THE FREE SPLIT OPFIBRATION ON A DELTA LENS

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MOTIVATION

$$\frac{\text{Split opfibrations over } B}{\text{Functors } B \longrightarrow \mathcal{C}at}$$

- **Delta lenses** capture the underlying structure of s. opfs.
- Lenses model **bidirectional transformations** — often want these to be "least change".

Many similarities:

- Admit Grothendieck constructions
- Right class of AWFS
- (Co)algebras for a comonad

How may we complete a delta lens to a split opfibration?

$$\begin{tikzcd} \mathsf{SOpf} \arrow[rr, hook, shift right=1.5ex] \arrow[rr, phantom, "\perp"] & & \mathcal{L}ens \arrow[ll, shift right=1.5ex] \end{tikzcd}$$

OUTLINE OF THE TALK

1 Warming up: Chosen (initial) objects

2. Delta lenses & split opfibrations

3. Free delta lenses & split opfibrations

4. Split opfibrations are reflective in delta lenses

WARMING UP:

CHOSEN (INITIAL) OBJECTS

ADJOINT TRIANGLE THEOREM

$$
\begin{tikzcd}
A \arrow[rr, "J"] \arrow[rr, phantom, shift left=3, "\perp"] \arrow[dr, shift left, "U_2"] & & B \arrow[ll, dashed, bend right=50, "L"'] \arrow[dl, shift right, "U_1"'] \\
& C \arrow[ul, shift left, "F_2"] \arrow[ul, phantom, "\dashv"] \arrow[ur, shift right, "F_1"'] \arrow[ur, phantom, "\vdash"] &
\end{tikzcd}
\qquad U_2 = U_1 J
$$

If A has reflexive coequalisers &

$$
\begin{tikzcd}
F_1U_1F_1U_1 \arrow[r, shift left, "F_1U_1\varepsilon"] \arrow[r, shift right, "\varepsilon F_1U_1"'] & F_1U_1 \arrow[r, "\varepsilon"] & 1_B
\end{tikzcd}
$$

is a pointwise coequaliser, then J admits a left adjoint.

$$
\begin{tikzcd}
F_2U_1F_1U_1 \arrow[d, "F_2U_1F_1\cdot\eta_2\cdot U_1"'] \arrow[ddddr, bend left, "F_2U_1\cdot\varepsilon_1"] & \\
F_2U_1F_1U_2F_2U_1 \arrow[d, equal] & \\
F_2U_1F_1U_1JF_2U_1 \arrow[d, "F_2U_1\cdot\varepsilon_1\cdot JF_2U_1"'] & \\
F_2U_1JF_2U_1 \arrow[d, equal] & \\
F_2U_2F_2U_1 \arrow[r, bend right, "\varepsilon_2\cdot F_2U_1"'] & F_2U_1 \arrow[r] & L
\end{tikzcd}
$$

CATEGORIES WITH A CHOSEN OBJECT

Let (A, x) denote a category A with a chosen object $x \in A$.

Let $\mathbf{Cat}_*$ be the category of (small) categories with a chosen object.

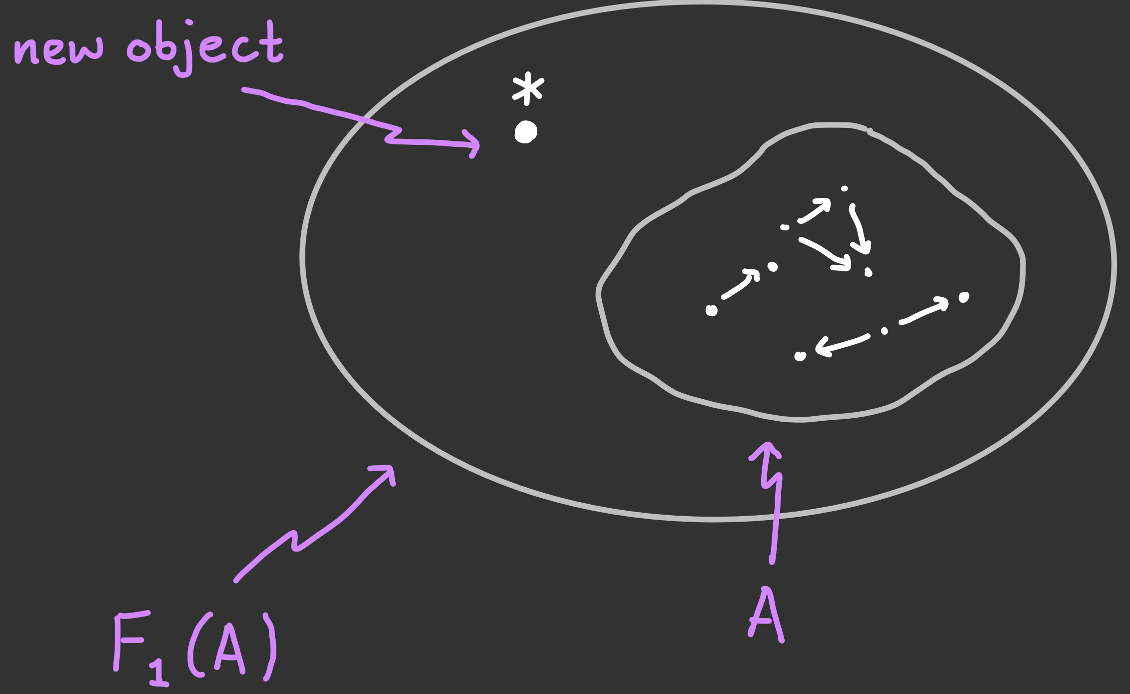
The forgetful functor

$$U_1 : \mathbf{Cat}_* \longrightarrow \mathbf{Cat}$$

is monadic.

Its left adjoint F_1 freely adjoins an object: $F_1(A) = (A + \underline{1}, *)$

coproduct with terminal category



CATEGORIES WITH A CHOSEN INITIAL OBJECT

Let $\mathbf{Cat}_\perp$ be the full subcategory

$$\mathbf{Cat}_\perp \overset{J}{\hookrightarrow} \mathbf{Cat}_*$$

of categories with chosen initial object.

The forgetful functor

$$\mathbf{Cat}_\perp \xrightarrow{U_2 = U_1 J} \mathbf{Cat}_*$$

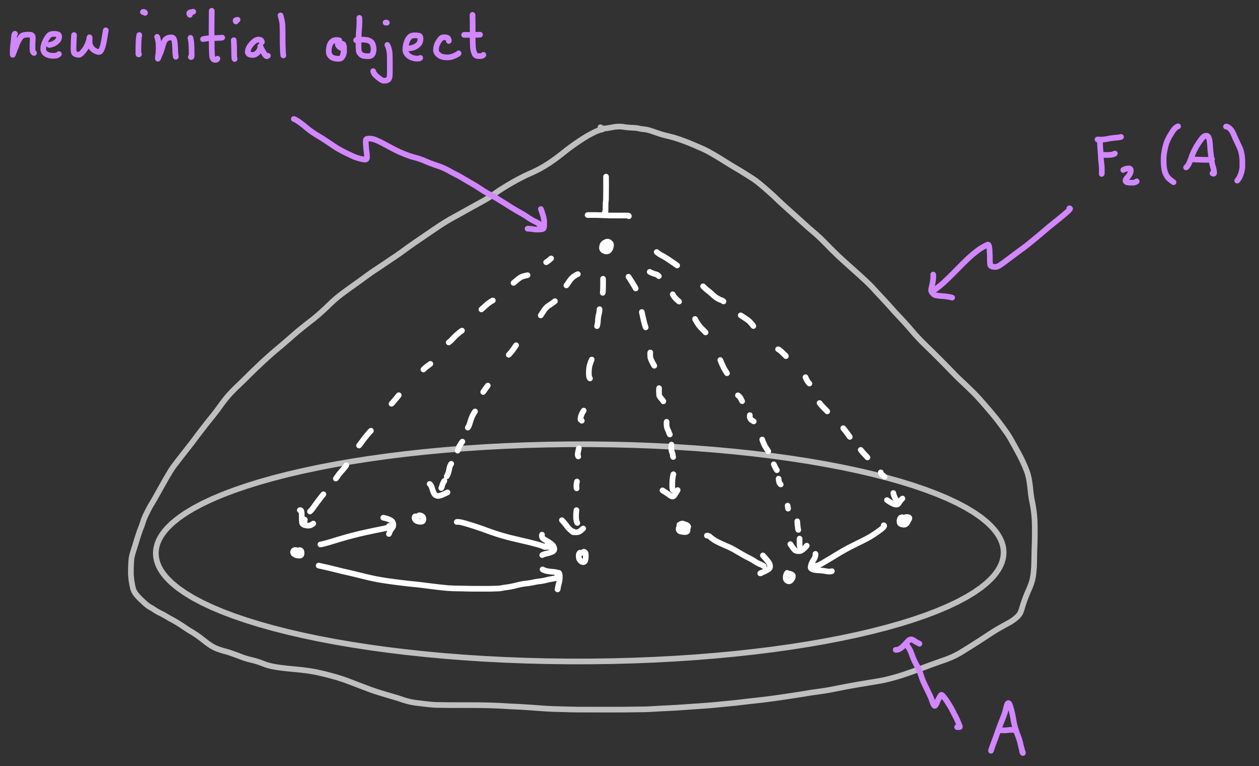
is monadic.

$$\underline{1} \overset{T}{\nrightarrow} A \quad \rightsquigarrow \quad \underline{1}^{op} \times A \longrightarrow \mathbf{Set}$$

$$a \longmapsto \{*\}$$

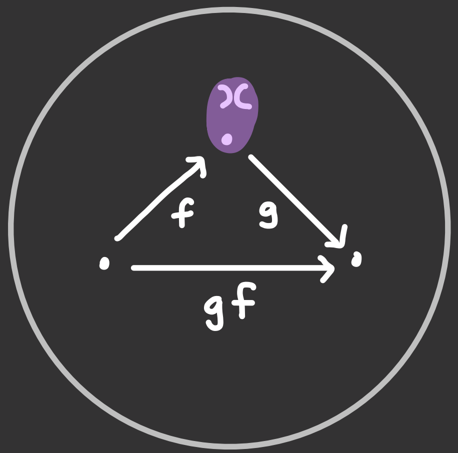
Its left adjoint F_2 freely adjoins an initial object: $F_2(A) = (\mathrm{Coll}(T), \perp)$

collage/cotabulator of terminal profunctor



TURNING A CHOSEN OBJECT INTO AN INITIAL OBJECT

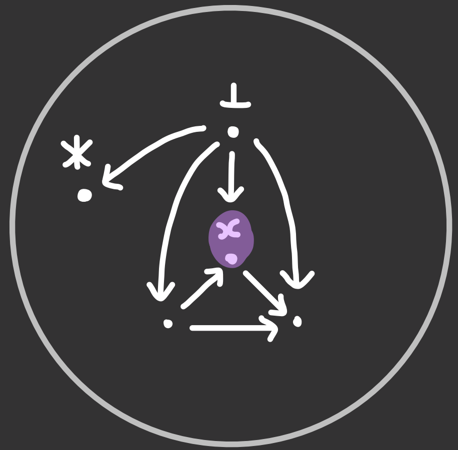
$$\begin{tikzcd} \mathrm{Cat}_\perp \arrow[rr, "J"', ""{name=J}] \arrow[dr, "U_2"] & & \mathrm{Cat}_* \arrow[ll, dashed, bend right=40, "L"', ""{name=L}] \arrow[dl, "U_1"'] \arrow[from=L, to=J, phantom, "\perp"] \\ & \mathrm{Cat} \arrow[ul, shift left=3, "F_2"] \arrow[ur, shift right=3, "F_1"'] \arrow[ul, phantom, shift left=1.5, "\top"] \arrow[ur, phantom, shift right=1.5, "\top"] & \end{tikzcd}$$



A

$\eta_A \downarrow$

$F_2 U_1 F_1 U_1(A)$



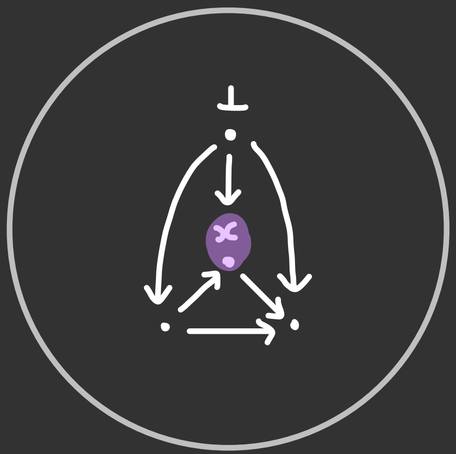
$S(*) = x$

$\xrightarrow{S}$

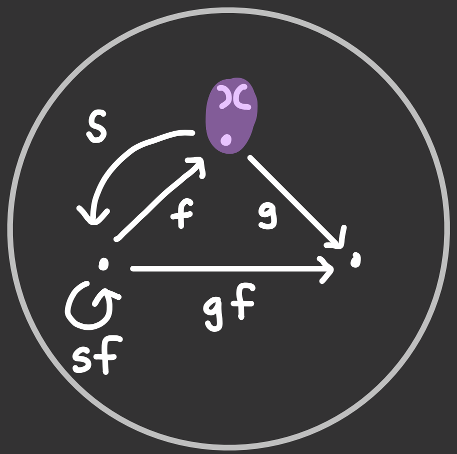
$\xrightarrow[T]{}$

$T(*) = \perp$

$F_2 U_1(A)$



$\xrightarrow{Q}$



$JL(A)$

$\|$

$L(A)$

DELTA LENSES &

SPLIT OPFIBRATIONS

DELTA LENSES

A **delta lens** $(f,\varphi) \colon A \nrightarrow B$ is a functor equipped with a **choice of lifts**

$$\begin{tikzcd} A \arrow[d, "f"'] & a \arrow[rr, "\varphi(a,u)"] \arrow[d, dotted, no head] & & \overline{\varphi}(a,u) \arrow[d, dotted, no head] \\ B & fa \arrow[rr, "u"] & & b \end{tikzcd}$$

satisfying the axioms:

1. $f\varphi(a,u) = u$
2. $\varphi(a, id_{fa}) = id_a$
3. $\varphi(a, v \circ u) = \varphi(\overline{\varphi}(a,u), v) \circ \varphi(a,u)$

Let $\mathcal{L}ens$ be the category of delta lenses whose morphisms a pairs of functors

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{(f,\varphi)}"'] & C \arrow[d, "{(g,\psi)}"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

such that $kf = gh$ & $h\varphi(a,u) = \psi(ha, ku)$.

SPLIT OPFIBRATIONS VIA PROPERTY

A delta lens $(f, \varphi) : A \nrightarrow B$ is a split opfibration if each $\varphi(a, u)$ is:

opcartesian

$$
\begin{tikzcd}
A \arrow[d, "f"'] & a \arrow[r, "\varphi(a,u)"'] \arrow[rr, bend left, "w"] & \overline{\varphi}(a,u) \arrow[r, dashed, "\exists!"'] & a' \\
B & fa \arrow[r, "u"] \arrow[rr, bend right, "fw"'] & b \arrow[r, "v"] & fa'
\end{tikzcd}
$$

weakly opcartesian

$$
\begin{tikzcd}
 & & \overline{\varphi}(a,fw) \arrow[d, dashed, "\exists!"] \\
A \arrow[d, "f"'] & a \arrow[ur, "\varphi(a,fw)"] \arrow[r, "w"'] & a' \\
B & fa \arrow[r, "fw"'] & fa'
\end{tikzcd}
$$

Let $\mathsf{SOpf} \hookrightarrow \mathcal{L}\mathsf{ens}$ denote the full subcategory of split opfibrations.

BASIC EXAMPLES

- A discrete opfibration is a functor with a unique choice of lifts.
- Construct $F_2(\mathbb{C}) \xrightarrow{F_2(!)} F_2(\underline{1})$

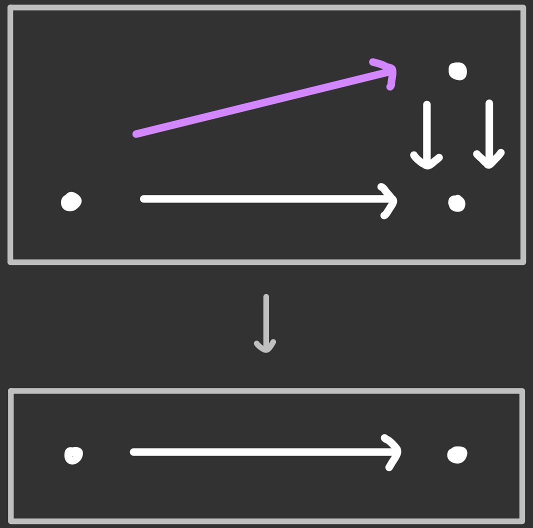
chosen object $\simeq$ delta lens

chosen initial object $\simeq$ split opfibration

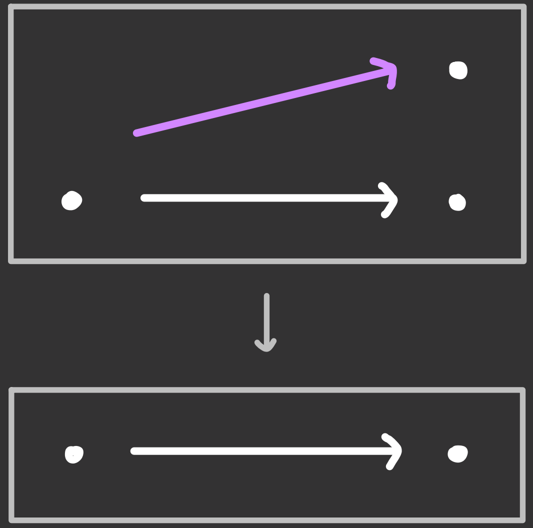
$$\begin{tikzcd} \mathcal{C}at_{\perp} \arrow[r] \arrow[d, hook] & \mathcal{SO}pf \arrow[d, hook] \\ \mathcal{C}at_{*} \arrow[r] & \mathcal{L}ens \end{tikzcd}$$

Delta lenses but not split opfibrations

Failure of uniqueness



Failure of existence



DIAGRAMMATIC APPROACH

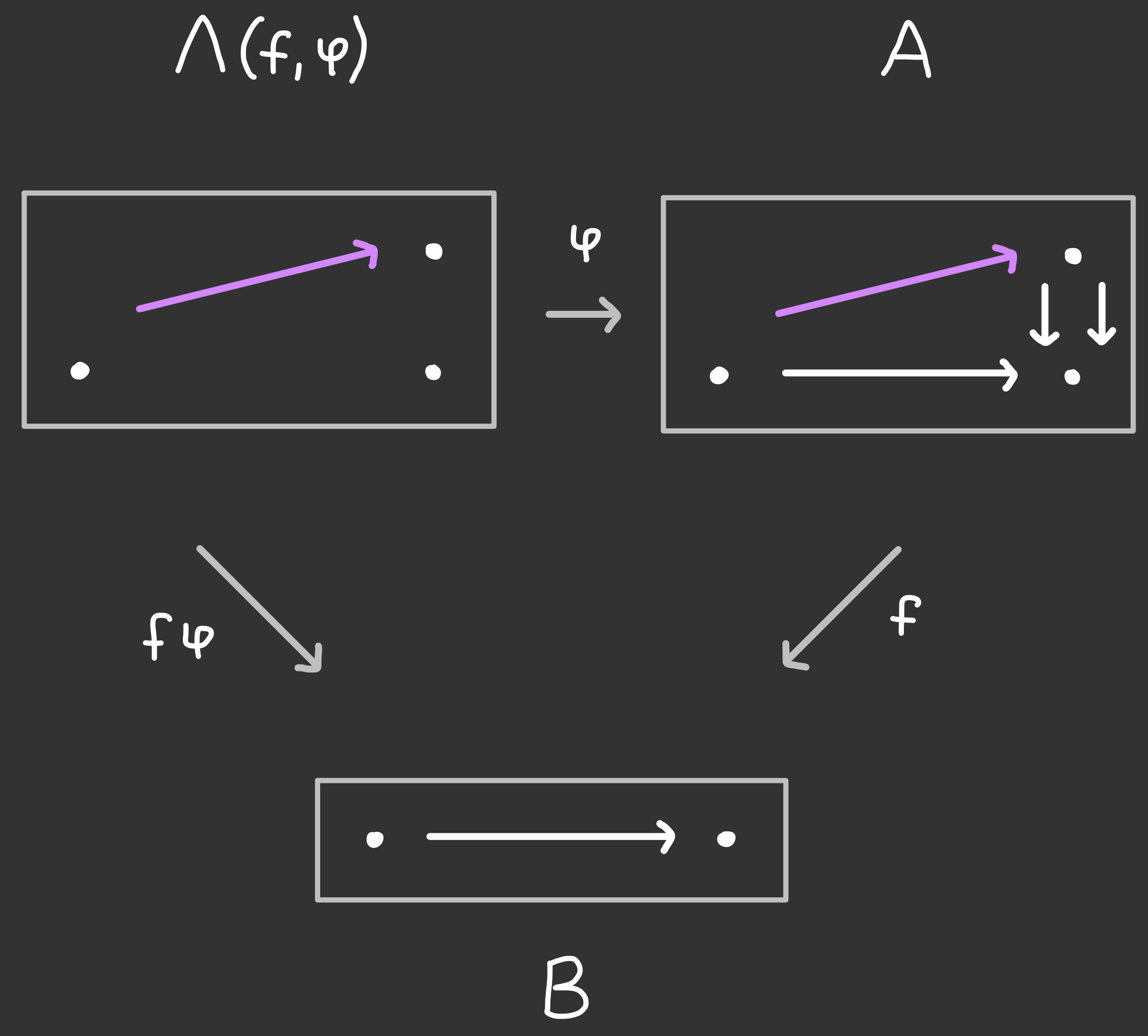
A delta lens $(f, \varphi) : A \to B$ determines a wide subcategory $\Lambda(f,\varphi) \rightarrowtail A$ whose morphisms are the chosen lifts $\varphi(a,u)$.

A delta lens is equivalent to a commutative diagram in $\mathcal{C}at$ s.t.

bijective-on-objects

$$\begin{tikzcd} \Lambda(f,\varphi) \simeq X \arrow[rr, "\varphi"] \arrow[dr, "f\varphi"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

discrete opfibration



FREE DELTA LENSES &

SPLIT OPFIBRATIONS

DELTA LENSES AS ALGEBRAS FOR A MONAD

The forgetful functor

$$\mathcal{L}ens \xrightarrow{U_1} \mathcal{C}at^{\mathbf{2}}$$

is monadic.

The left adjoint $F_1 \colon \mathcal{C}at^{\mathbf{2}} \longrightarrow \mathcal{L}ens$ is constructed using:

- discrete category comonad
- comprehensive factorisation system
- pushouts

$$\begin{tikzcd}
A_0 \arrow[r, "\text{b.o.o.}"] \arrow[d, "\text{initial}"'] & A \arrow[d, "\eta_1(f)"] \arrow[ddr, bend left, "f"] & \\
\sum_{a \in A_0} fa/B \arrow[r, "\text{b.o.o.}"] \arrow[drr, bend right, "\text{d.opf.}"'] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] \arrow[d, phantom, "\circledast"] & \\
& & B
\end{tikzcd}$$

FREE DELTA LENSES

The free delta lens $F_1 f : E_1(f) \to B$ on a functor $f : A \to B$ has domain whose:

- objects are pairs $(a \in A,\ u : fa \to b \in B)$
- morphisms $(a_1, u_1) \longrightarrow (a_2, u_2)$ are given by the following two sorts:

$$\begin{tikzcd}
a_1 \arrow[r, equal] & a_2 \\
fa_1 \arrow[r, equal] \arrow[d, "u_1"'] \arrow[dr, phantom, "\circlearrowright"] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[r, "v"'] & b_2
\end{tikzcd}
\qquad
\boxed{\begin{array}{c} a_1 = a_2 \\ v \circ u_1 = u_2 \end{array}}
\qquad\qquad
\begin{tikzcd}
a_1 \arrow[r, "w"] & a_2 \\
fa_1 \arrow[r, "fw"] \arrow[d, shift right, "u_1"'] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[u, shift right, "v"'] \arrow[r, "u_2 \circ fw \circ v"'] & b_2
\end{tikzcd}
\qquad
\boxed{\begin{array}{c} w \neq 1 \\ v \circ u_1 = 1 \end{array}}$$

The functor $F_1 f$ sends these to $v : b_1 \to b_2$ and $u_2 \circ fw \circ v : b_1 \to b_2$, respectively.

The chosen lifts are morphisms of the first sort.

SPLIT OPFIBRATIONS AS ALGEBRAS FOR A MONAD

The forgetful functor

$$\mathsf{SOpf} \xrightarrow{U_2} \mathsf{Cat}^2$$

is monadic.

The left adjoint $F_2 : \mathsf{Cat}^2 \longrightarrow \mathsf{SOpf}$ is constructed using comma cats. or cat. of elements / tabulator of:

$$A \overset{f_*}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B \quad \rightsquigarrow \quad A^{op} \times B \longrightarrow \mathsf{Set}$$
$$(a, b) \longmapsto B(fa, b)$$

$$\begin{tikzcd}
 & A \arrow[ddl, bend right, "1_A"'] \arrow[ddr, bend left, "f"] \arrow[d, dashed, "\eta_2(f)"] & \\
 & \bullet \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Rightarrow"] & \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}$$

The forgetful functor is also comonadic — see "A comonad for Grothendieck fibrations", 2024. Emmenegger, Et al.

FREE SPLIT OPFIBRATIONS

The free split opfibration $F_2 f : E_2(f) \longrightarrow B$ on a functor $f : A \to B$ has domain with:

- objects are pairs $(a \in A,\ u : fa \to b \in B)$
- morphisms $(a_1, u_1) \longrightarrow (a_2, u_2)$ are given by pairs $\langle w, v \rangle$

$$a_1 \xrightarrow{w} a_2$$

$$\begin{tikzcd}
fa_1 \arrow[r, "fw"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\circlearrowleft"] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[r, "v"'] & b_2
\end{tikzcd}$$

$$\boxed{v \circ u_1 = u_2 \circ fw}$$

The underlying functor $F_2 f$ sends these to $v : b_1 \to b_2$.

The chosen lifts are morphisms $\langle \mathrm{id}, v \rangle$.

SPLIT OPFIBRATIONS ARE REFLECTIVE IN DELTA LENSES

MAIN THEOREM

Thm: $\mathcal{SO}pf \hookrightarrow \mathcal{L}ens$ has a left adjoint.

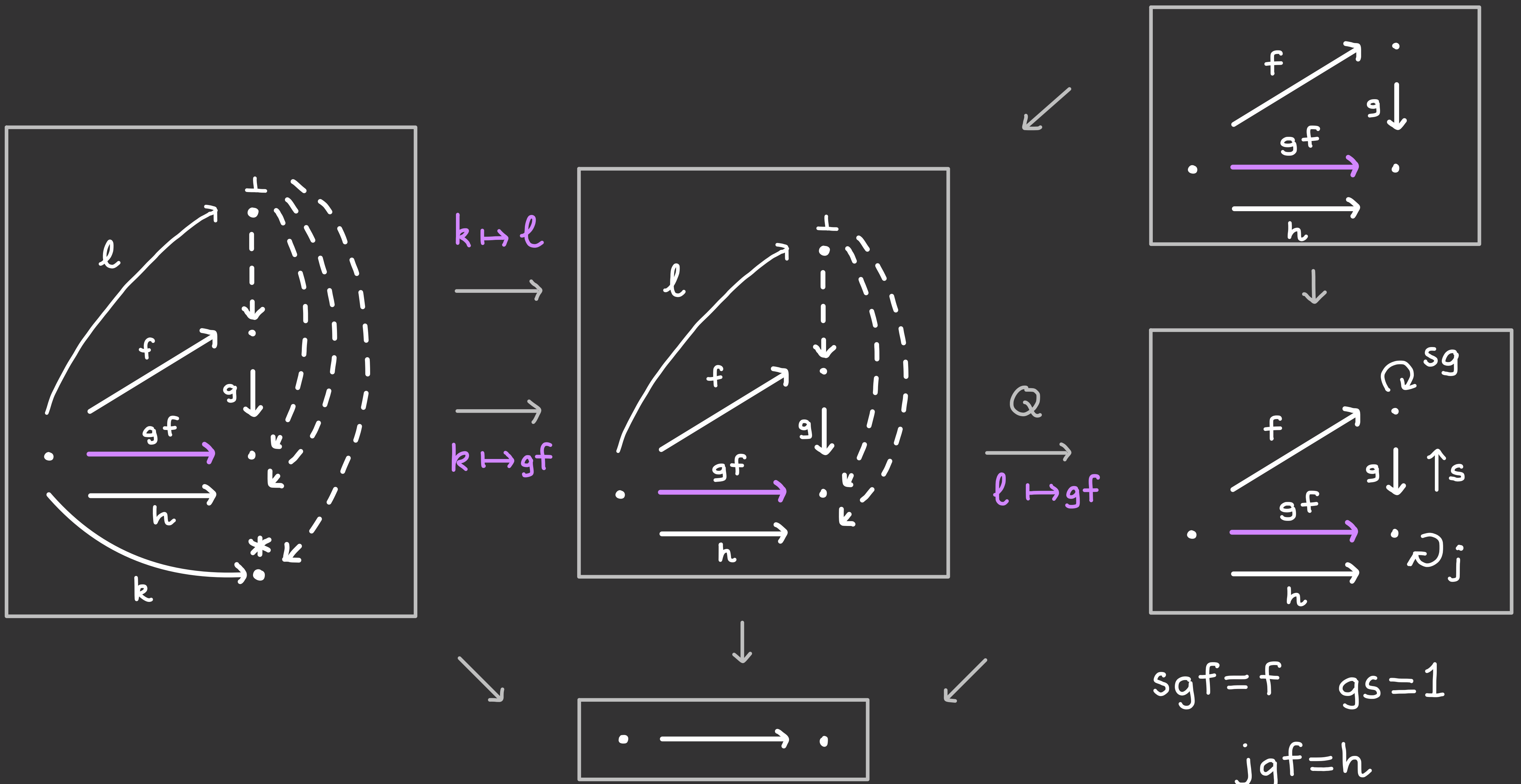
$$\begin{tikzcd}
\mathcal{SO}pf \arrow[rr, hook, bend right, "J"'] \arrow[rr, phantom, "\perp"] \arrow[dr, "U_2"] & & \mathcal{L}ens \arrow[ll, bend right, "L"'] \arrow[dl, "U_1"'] \\
& \mathcal{C}at^{\underline{2}} \arrow[ul, bend left, "F_2"] \arrow[ul, phantom, "\dashv"] \arrow[ur, bend right, "F_1"'] \arrow[ur, phantom, "\vdash"] &
\end{tikzcd}$$

Since U_1 is monadic & $\mathcal{SO}pf$ has reflexive coequalisers, we may construct L via the adjoint triangle theorem.

- The left adjoint adds the property of being a split opfibration to the structure of being a delta lens.
- Fixes the codomain/base and the set of objects of the domain.
- Have morphism of adjunctions:

$$\begin{tikzcd}
\mathcal{C}at_{\perp} \arrow[r] \arrow[d, shift right] \arrow[d, phantom, "\vdash"] & \mathcal{SO}pf \arrow[d, shift right] \arrow[d, phantom, "\vdash"] \\
\mathcal{C}at_{*} \arrow[r] \arrow[u, shift right] & \mathcal{L}ens \arrow[u, shift right]
\end{tikzcd}$$

TURNING A CHOSEN LIFT INTO AN OPCARTESIAN LIFT



$sgf = f \qquad gs = 1$

$jgf = h$

SUMMARY & FURTHER WORK

$$
\begin{tikzcd}
 & \mathcal{C}at \arrow[dl, shift right] \arrow[dr, shift left] & \\
\mathcal{C}at_{\perp} \arrow[ur, shift right] \arrow[ul, phantom] \arrow[rr, hook, shift right, ""] \arrow[d] & & \mathcal{C}at_{*} \arrow[ul, shift left] \arrow[ll, shift right] \arrow[ll, phantom, "\perp"] \arrow[d] \\
\mathrm{SOpf} \arrow[rr, hook, shift right, "J"'] \arrow[dr, shift left, "U_2"] & & \mathcal{L}ens \arrow[ll, shift right, "L"'] \arrow[ll, phantom, "\perp"] \arrow[dl, shift right, "U_1"'] \\
 & \mathcal{C}at^{\mathbf{2}} \arrow[ul, shift left, "F_2"] \arrow[ur, shift right, "F_1"'] &
\end{tikzcd}
$$

- Restricting to monoids allows us to construct the free Schreier split epimorphism on a split epi. in Mon.
- Developing a nice syntax for L.
- Does J admit a right adjoint?
- How may we better understand the relationship between structure and structure with property?