DRAFT NOTES FOR BOARD TALK

Limits and colimits in double categories, by example

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A <u>double category</u> consists of 4 kinds of things:

- objects $A, B, C, D, \ldots$
- tight morphisms $A \longrightarrow B$
- loose morphisms $A \rightarrow\!\!\!\!\!+\; C$
- cells

$$\begin{tikzcd} A \arrow[r, "P", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "Q"', "|" marking] & D \end{tikzcd}$$

We have both a tight composition (vertically) and a loose composition (horizontally).

The latter may only be unital & associative up to specified isomorphism (like a bicategory).

<u>Examples</u>:

- For each category $\mathcal{C}$, let $\mathbb{S}\mathrm{q}(\mathcal{C})$ be the double category whose cells are commutative squares in $\mathcal{C}$.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \end{tikzcd} \qquad gh = kf$$

- For each 2-category $\mathcal{K}$, let $\mathbb{Q}(\mathcal{K})$ be the "double category of quintets" with cells given by the 2-cells.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \end{tikzcd} \qquad \alpha : gh \Rightarrow kf$$

- Let $\mathbb{S}\mathrm{pan}$ be the double category of sets, functions, and spans. A cell is given by a diagram:

$$
\begin{tikzcd}
A \arrow[d, "f"'] & X \arrow[l, "h_1"'] \arrow[r, "h_2"] \arrow[d, "\alpha"] & C \arrow[d, "g"] \\
B & Y \arrow[l, "k_1"] \arrow[r, "k_2"'] & D
\end{tikzcd}
$$

Composition of loose morphisms is given by pullback.

$$
\begin{tikzcd}
 & & \bullet \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & X \arrow[dl] \arrow[dr] & {} & Y \arrow[dl] \arrow[dr] & \\
A & & B & & C
\end{tikzcd}
$$

This extends to loose composition of cells via universal property.

- Let $\mathbb{P}\mathrm{rof}$ be the double category of categories, functors, and profunctors.

$$
\frac{A \overset{P}{\longrightarrow\!\!\!\!\!\!\!\!\!\!\mid\;\;\;\;\;} C}{P : C^{op} \times A \longrightarrow \mathrm{Set}}
$$

A cell is given by a natural transformation

$$
\begin{tikzcd}
A \arrow[r, "P", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\
B \arrow[r, "Q"', "|" marking] & D
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
C^{op} \times A \arrow[rd, bend left, "P"] \arrow[d, "g \times f"'] \arrow[r, phantom, "\Downarrow \alpha"] & {} \\
D^{op} \times B \arrow[r, bend right, "Q"'] & \mathrm{Set}
\end{tikzcd}
$$

$$\alpha_{a,c} : P(c,a) \longrightarrow Q(gc, fa)$$

Composition of loose morphisms is given by coends.

$$
\frac{A \overset{P}{\longrightarrow\!\!\!\!\!\!\!\!\!\!\mid\;\;\;\;\;} B \overset{Q}{\longrightarrow\!\!\!\!\!\!\!\!\!\!\mid\;\;\;\;\;} C}{P \bullet Q : C^{op} \times A \longrightarrow \mathrm{Set}}
$$

$$(c, a) \longmapsto \int^{b \in B} Q(c,b) \times P(b,a)$$

$$c \xrightarrow{q} b \xrightarrow{p} a \sim c \xrightarrow{q'} b' \xrightarrow{p'} a$$

if $\exists u : b \to b'$ in B such that $q \otimes u = q'$, $u \otimes p' = p$

Terminal object (in $\mathbb{D}$)

A terminal object is an object T such that:

(i) for each object A there exists a unique tight morphism $t : A \to T$.

(ii) for each loose morphism $P : A \nrightarrow B$, there exists a unique cell

$$\begin{tikzcd}
A \arrow[r, "P", "|" marking] \arrow[d, "t"'] \arrow[dr, phantom, "\tau"] & B \arrow[d, "t"] \\
T \arrow[r, "\mathrm{id}_T"', "|" marking] & T
\end{tikzcd}$$

An initial object I is defined dually.

- In $\mathbb{S}\mathrm{pan}$, the singleton set is a terminal object

$$\begin{tikzcd}
A \arrow[d, "!"'] & X \arrow[l] \arrow[r] \arrow[d, "!"] & B \arrow[d, "!"] \\
\{*\} \arrow[r, equal] & \{*\} \arrow[r, equal] & \{*\}
\end{tikzcd}$$

- In $\mathbb{S}\mathrm{pan}$, the empty set is an initial object.

$$\begin{tikzcd}
\emptyset \arrow[r, equal] \arrow[d, "!"'] & \emptyset \arrow[r, equal] \arrow[d, "!"'] & \emptyset \arrow[d, "!"'] \\
A & X \arrow[l] \arrow[r] & B
\end{tikzcd}$$

- In $\mathbb{P}\mathrm{rof}$, the category $\underline{1}$ is a terminal object.

$$\begin{tikzcd}
A \arrow[r, "P", "|" marking] \arrow[d, "!"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "!"] \\
\underline{1} \arrow[r, "\mathrm{Hom}"', "|" marking] & \underline{1}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
C^{op} \times A \arrow[dr, bend left, "P"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \\
\underline{1}^{op} \times \underline{1} \arrow[r, bend right, "\mathrm{Hom}"'] & \mathcal{S}et
\end{tikzcd}$$

$$\alpha_{a,c} : P(c,a) \longrightarrow \{*\}$$

- In $\mathbb{P}\mathrm{rof}$, the empty category is an initial object.

$$\begin{tikzcd}
\underline{0} \arrow[r, "\mathrm{Hom}", "|" marking] \arrow[d, "!"'] \arrow[dr, phantom, "\alpha"] & \underline{0} \arrow[d, "!"] \\
B \arrow[r, "Q"', "|" marking] & D
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\underline{0}^{op} \times \underline{0} \arrow[dr, bend left, "\mathrm{Hom}"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \\
D^{op} \times B \arrow[r, bend right, "Q"'] & \mathcal{S}et
\end{tikzcd}$$

Products

Given a pair of objects A and B, their <u>product</u> is an object $A \times B$ together with tight morphisms $\pi_A : A \times B \longrightarrow A$ and $\pi_B : A \times B \longrightarrow B$ such that:

(i) For each span of tight morphisms

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, dashed, "\exists! \langle f,g \rangle"] & \\ A & & B \\ & A \times B \arrow[ul, "\pi_A"] \arrow[ur, "\pi_B"'] & \end{tikzcd}$$

there exists a unique tight morphism $\langle f,g \rangle$ such that above diagram commutes.

(ii) Given cells

$$\begin{tikzcd} X \arrow[r, "p", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, "g"] \\ A \arrow[r, "\mathrm{id}_A"', "|" marking] & A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "p", "|" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\beta"] & Y \arrow[d, "k"] \\ B \arrow[r, "\mathrm{id}_B"', "|" marking] & B \end{tikzcd}$$

there exists a unique cell

$$\begin{tikzcd} X \arrow[r, "p", "|" marking] \arrow[d, "\langle f,h \rangle"'] \arrow[dr, phantom, "\langle \alpha,\beta \rangle"] & Y \arrow[d, "\langle g,k \rangle"] \\ A \times B \arrow[r, "\mathrm{id}_{A \times B}"'] & A \times B \end{tikzcd}$$

such that

$$\frac{\langle \alpha,\beta \rangle}{\mathrm{id}_{\pi_A}} = \alpha \qquad \frac{\langle \alpha,\beta \rangle}{\mathrm{id}_{\pi_B}} = \beta$$

The coproduct $A+B$ of a pair of objects is defined dually.

- In Span, the product is just the usual product of sets. Given cells

$$\begin{tikzcd} C \arrow[d, "f"'] & X \arrow[l] \arrow[r] \arrow[d, "\alpha"] & D \arrow[d, "g"] \\ A \arrow[r, equal] & A \arrow[r, equal] & A \end{tikzcd} \qquad \begin{tikzcd} C \arrow[d, "h"'] & X \arrow[l] \arrow[r] \arrow[d, "\beta"] & D \arrow[d, "k"] \\ B \arrow[r, equal] & B \arrow[r, equal] & B \end{tikzcd}$$

there exists a unique cell

$$
\begin{tikzcd}
C \arrow[d, "{\langle f,h\rangle}"'] & X \arrow[l] \arrow[r] \arrow[d, "{\langle \alpha,\beta\rangle}"] & D \arrow[d, "{\langle g,k\rangle}"] \\
A\times B \arrow[r, equal] & A\times B \arrow[r, equal] & A\times B
\end{tikzcd}
$$

- In **Span**, the coproduct is just the usual coproduct of sets. Given cells,

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "\alpha"] & A \arrow[d, "g"] \\
C & X \arrow[l] \arrow[r] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "h"'] & B \arrow[r, equal] \arrow[d, "\beta"] & B \arrow[d, "k"] \\
C & X \arrow[l] \arrow[r] & D
\end{tikzcd}
$$

there exists a unique cell

$$
\begin{tikzcd}
A+B \arrow[r, equal] \arrow[d, "{[f,h]}"'] & A+B \arrow[r, equal] \arrow[d, "{[\alpha,\beta]}"] & A+B \arrow[d, "{[g,k]}"] \\
C & X \arrow[l] \arrow[r] & D
\end{tikzcd}
$$

- In **Prof**, the product is the usual product of categories.

Given cells

$$
\begin{tikzcd}
C^{op}\times A \arrow[dr, bend left, "P"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \\
A^{op}\times A \arrow[r, bend right, "Hom"'] & Set
\end{tikzcd}
$$

$$\alpha_{a,c} : P(c,a) \longrightarrow A(fc, ga)$$

$$
\begin{tikzcd}
C^{op}\times A \arrow[dr, bend left, "P"] \arrow[d, "h\times k"'] \arrow[dr, phantom, "\Downarrow\beta"] & \\
B^{op}\times B \arrow[r, bend right, "Hom"'] & Set
\end{tikzcd}
$$

$$\beta_{a,c} : P(c,a) \longrightarrow B(hc, ka)$$

there exists a unique cell

$$
\begin{tikzcd}
C^{op}\times A \arrow[dr, bend left, "P"] \arrow[d, "{\langle f,h\rangle \times \langle g,k\rangle}"'] \arrow[dr, phantom, "\Downarrow"] & \\
(A\times B)^{op}\times (A\times B) \arrow[r, bend right, "Hom"'] & Set
\end{tikzcd}
$$

$$\langle \alpha_{a,c}, \beta_{a,c}\rangle : P(c,a) \longrightarrow A(fc, ga)\times B(hc, ka)$$

- In Prof, the coproduct is the usual coproduct of categories. Given cells

$$\begin{tikzcd} A^{op}\times A \arrow[rd, bend left, "\mathrm{Hom}"] \arrow[d, "f\times g"'] \arrow[rd, phantom, "\Downarrow\alpha"] & \\ D^{op}\times B \arrow[r, bend right, "Q"'] & \mathrm{Set} \end{tikzcd}$$

$$\alpha_{a,a'} : A(a,a') \longrightarrow Q(fa, ga')$$

$$\begin{tikzcd} C^{op}\times C \arrow[rd, bend left, "\mathrm{Hom}"] \arrow[d, "h\times k"'] \arrow[rd, phantom, "\Downarrow\beta"] & \\ D^{op}\times B \arrow[r, bend right, "Q"'] & \mathrm{Set} \end{tikzcd}$$

$$\beta_{c,c'} : C(c,c') \longrightarrow Q(hc, kc')$$

there exists a unique cell

$$A^{op}\times A + C^{op}\times A + A^{op}\times C + C^{op}\times C \cong$$

$$\begin{tikzcd} (A+C)^{op}\times(A+C) \arrow[rd, bend left, "\mathrm{Hom}"] \arrow[d, "{[f,h]\times[g,k]}"'] \arrow[rd, phantom, "\Downarrow\gamma"] & \\ D^{op}\times B \arrow[r, bend right, "Q"'] & \mathrm{Set} \end{tikzcd}$$

where

$$\gamma_{x,y} = \begin{cases} \emptyset \xrightarrow{!} Q(fa, kc) & x=a \quad y=c \\ \emptyset \xrightarrow{!} Q(hc, ga) & x=c \quad y=a \\ \alpha_{a,a'} & x=a \quad y=a' \\ \beta_{c,c'} & x=c \quad y=c' \end{cases}$$

Equalisers

Given a parallel pair of tight morphisms $f,g : A \rightrightarrows B$, their equaliser is an object E and a cone of tight morphisms

$$\begin{tikzcd} & E \arrow[dl, "e"'] \arrow[dr, "fe=ge"] & \\ A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B \end{tikzcd}$$

such that

(i) For each cone

$$
\begin{tikzcd}
 & X \arrow[dl, bend right, "x"'] \arrow[d, dashed, "\exists!"'] \arrow[dr, bend left] & \\
 & E \arrow[dl, "e"'] \arrow[dr] & \\
A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B
\end{tikzcd}
$$

there exists a unique tight morphism $X \to E$ such that the diagram commutes.

(ii) For each cell α such that

$$
\begin{tikzcd}
X \arrow[r, "P", "\shortmid"{marking}] \arrow[d, "x"'] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, "y"] \\
A \arrow[r, "\mathrm{id}_A"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{id}_f"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathrm{id}_A"', "\shortmid"{marking}] & B
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "P", "\shortmid"{marking}] \arrow[d, "x"'] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, "y"] \\
A \arrow[r, "\mathrm{id}_A"', "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\mathrm{id}_g"] & A \arrow[d, "g"] \\
B \arrow[r, "\mathrm{id}_B"', "\shortmid"{marking}] & B
\end{tikzcd}
$$

there exists a unique cell

$$
\begin{tikzcd}
X \arrow[r, "P", "\shortmid"{marking}] \arrow[d, "t"'] \arrow[dr, phantom, "\gamma"] & Y \arrow[d, "s"] \\
E \arrow[r, "\mathrm{id}_E"', "\shortmid"{marking}] & E
\end{tikzcd}
$$

such that $\frac{\gamma}{\mathrm{id}_e} = \alpha$

The coequaliser is defined dually.

- **Span** has all (co)equalisers.
- **Prof** has all (co)equalisers

Tabulators

Given a loose morphism $P : A \to B$, its <u>tabulator</u> is an object TP together with a cell

$$
\begin{tikzcd}
TP \arrow[r, "\mathrm{id}_{TP}", "\shortmid"{marking}] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\tau_P"] & TP \arrow[d, "\pi_B"] \\
A \arrow[r, "P"', "\shortmid"{marking}] & B
\end{tikzcd}
$$

such that:

(i) For each cell

$$
\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & X \arrow[d, "g"] \\
A \arrow[r, "P"', "\shortmid"{marking}] & B
\end{tikzcd}
$$

there exists a unique tight morphism

$t : X \longrightarrow TP$ such that

$$\frac{id_t}{\tau_P} = \alpha .$$

(ii) Given cells $\alpha, \beta, \gamma, \delta$ such that

$$\begin{tikzcd} X \arrow[r, "id_X", "|" marking] \arrow[d, "f"'] & X \arrow[r, "Q", "|" marking] \arrow[d, "g"] & Y \arrow[d, "h"] \\ A \arrow[r, "P"', "|" marking] \arrow[ur, phantom, "\alpha"] & B \arrow[r, "id_B"', "|" marking] \arrow[ur, phantom, "\gamma"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, "Q", "|" marking] \arrow[d, "f"'] & Y \arrow[r, "id_Y", "|" marking] \arrow[d, "k"] & Y \arrow[d, "h"] \\ A \arrow[r, "id_A"', "|" marking] \arrow[ur, phantom, "\delta"] & A \arrow[r, "P"', "|" marking] \arrow[ur, phantom, "\beta"] & B \end{tikzcd}$$

there exists a unique cell

$$\begin{tikzcd} X \arrow[r, "Q", "|" marking] \arrow[d, "e_1"'] & Y \arrow[d, "e_2"] \\ TP \arrow[r, "id_{TP}"', "|" marking] \arrow[ur, phantom, "\varepsilon"] & TP \end{tikzcd}$$

such that

$$\frac{\varepsilon}{id_{\pi_A}} = \delta \qquad \frac{\varepsilon}{id_{\pi_B}} = \gamma \qquad \frac{id_{e_1}}{\tau_P} = \alpha \qquad \frac{id_{e_2}}{\tau_P} = \beta$$

- In Span the tabulator is given by the cell:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "g"] \\ A & X \arrow[l, "f"] \arrow[r, "g"'] & B \end{tikzcd}$$

The cotabulator is given by the cell:

$$\begin{tikzcd} A \arrow[d, "\imath_A"'] & X \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & B \arrow[d, "\imath_B"] \\ A +_X B \arrow[r, equal] & A +_X B \arrow[r, equal] & A +_X B \end{tikzcd}$$

pushout of f & g.

- In $\mathbb{P}$rof, the tabulator of $P : B^{op} \times A \longrightarrow Set$ is category of elements TP whose:
  - objects are elements $p : b \nrightarrow a$ in $\sum_{b \in B, a \in A} P(b, a)$.
  - morphisms are given by

$$\begin{tikzcd}
b \arrow[r, "p"] \arrow[d, "B(b,b') \ni u"'] & a \arrow[d, "v \in A(a,a')"] \\
b' \arrow[r, "p'"'] & a'
\end{tikzcd}$$

such that

$$p \otimes v = u \otimes p'$$

Given a cell in $\mathbb{P}\mathrm{rof}$

$$\begin{tikzcd}
X \arrow[r, "\mathrm{Hom}"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & X \arrow[d, "g"] \\
A \arrow[r, "P"'] & B
\end{tikzcd}$$

$$\alpha_{x,x'} : X(x,x') \longrightarrow P(gx, fx')$$

there is a unique functor.

$$X \longrightarrow TP$$

$$x \longmapsto \alpha_{x,x}(id_x) : gx \to fx$$

- In $\mathbb{P}\mathrm{rof}$, the cotabulator $P : B^{op} \times A \longrightarrow Set$ is category $Coll(P)$ whose:

- set of objects is $obj(A) + obj(B)$
- homset

$$Coll(P)(x,y) = \begin{cases} A(x,y) & \text{if } x,y \in A \\ B(x,y) & \text{if } x,y \in B \\ P(x,y) & \text{if } x \in B,\ y \in A \\ \emptyset & \text{otherwise} \end{cases}$$

- composition is determined by composition in A, B, and actions of P.

Given a cell in $\mathbb{P}\mathrm{rof}$

$$\begin{tikzcd}
A \arrow[r, "P"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\
X \arrow[r, "id_X"'] & X
\end{tikzcd}$$

$$\alpha_{b,a} : P(b,a) \longrightarrow X(gb, fa)$$

there is a unique functor

$$Coll(P) \longrightarrow X$$

$$p : b \to a \longmapsto \alpha_{b,a}(p)$$

General theory of limits

A *lax double functor* $F : \mathbb{C} \to \mathbb{D}$ is given by an assignment

$$
\begin{tikzcd}
A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\
B \arrow[r, "Q"', "\shortmid"{marking}] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \arrow[r, "Fp", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\alpha"] & FC \arrow[d, "Fg"] \\
FB \arrow[r, "FQ"', "\shortmid"{marking}] & FD
\end{tikzcd}
$$

that preserves the tight direction strictly, and the loose direction up to comparison cells:

$$
\begin{tikzcd}
FA \arrow[r, "\mathrm{id}_{FA}"] \arrow[d, equal] \arrow[dr, phantom, "\eta_A"] & FA \arrow[d, equal] \\
FA \arrow[r, "F(\mathrm{id}_A)"', "\shortmid"{marking}] & FA
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \arrow[r, "Ff", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\mu_{f,g}"] & FB \arrow[r, "Fg", "\shortmid"{marking}] & FC \arrow[d, equal] \\
FA \arrow[rr, "F(f \bullet g)"', "\shortmid"{marking}] & & FC
\end{tikzcd}
$$

A *tight transformation* between lax double functors $\alpha : F \to G : \mathbb{C} \to \mathbb{D}$ consists of:

(i) For each object $A \in \mathbb{C}$, a tight morphism $\alpha_A : FA \to GA$

(ii) For each loose morphism $P : A \nrightarrow B$ in $\mathbb{C}$, a cell

$$
\begin{tikzcd}
FA \arrow[r, "FP", "\shortmid"{marking}] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "\alpha_P"] & FB \arrow[d, "\alpha_B"] \\
GA \arrow[r, "GP"', "\shortmid"{marking}] & GB
\end{tikzcd}
$$

which is natural, and coherent with loose identities & composition.

A *pseudo loose transformation* between lax double functors $\sigma : F \nrightarrow G : \mathbb{C} \to \mathbb{D}$ consists of

(i) For each object $A \in \mathbb{C}$, a loose morphism $\sigma_A : FA \nrightarrow GA$

(ii) For each tight morphism $f : A \to B$ a cell

$$\begin{tikzcd} FA \arrow[r, "\sigma_A", "|" marking] \arrow[d, "Ff"'] \arrow[dr, phantom, "\sigma_f"] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\sigma_B"', "|" marking] & GB \end{tikzcd}$$

(iii) For loose morphism $P : A \nrightarrow B$ in $\mathbb{C}$, an isocell

$$\begin{tikzcd} FA \arrow[r, "FP", "|" marking] \arrow[d, equal] & FB \arrow[r, "\sigma_B", "|" marking] \arrow[d, phantom, "\sigma_P"] & GB \arrow[d, equal] \\ FA \arrow[r, "\sigma_A"', "|" marking] & GA \arrow[r, "GP"', "|" marking] & GB \end{tikzcd}$$

which is natural and satisfies 4 coherence laws.

A <u>modification</u>

pseudo loose trans.

tight trans.

$$\begin{tikzcd} F \arrow[r, "\sigma", "|" marking] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\theta"] & G \arrow[d, "\beta"] \\ F' \arrow[r, "\tau"', "|" marking] & G' \end{tikzcd}$$

has components θ_A for each object $A \in \mathbb{C}$

$$\begin{tikzcd} FA \arrow[r, "\sigma_A", "|" marking] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "\theta_A"] & GA \arrow[d] \\ F'A \arrow[r, "\tau_A"'] & G'A \end{tikzcd}$$

which is coherent in both tight and loose directions.

We have a double category $\mathbb{L}\mathrm{x}(\mathbb{C}, \mathbb{D})$ of lax double functors, tight transformations, pseudo loose transformations, and modifications for each pair of double categories $\mathbb{C}$ and $\mathbb{D}$.

If $\mathbb{C}$ is unitary, we have a diagonal $\Delta : \mathbb{D} \longrightarrow \mathbb{L}\mathrm{x}(\mathbb{C}, \mathbb{D})$

which sends each object to the constant double functor at that object.

The double category of cones over a lax double functor $F: \mathbb{C} \to \mathbb{D}$ is given by the comma double category

$$\begin{tikzcd}
\mathrm{Cone}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \mathbb{1} \arrow[d, "F"] \\
\mathbb{D} \arrow[r, "\Delta"'] & \mathbb{L}\mathrm{x}(\mathbb{C}, \mathbb{D})
\end{tikzcd}$$

The <u>limit</u> of the double functor $F: \mathbb{C} \to \mathbb{D}$ is a terminal object in $\mathrm{Cone}(F)$.

Theorem:

A double category has all limits if and only if it has products, equalisers, and tabulators.

Prop

$$\mathrm{Disc}: \begin{tikzcd}
\mathbb{S}\mathrm{pan} \arrow[r] & \mathbb{P}\mathrm{rof} \arrow[l, bend right=60, "\pi"', "\perp"] \arrow[l, bend left=60, "\mathrm{ob}", "\perp"']
\end{tikzcd}$$

unitary colax

lax

preserves all limits and coproducts / coequalisers.

References

Grandis & Paré (1999). Limits in double categories.

Grandis (2019). Higher Dimensional Categories.

The <u>limit</u> of a double functor $\mathbb{I} \xrightarrow{\Gamma} \mathbb{D}$ is an object $\lim\Gamma$ with projections

$$
\begin{tikzcd}
 & \lim\Gamma \arrow[dl, bend right, "\pi_C"'] \arrow[d, "\pi_A"] \arrow[dr, bend left, "\pi_B"] \arrow[dl, phantom, "\pi_p"] \arrow[dr, phantom, "\cong"] & \\
\Gamma C & \Gamma A \arrow[l, "|" marking, "\Gamma p"] \arrow[r, "\Gamma f"'] & \Gamma B
\end{tikzcd}
$$

such that

(i) for any other cone there exists a unique tight morphism

$$\bullet \longrightarrow \lim\Gamma$$

(ii) for any double cone

$$
\begin{tikzcd}
X \arrow[r, "|" marking, "id_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & X \arrow[r, "|" marking, "Q"] \arrow[d, "g"] \arrow[dr, phantom, "\gamma"] & Y \arrow[d, "h"] \\
\Gamma A \arrow[r, "|" marking, "\Gamma p"'] & \Gamma B \arrow[r, "|" marking, "id_{\Gamma B}"'] & \Gamma_B
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "|" marking, "Q"] \arrow[d, "f"'] \arrow[dr, phantom, "\delta"] & Y \arrow[r, "|" marking, "id_Y"] \arrow[d, "k"] \arrow[dr, phantom, "\beta"] & Y \arrow[d, "h"] \\
\Gamma A \arrow[r, "|" marking, "id_{\Gamma A}"'] & \Gamma A \arrow[r, "|" marking, "\Gamma p"'] & \Gamma B
\end{tikzcd}
$$

Factors uniquely as

$$
\begin{tikzcd}
X \arrow[rr, "|" marking, "Q"] \arrow[dr] \arrow[rr, phantom, shift right=4ex, "\exists!"] & & Y \arrow[dl] \\
 & \lim\Gamma \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] \arrow[d, phantom, "\pi_p"] & \\
\Gamma A \arrow[rr, "|" marking, "\Gamma p"'] & & \Gamma B
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[r, "id_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & X \arrow[r, "Q"] \arrow[d, "g"] \arrow[dr, phantom, "\gamma"] & Y \arrow[d, "h"] \\
A \arrow[r, "p"'] & B \arrow[r, "id_B"'] & B
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "Q"] \arrow[d, "f"'] \arrow[dr, phantom, "\delta"] & Y \arrow[r, "id_Y"] \arrow[d, "k"] \arrow[dr, phantom, "\beta"] & Y \arrow[d, "h"] \\
A \arrow[r, "id_A"'] & A \arrow[r, "p"'] & B
\end{tikzcd}
$$

there exists a unique cell

$$
\begin{tikzcd}
X \arrow[r, "Q"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\varepsilon"] & Y \arrow[d, "e_2"] \\
TP \arrow[r, "id_{TP}"'] & TP
\end{tikzcd}
$$

2-dimensional universal property in Prof

$$
\begin{tikzcd}
Y^{op} \times X \arrow[rr, bend left, "Q"] \arrow[d] \arrow[r, phantom, "\Downarrow"] & & Set \\
el(P)^{op} \times el(P) \arrow[urr] & &
\end{tikzcd}
\qquad (y,x) \mapsto
$$

$HOM(\beta_{y,y}(id_y) : hy \nrightarrow ky \,,\, \alpha_{x,x}(id_x) : gx \nrightarrow fx)$

$$
Q(y,x) \qquad m : y \nrightarrow x
\qquad
\begin{tikzcd}
hy \arrow[r, "\beta(id_y)"] \arrow[d, "\gamma(m)"'] & ky \arrow[d, "\delta(m)"] \\
gx \arrow[r, "\alpha(id_x)"'] & fx
\end{tikzcd}
$$

$$
\begin{tikzcd}
X^{op} \times X \arrow[rr, bend left, "Hom"] \arrow[d, "g \times f"'] \arrow[r, phantom, "\Downarrow \alpha"] & & Set \\
B^{op} \times A \arrow[urr, bend right, "p"'] & &
\end{tikzcd}
\qquad
\begin{array}{c} \alpha_{x,x'} \\ X(x,x') \\ \downarrow \\ P(gx, fx') \end{array}
$$

$$
\begin{tikzcd}
Y^{op} \times X \arrow[rr, bend left, "Q"] \arrow[d, "h \times g"'] \arrow[r, phantom, "\Downarrow \gamma"] & & Set \\
B^{op} \times B \arrow[urr, bend right, "Hom"'] & &
\end{tikzcd}
\qquad
\begin{array}{c} \gamma_{y,x} \\ Q(y,x) \\ \downarrow \\ B(hy, gx) \end{array}
$$

$$
\begin{tikzcd}
Y^{op} \times X \arrow[rr, bend left, "Q"] \arrow[d, "k \times f"'] \arrow[r, phantom, "\Downarrow \delta"] & & Set \\
A^{op} \times A \arrow[urr, bend right, "Hom"'] & &
\end{tikzcd}
\qquad
\begin{array}{c} \delta_{y,x} \\ Q(y,x) \\ \downarrow \\ A(ky, fx) \end{array}
$$

$$
\begin{tikzcd}
Y^{op} \times Y \arrow[rr, bend left, "Hom"] \arrow[d, "h \times k"'] \arrow[r, phantom, "\Downarrow \beta"] & & Set \\
B^{op} \times A \arrow[urr, bend right, "p"'] & &
\end{tikzcd}
\qquad
\begin{array}{c} \beta_{y,y'} \\ Y(y,y') \\ \downarrow \\ P(hy, ky') \end{array}
$$