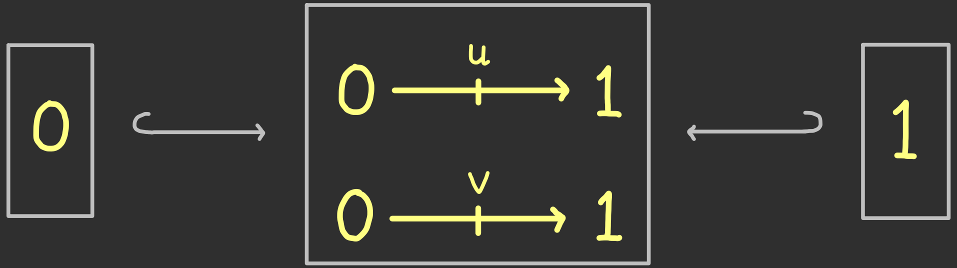
$$\begin{tikzcd}
A \times C \arrow[d, "f \times g"'] \arrow[dr, dashed, bend left, "R(f-, g-)"] & \\
B \times D \arrow[r, "R"'] & \{\bot \to \top\}
\end{tikzcd}$$

- Restrictions, etc. are preserved by any unitary lax functor.

LOCAL LIMITS ARE LIMITS

- A local limit is a limit whose shape is $\mathbb{1} \overset{p}{\nrightarrow} \mathbb{1}$.

- For example, a local product has the shape



- The local product of $p_1, p_2 : A \nrightarrow B$ is a loose morphism $p_1 \wedge p_2 : A \nrightarrow B$ and projection cells with the universal property:

$$\begin{tikzcd}
X \arrow[r, "r", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\forall \alpha_i"] & Y \arrow[d, "g"] & & X \arrow[r, "r", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\exists!"] & Y \arrow[d, "g"] \\
A \arrow[r, "p_i"', "|" marking] & B & = & A \arrow[r, "p_1 \wedge p_2", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\pi_i"] & B \arrow[d, equal] \\
 & & \boxed{i = 1, 2} & A \arrow[r, "p_i"', "|" marking] & B
\end{tikzcd}$$

- Local limits are not necessarily preserved by composition with loose morphisms.

$$\begin{tikzcd}
A \arrow[r, "p_1 \wedge p_2", "|" marking] \arrow[d, equal] & B \arrow[r, "q", "|" marking] \arrow[d, phantom, "\exists!"] & C \arrow[d, equal] \\
A \arrow[rr, "(p_1 \odot q) \wedge (p_2 \odot q)"', "|" marking] & & C
\end{tikzcd}$$

← Not necessarily invertible

- A double category admits local limits if and only if it admits local products and local equalisers.

- $\mathbb{R}\mathrm{el}$, $\mathbb{S}\mathrm{pan}$, and $\mathbb{D}\mathrm{ist}$ admit all local limits.

$$\begin{tikzcd}
A \times B \arrow[r, "\langle R, S \rangle"] \arrow[dr, dashed, "R \wedge S"'] & \{\bot \to \top\} \times \{\bot \to \top\} \arrow[d, "\wedge"] \\
 & \{\bot \to \top\}
\end{tikzcd}$$

local product in $\mathbb{R}\mathrm{el}$

HOMOLOGOUS & TIGHT PARALLEL LIMITS

- A homologous limit is a limit whose shape is

$$\mathbb{T}\mathrm{i}(I) \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{T}\mathrm{i}(J) \qquad I, J \text{ categories}$$

- An alteration of this shape into $\mathbb{D}$ is precisely

$$\begin{tikzcd} I \arrow[d, "F"'] & P \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "\Phi"] & J \arrow[d, "G"] \\ D_0 & D_1 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & D_0 \end{tikzcd} \quad \text{in } \mathrm{CAT}$$

$\mathbb{T}\mathrm{i}(I) \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{T}\mathrm{i}(J)$ is a span

- Restrictions and local limits are examples.

- A tight parallel limit is a limit whose shape is

$$\mathbb{T}\mathrm{i}(I) \xrightarrow[\;\mapstochar\;]{\mathrm{Hom}} \mathbb{T}\mathrm{i}(I) \qquad I \text{ category}$$

- A double category has tight parallel limits if and only if it admits parallel products & parallel equalisers.

- E.g. the parallel product of $p : A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ and $q : C \relbar\joinrel\mapstochar\joinrel\rightarrow D$ is

$$\begin{tikzcd} A \times C \arrow[r, "p \times q", "|" marking] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & B \times D \arrow[d, "\pi_B"] \\ A \arrow[r, "p"', "|" marking] & B \end{tikzcd} \qquad \begin{tikzcd} A \times C \arrow[r, "p \times q", "|" marking] \arrow[d, "\pi_C"'] \arrow[dr, phantom, "\pi_q"] & B \times D \arrow[d, "\pi_D"] \\ C \arrow[r, "q"', "|" marking] & D \end{tikzcd}$$

Theorem: A double category admits homologous limits if and only if it admits tight parallel limits and restrictions.

Corollary: $\mathbb{D}$ has homologous limits if and only if D_0 and D_1 have limits preserved by $\mathrm{dom}, \mathrm{cod} : D_1 \rightrightarrows D_0$ and $\langle \mathrm{dom}, \mathrm{cod} \rangle : D_1 \longrightarrow D_0 \times D_0$ is a fibration.

Corollary: A double category admits local XX if it admits parallel XX and restrictions.
XX = products, equalisers, etc.

PARALLEL TABULATORS & MAIN THEOREM

- A parallel tabulator is a limit whose shape is

$$\mathbb{2} \xrightarrow[\quad]{\text{Hom}} \mathbb{2}$$

- An alteration with this shape determines cells in $\mathbb{D}$

$$\begin{tikzcd}
A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "r_2", "\shortmid"{marking}] & D \arrow[d, equal] \\
A \arrow[rr, "r_3"', "\shortmid"{marking}] \arrow[rru, phantom, "\alpha"] & & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "r_1", "\shortmid"{marking}] \arrow[d, equal] & C \arrow[r, "q", "\shortmid"{marking}] & D \arrow[d, equal] \\
A \arrow[rr, "r_3"', "\shortmid"{marking}] \arrow[rru, phantom, "\beta"] & & D
\end{tikzcd}$$

whose parallel tabulator is a loose morphism $Tp \xrightarrow{\shortmid} Tq$ between tabulators and a cone given by cells

$$\begin{tikzcd}
Tp \arrow[r, "\shortmid"{marking}] \arrow[d, "\pi_A"'] \arrow[rd, phantom, "\pi_{r_1}"] & Tq \arrow[d, "\pi_C"] \\
A \arrow[r, "r_1"', "\shortmid"{marking}] & C
\end{tikzcd}
\quad
\begin{tikzcd}
Tp \arrow[r, "\shortmid"{marking}] \arrow[d, "\pi_B"'] \arrow[rd, phantom, "\pi_{r_2}"] & Tq \arrow[d, "\pi_D"] \\
B \arrow[r, "r_2"', "\shortmid"{marking}] & D
\end{tikzcd}
\quad
\begin{tikzcd}
Tp \arrow[r, "\shortmid"{marking}] \arrow[d, "\pi_A"'] \arrow[rd, phantom, "\pi_{r_3}"] & Tq \arrow[d, "\pi_D"] \\
A \arrow[r, "r_3"', "\shortmid"{marking}] & D
\end{tikzcd}$$

which are suitably compatible with α and β.

- A parallel limit is a limit whose shape is

$$\mathbb{I} \xrightarrow[\quad]{\text{Hom}} \mathbb{I}$$

Theorem (Grandis-Paré, 99)
A double category admits parallel limits if and only if it admits parallel tabulators and tight parallel limits.

Theorem: A double category $\mathbb{D}$ admits limits indexed by loose distributors if and only if

(1) $\mathbb{D}$ admits parallel limits and restrictions

if and only if

(2) $\mathbb{D}$ admits parallel tabulators and homologous limits.

SUMMARY & FURTHER WORK

- Introduced a new framework for limits in double categories indexed by loose distributors $\mathbb{I} \overset{P}{\twoheadrightarrow} \mathbb{J}$.
- Captures many well-known concepts as examples:
  - * Parallel limits $\mathbb{I} \overset{\mathrm{Hom}}{\twoheadrightarrow} \mathbb{I}$ and parallel tabulators $\mathbb{2} \overset{\mathrm{Hom}}{\twoheadrightarrow} \mathbb{2}$.
  - * Restrictions $\mathbb{T}\mathrm{i}(2) \twoheadrightarrow \mathbb{T}\mathrm{i}(2)$, companions and conjoints.
  - * Local limits $\mathbb{1} \overset{P}{\twoheadrightarrow} \mathbb{1}$, including local products
  - * Homologous limits $\mathbb{T}\mathrm{i}(\mathcal{C}) \overset{P}{\twoheadrightarrow} \mathbb{T}\mathrm{i}(\mathcal{C})$.

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Many current and future research directions.

- Sufficient conditions for completeness of $\mathbb{S}\mathrm{pan}(\mathcal{E})$, $\mathbb{R}\mathrm{el}(\mathcal{E})$, $\mathbb{M}\mathrm{at}(\mathbb{D})$, $\mathbb{M}\mathrm{od}(\mathbb{D})$, etc.
- Interactions between limits indexed by double categories and limits indexed by loose distributors.
- Relationship with bicategorical (co)limits.
- Constructing (co)completions of double categories.
- Extending Lambert-Patterson's Cartesian double theories to a general framework of double-categorical sketches.
- Characterising the class of absolute (co)limits.
- Generalisation to virtual double categories.

BONUS SLIDE: LAX BICATEGORICAL COLIMITS

- Each $\mathbb{I} \xrightarrow{!} \mathbb{1}$ determines canonical loose distributors:

$$\begin{tikzcd}
\mathbb{I} \arrow[r, "\shortparallel"{marking}, "\top"] & \mathbb{1} & & \mathbb{1} \arrow[r, "\shortparallel"{marking}, "\top"] & \mathbb{I}
\end{tikzcd}$$

- Example: consider the colimit of $\{1\ 2\} \xrightarrow{\top} \mathbb{1}$ whose diagram is a pair of loose morphisms $A_1 \xrightarrow{p_1} X \xleftarrow{p_2} A_2$

$$\begin{tikzcd}
A_i \arrow[r, "\shortmid"{marking}, "p_i"] \arrow[d, "\amalg_i"'] \arrow[dr, phantom, "\theta_i"] & X \arrow[d, equal] \\
A_1 + A_2 \arrow[r, "\shortmid"{marking}, "\mathrm{colim}(p)"'] & X
\end{tikzcd}$$

and the colimit of $\mathbb{1} \xrightarrow{\top} \{1\ 2\}$ given by:

$$\begin{tikzcd}
Y \arrow[r, "\shortmid"{marking}, "q_i"] \arrow[d, equal] \arrow[dr, phantom, "\omega_i"] & A_i \arrow[d, "\amalg_i"] \\
Y \arrow[r, "\shortmid"{marking}, "\mathrm{colim}(q)"'] & A_1 + A_2
\end{tikzcd}$$

- If $\mathbb{D}$ has companions and conjoints we obtain cells:

$$\begin{tikzcd}
A \arrow[r, "\shortmid"{marking}, "id"] \arrow[d, equal] & A_i \arrow[r, "\shortmid"{marking}, "p_i"] \arrow[d, "\amalg_i"'] \arrow[dr, phantom, "\theta_i"] & X \arrow[d, equal] & Y \arrow[r, "\shortmid"{marking}, "q_i"] \arrow[d, equal] \arrow[dr, phantom, "\omega_i"] & A_i \arrow[r, "\shortmid"{marking}, "id"] \arrow[d, "\amalg_i"] & A_i \arrow[d, equal] \\
A \arrow[r, "\shortmid"{marking}, "(\amalg_i)_*"'] & A_1 + A_2 \arrow[r, "\shortmid"{marking}, "\mathrm{colim}(p)"'] & X & Y \arrow[r, "\shortmid"{marking}, "\mathrm{colim}(q)"'] & A_1 + A_2 \arrow[r, "\shortmid"{marking}, "(\amalg_i)^*"'] & A_i
\end{tikzcd}$$

- In $\mathbb{R}$el, $\mathbb{S}$pan, and $\mathbb{D}$ist these cells are invertible, and describe the coproduct and product in the underlying bicategory of $\mathbb{D}$ — which coincide!

- Takeaway: biproducts in Rel are colimits in $\mathbb{R}$el.

Conjecture: Let $\mathbb{D}$ have companions & conjoints. A (unitary colax) functor $\mathbb{J}$ admits a lax colimit if and only if alteration from $\mathbb{J} \overset{\bot}{\twoheadrightarrow} \mathbb{1}$ admits a colimit.