REFLECTING ON LENSES & SPLIT OPFIBRATIONS

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MOTIVATION

$$\frac{\text{Split opfibrations over } B}{\text{Functors } B \longrightarrow \mathcal{C}at}$$

- **Delta lenses** capture the underlying structure of s. opfs.
- Lenses model **bidirectional transformations** — often want these are "least change".

Many similarities:

- Right class of AWFS
- Admit Grothendieck constructions
- Coalgebras for a comonad*

Goal for today:

$$\begin{tikzcd} \mathsf{SOpf} \arrow[rr, hook, shift right=2] \arrow[rr, phantom, "\perp"] & & \mathcal{L}\mathsf{ens} \arrow[ll, shift right=2] \end{tikzcd}$$

OUTLINE OF THE TALK

1. Warming up: Pointed categories & initial objects
2. Main characters: Delta lenses & split opfibrations
3. Free things: The complexity of computing coequalisers
4. Coming together: Split opfibrations are reflective in delta lenses

PART 1

Pointed categories & initial objects

POINTED CATEGORIES

A pointed category $(\mathcal{C}, x)$ is a category $\mathcal{C}$ with a chosen object x.

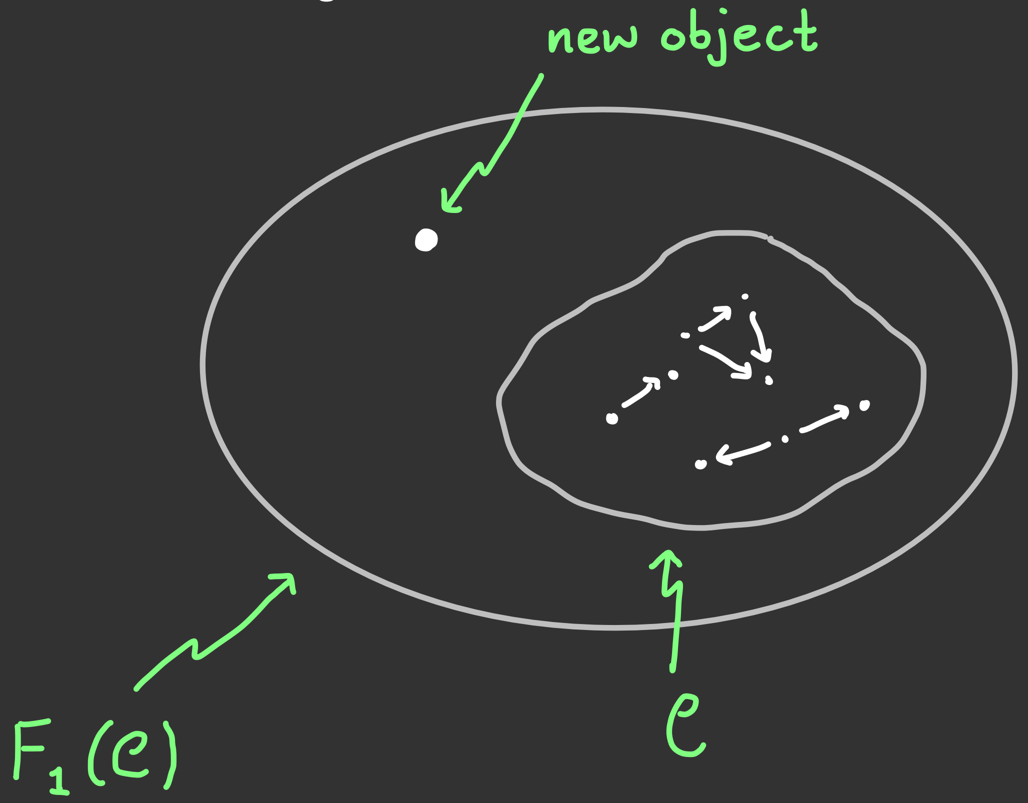
Let $\mathbf{Cat}_*$ be the category of (small) pointed categories.

The forgetful functor

$$\mathbf{Cat}_* \xrightarrow{\mathcal{U}_1} \mathbf{Cat}$$

is monadic.

The left adjoint $F_1 : \mathbf{Cat} \longrightarrow \mathbf{Cat}_*$ freely adds an object to each category.



CATEGORIES WITH (CHOSEN) INITIAL OBJECTS

Let $\mathsf{Cat}_\perp$ be the full subcategory

$$\mathsf{Cat}_\perp \xhookrightarrow{V} \mathsf{Cat}_*$$

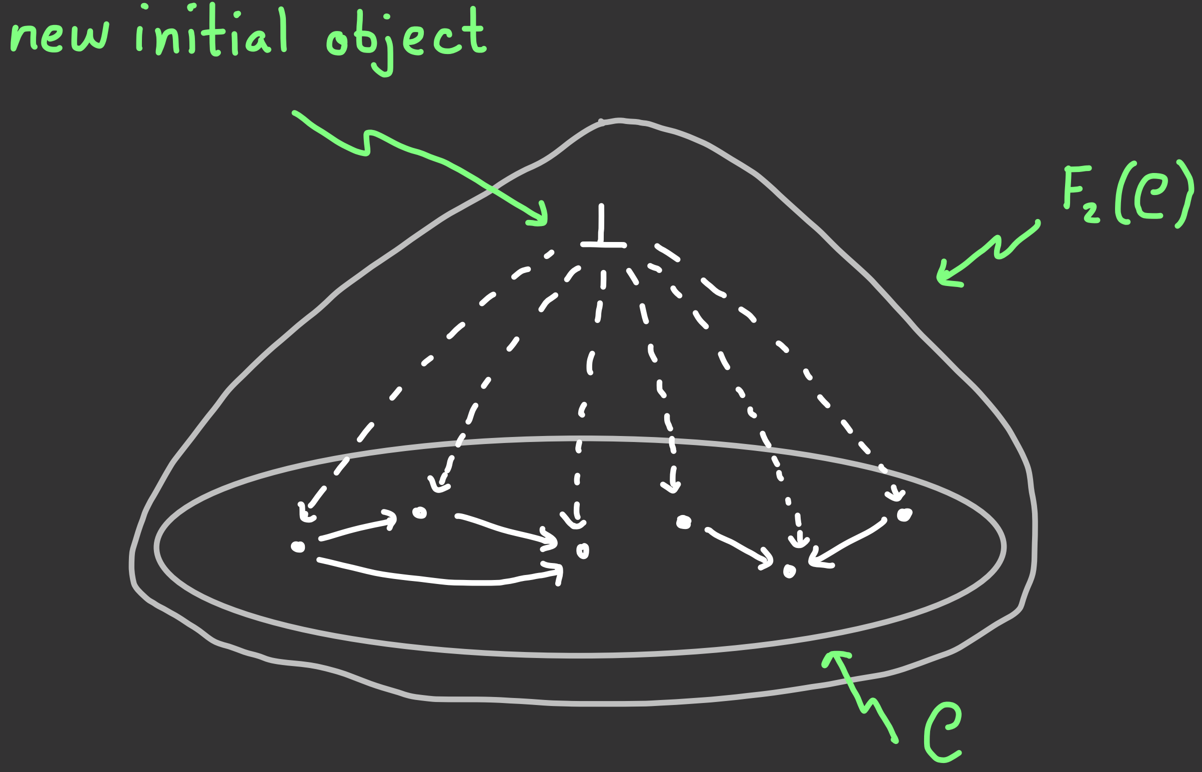
of categories with chosen initial object.

The forgetful functor

$$\mathsf{Cat}_\perp \xrightarrow{\mathcal{U}_2 = U_1 V} \mathsf{Cat}$$

is monadic.

The left adjoint $F_2 : \mathsf{Cat} \longrightarrow \mathsf{Cat}_\perp$ freely adds an initial object to each category.



TURNING A CHOSEN OBJECT INTO AN INITIAL OBJECT

$$\begin{tikzcd}
\mathrm{Cat}_\perp \arrow[rr, hook, "V" description] \arrow[dr, shift left] \arrow[rr, phantom, shift left=3, "\perp"] \arrow[rr, phantom, shift right=3, "\perp"] & & \mathrm{Cat}_* \arrow[ll, dashed, shift right=6, "L"'] \arrow[ll, dashed, shift left=6, "R"] \arrow[dl, shift left] \\
& \mathrm{Cat} \arrow[ul, shift left, "\top"] \arrow[ur, shift left, "\top"'] &
\end{tikzcd}$$

The category $L(\mathcal{C}, x)$ has:

- same objects as $\mathcal{C}$ & x chosen obj.
- $\mathrm{Hom}(c,d) = \begin{cases} 1 & \text{if } c = x \\ \mathcal{C}(c,d) & \text{otherwise} \end{cases}$

The category $R(\mathcal{C}, x) = {}^{x}/\mathcal{C}$

In detail, it has:

- objects given by morphisms $x \longrightarrow c$ in $\mathcal{C}$ & id_x chosen
- morphisms given by commutative triangles in $\mathcal{C}$:

$$\begin{tikzcd}
& x \arrow[dl] \arrow[dr] & \\
c \arrow[rr] & & d
\end{tikzcd}$$

PART 2

Delta lenses & split opfibrations

DELTA LENSES

A **delta lens** $(f,\varphi) : A \nrightarrow B$ is a functor equipped with a **choice of lifts**

$$\begin{tikzcd} A \arrow[d, "f"'] & a \arrow[r, "\varphi(a,u)"] \arrow[d, dotted, no head] & \overline{\varphi}(a,u) \arrow[d, dotted, no head] \\ B & fa \arrow[r, "u"] & b \end{tikzcd}$$

satisfying the axioms:

1. $f\varphi(a,u) = u$
2. $\varphi(a, \mathrm{id}_{fa}) = \mathrm{id}_a$
3. $\varphi(a, v \circ u) = \varphi(\overline{\varphi}(a,u), v) \circ \varphi(a,u)$

Let $\mathcal{L}ens$ be the category of delta lenses whose morphisms a pairs of functors

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{(f,\varphi)}"'] & C \arrow[d, "{(g,\psi)}"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

such that $kf = gh$ & $h\varphi(a,u) = \psi(ha,ku)$.

SPLIT OPFIBRATIONS VIA PROPERTY

A delta lens $(f,\varphi) : A \nrightarrow B$ is a split opfibration if each $\varphi(a,u)$ is:

opcartesian

$$
\begin{tikzcd}
A \arrow[d, "f"'] & a \arrow[r, "\varphi(a,u)"'] \arrow[rr, bend left=40, "w"] \arrow[d, dotted, no head] & \overline{\varphi}(a,u) \arrow[r, dashed, "\exists!"'] \arrow[d, dotted, no head] & a' \arrow[d, dotted, no head] \\
B & fa \arrow[r, "u"] \arrow[rr, bend right=40, "fw"'] & b \arrow[r, "v"] & fa'
\end{tikzcd}
$$

weakly opcartesian

$$
\begin{tikzcd}
 & & \overline{\varphi}(a,fw) \arrow[d, dashed, "\exists!"] \\
A \arrow[d, "f"'] & a \arrow[ur, "\varphi(a,fw)"] \arrow[r, "w"'] \arrow[d, dotted, no head] & a' \arrow[d, dotted, no head] \\
B & fa \arrow[r, "fw"'] & fa'
\end{tikzcd}
$$

Let $\mathsf{SOpf} \hookrightarrow \mathcal{L}\mathsf{ens}$ denote the full subcategory of split opfibrations.

BASIC EXAMPLES

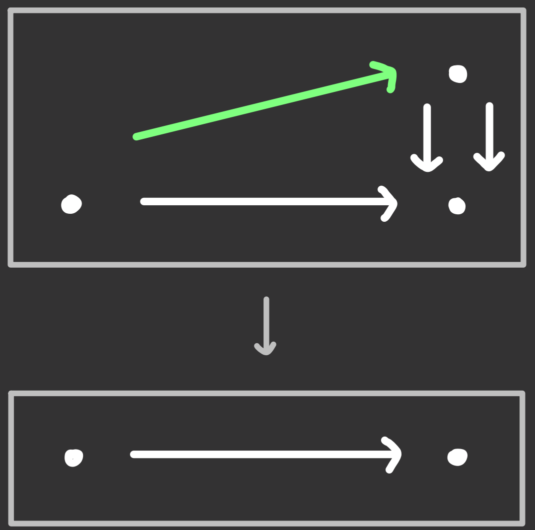
- A discrete opfibration is a functor with a unique choice of lifts.
- A split epimorphism of monoids is same as a delta lens / Schreier split epimorphism $\simeq$ split opfibration
- Construct $F_2(\mathcal{C}) \xrightarrow{F_2(!)} F_2(\mathbb{1})$

  Pointed category $\simeq$ delta lens

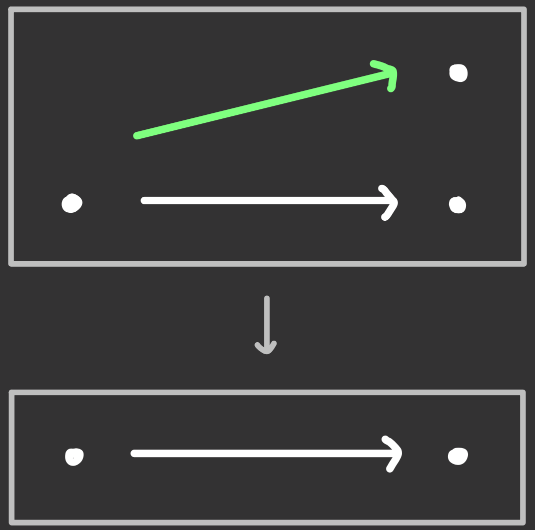
  chosen initial object $\simeq$ split opfibration

Delta lenses but not split opfibrations

Failure of uniqueness



Failure of existence



DIAGRAMMATIC APPROACH

A delta lens $(f,\varphi) : A \nrightarrow B$ determines a wide subcategory $\Lambda(f,\varphi) \rightarrowtail A$ whose morphisms are the chosen lifts $\varphi(a,u)$.

A delta lens is equivalent to a commutative diagram in $\mathcal{C}at$ s.t.

$$\begin{tikzcd}
& \Lambda(f,\varphi) \arrow[d, phantom, "\cong" rotate=90] & \\
& X \arrow[dl, "\varphi"'] \arrow[dr, "f\varphi"] & \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

bijective-on-objects

discrete opfibration

Decalage is a comonad on $\mathcal{C}at$

$$\mathcal{C} \longmapsto \mathrm{Dec}(\mathcal{C}) = \sum_{x \in \mathcal{C}} \mathcal{C}/x$$

Split opfibration is delta lens s.t.

discrete opfibration

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left] & \mathrm{Dec}(A) \arrow[r] \arrow[d, "\varepsilon_A"] & \mathrm{Dec}(B) \arrow[d, "\varepsilon_B"] \\
\Lambda(f,\varphi) \arrow[r, tail] \arrow[rr, bend right] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

PART 3

Free things: The complexity of computing coequalisers

DELTA LENSES AS ALGEBRAS FOR A MONAD

The forgetful functor

$$\mathcal{L}\mathrm{ens} \xrightarrow{U_1} \mathcal{C}\mathrm{at}^2$$

is monadic.

The left adjoint $F_1 : \mathcal{C}\mathrm{at}^2 \longrightarrow \mathcal{L}\mathrm{ens}$ is constructed using:

- discrete category comonad
- comprehensive factorisation system
- pushouts

$$\begin{tikzcd}
A_0 \arrow[r, "\text{b.o.o.}"] \arrow[d, "\text{initial}"'] & A \arrow[d, "\eta_1(f)"] \arrow[ddr, bend left, "f"] & \\
\sum\limits_{a \in A_0} fa/B \arrow[r, "\text{b.o.o.}"] \arrow[ddrr, phantom, "\ulcorner" very near end] \arrow[drr, bend right, "\text{d.opf.}"'] & \bullet \arrow[dr, dashed] \arrow[d, phantom, "\circledast"] & \\
 & & B
\end{tikzcd}$$

FREE DELTA LENSES

The **free delta lens** $\underline{\mathrm{cod}} : F_1 f \longrightarrow B$ on a functor $f : A \longrightarrow B$ has domain whose:

- objects are pairs $(a \in A,\ u : fa \to b \in B)$
- morphisms are generated by the following:

$$\begin{tikzcd} a_1 \arrow[r, equal] & a_2 \\ fa_1 \arrow[r, equal] \arrow[d, "u_1"'] \arrow[dr, phantom, "\circlearrowright"] & fa_2 \arrow[d, "u_2"] \\ b_1 \arrow[r, "v"'] & b_2 \end{tikzcd} \qquad \begin{tikzcd} a_1 \arrow[r, "w"] & a_2 \\ fa_1 \arrow[r, "fw"] \arrow[d, equal] & fa_2 \arrow[d, equal] \\ fa_1 \arrow[r, "fw"'] & fa_2 \end{tikzcd}$$

The functor $\underline{\mathrm{cod}}$ sends these to $v : b_1 \to b_2$ and $fw : fa_1 \to fa_2$, respectively.

The unit $\eta(f) : A \to F_1 f$ sends $w : a_1 \to a_2$ to the 2nd generator.

FREE DELTA LENSES

The **free delta lens** cod $: F_1 f \longrightarrow B$ on a functor $f : A \longrightarrow B$ has domain whose:

- objects are pairs $(a \in A,\ u : fa \to b \in B)$
- morphisms $(a_1, u_1) \longrightarrow (a_2, u_2)$ are given by the following two sorts:

$$\begin{tikzcd}
a_1 \arrow[r, equal] & a_2 \\
fa_1 \arrow[r, equal] \arrow[d, "u_1"'] \arrow[dr, phantom, "\circlearrowright"] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[r, "v"'] & b_2
\end{tikzcd}
\qquad
\begin{tikzcd}
a_1 \arrow[r, equal] & a_1 \arrow[r, "w"] & a_2 \arrow[r, equal] & a_2 \\
fa_1 \arrow[r, equal] \arrow[d, "u_1"'] \arrow[dr, phantom, "\circlearrowright"] & fa_1 \arrow[r, "fw"] \arrow[d, equal] & fa_2 \arrow[r, equal] \arrow[d, equal] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[r, "v"'] & fa_1 \arrow[r, "fw"'] & fa_2 \arrow[r, "u_2"'] & b_2
\end{tikzcd}$$

The functor cod sends these to $v : b_1 \to b_2$ and $u_2 \circ fw \circ v : b_1 \to b_2$, respectively.

The chosen lifts are morphisms of the first sort.

SPLIT OPFIBRATIONS AS ALGEBRAS FOR A MONAD

The forgetful functor

$$\mathrm{SOpf} \xrightarrow{U_2} \mathrm{Cat}^2$$

is monadic.

The left adjoint $F_2 : \mathrm{Cat}^2 \longrightarrow \mathrm{SOpf}$ is constructed using comma categories.

$$\begin{tikzcd}
 & A \arrow[dl, bend right, "1_A"'] \arrow[dr, bend left, "f"] \arrow[d, dashed, "\eta_2(f)"] & \\
 & \bullet \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] \arrow[d, phantom, "\Rightarrow"] & \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}$$

The forgetful functor is also comonadic – see "A comonad for Grothendieck fibrations", 2024. Emmenegger, Et al.

FREE SPLIT OPFIBRATIONS

The **free split opfibration** $\underline{\mathrm{cod}} : F_2 f \longrightarrow B$ on a functor $f : A \longrightarrow B$ has domain with:

- objects are pairs $(a \in A,\ u : fa \to b \in B)$
- morphisms $(a_1, u_1) \longrightarrow (a_2, u_2)$ are given by pairs $\langle w, v \rangle$

$$\begin{tikzcd} a_1 \arrow[r, "w"] & a_2 \end{tikzcd}$$

$$\begin{tikzcd} fa_1 \arrow[r, "fw"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\circlearrowright"] & fa_2 \arrow[d, "u_2"] \\ b_1 \arrow[r, "v"'] & b_2 \end{tikzcd}$$

The underlying functor $\underline{\mathrm{cod}}$ sends these to $v : b_1 \to b_2$.

The chosen lifts are morphisms $\langle \mathrm{id}, v \rangle$.

APPLYING THE ADJOINT TRIANGLE THEOREM

$$\begin{tikzcd}
\mathrm{SOpf} \arrow[rr, hook, shift right=2, "V"'] \arrow[dr, shift left, "U_2"] \arrow[rr, phantom, "\perp"] & & \mathrm{Lens} \arrow[ll, shift right=2, "L"'] \arrow[dl, shift right, "U_1"'] \\
& \mathrm{Cat}^{\underline{2}} \arrow[ul, shift left, "F_2"] \arrow[ur, shift right, "F_1"'] \arrow[ul, phantom, "\dashv"] \arrow[ur, phantom, "\vdash"] &
\end{tikzcd}$$

Since U_1 is monadic & SOpf has reflexive coequalisers, we may construct L via the adjoint triangle theorem as a pointwise coequaliser:

$$\begin{tikzcd}
F_2U_1F_1U_1 \arrow[d, "F_2U_1F_1\cdot\eta_2\cdot U_1"'] \arrow[ddr, bend left, "F_2U_1\cdot\varepsilon_1"] & & \\
F_2U_1F_1U_2F_2U_1 \arrow[d, equal] & & \\
F_2U_1F_1U_1VF_2U_1 \arrow[d, "F_2U_1\cdot\varepsilon_1\cdot VF_2U_1"'] & F_2U_1 \arrow[r] & L \\
F_2U_1VF_2U_1 \arrow[d, equal] & & \\
F_2U_2F_2U_1 \arrow[uur, bend right, "\varepsilon_2\cdot F_2U_1"'] & &
\end{tikzcd}$$

This is hard to compute!

PART 4

Split opfibrations are reflective in delta lenses

DELTA LENSES & POINTED CATEGORIES

Let $(f,\varphi) : A \nrightarrow B$ be a delta lens.

For each $a \in A$ and $u : fa \to b$ in B, we have a category $\mathrm{CoSl}(a,u)$ whose:

- objects are morphisms $w : a \to a'$ in A such that $fw = u$.
- morphisms are comm. diagrams

$$\begin{tikzcd} & a_1 \arrow[d, "p"] \\ a \arrow[ur, "w_1"] \arrow[r, "w_2"'] & a_2 \end{tikzcd}$$

such that $fp = \mathrm{id}_b$.

- A chosen object $\varphi(a,u)$ which is initial $\iff$ $\varphi(a,u)$ is w. opcartesian.

If $\varphi(a,u) : a \to a'$ is a chosen lift, we obtain a functor of pointed categories by precomposition:

$$\begin{tikzcd} a' & fa' \arrow[dr, "v"] & & & \mathrm{CoSl}(a',v) \arrow[dd, "(-)\circ\varphi(a,u)"] \\ & & b \arrow[r, squiggly] & {} & \\ a \arrow[uu, "\varphi(a,u)"] & fa \arrow[uu, "u"] \arrow[ur, "v \circ u"'] & & & \mathrm{CoSl}(a, v \circ u) \end{tikzcd}$$

$\varphi(a,u)$ is opcartesian $\iff$ this functor is invertible.

INDUCING FUNCTORS INTO Cat_*

Let $(f,\varphi): A \nrightarrow B$ be a delta lens, and let $\Lambda(f,\varphi)/b$ be the category with:

- objects are pairs $(a \in A,\ u: fa \to b \in B)$
- morphisms are comm. diagrams

chosen lift

$$\begin{tikzcd} a' & fa' \arrow[dr, "v"] & \\ & & b \\ a \arrow[uu, "\varphi(a{,}u)"] & fa \arrow[uu, "u"'] \arrow[ur, "v \circ u"'] & \end{tikzcd}$$

PROP: A delta lens (f,φ) gives a functor

$$\sum_{b \in B} \Lambda(f,\varphi)/b \xrightarrow{\mathrm{CoSl}} \mathrm{Cat}_*^{op}$$

- (f,φ) is a split opfibration $\iff$ CoSl factors through $\mathrm{Cat}_\perp^{op}$
- A morphism $(f,\varphi) \xrightarrow{\langle h,k \rangle} (g,\psi)$ yields

$$\begin{tikzcd} \sum_{b \in B} \Lambda(f,\varphi)/b \arrow[d] \arrow[dr, bend left, "\mathrm{CoSl}"] \arrow[dr, phantom, "\Downarrow"] & \\ \sum_{d \in D} \Lambda(g,\psi)/d \arrow[r, bend right, "\mathrm{CoSl}"'] & \mathrm{Cat}_*^{op} \end{tikzcd}$$

TWO CONJECTURES

$$\begin{tikzcd} \mathrm{SOpf} \arrow[r, hook, shift right, "V"'] \arrow[r, phantom, "\perp"] & \mathcal{L}\mathrm{ens} \arrow[l, shift right, "L"'] \end{tikzcd}$$

The left adjoint L is induced by postcomposing

$$\sum_{b \in B} \Lambda(f,\varphi)/b \xrightarrow{\mathrm{CoSl}} \mathrm{Cat}_*^{op}$$

by the left adjoint

$$\begin{tikzcd} \mathrm{Cat}_\perp \arrow[r, hook, shift right, "V"'] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_* \arrow[l, dashed, shift right] \end{tikzcd}$$

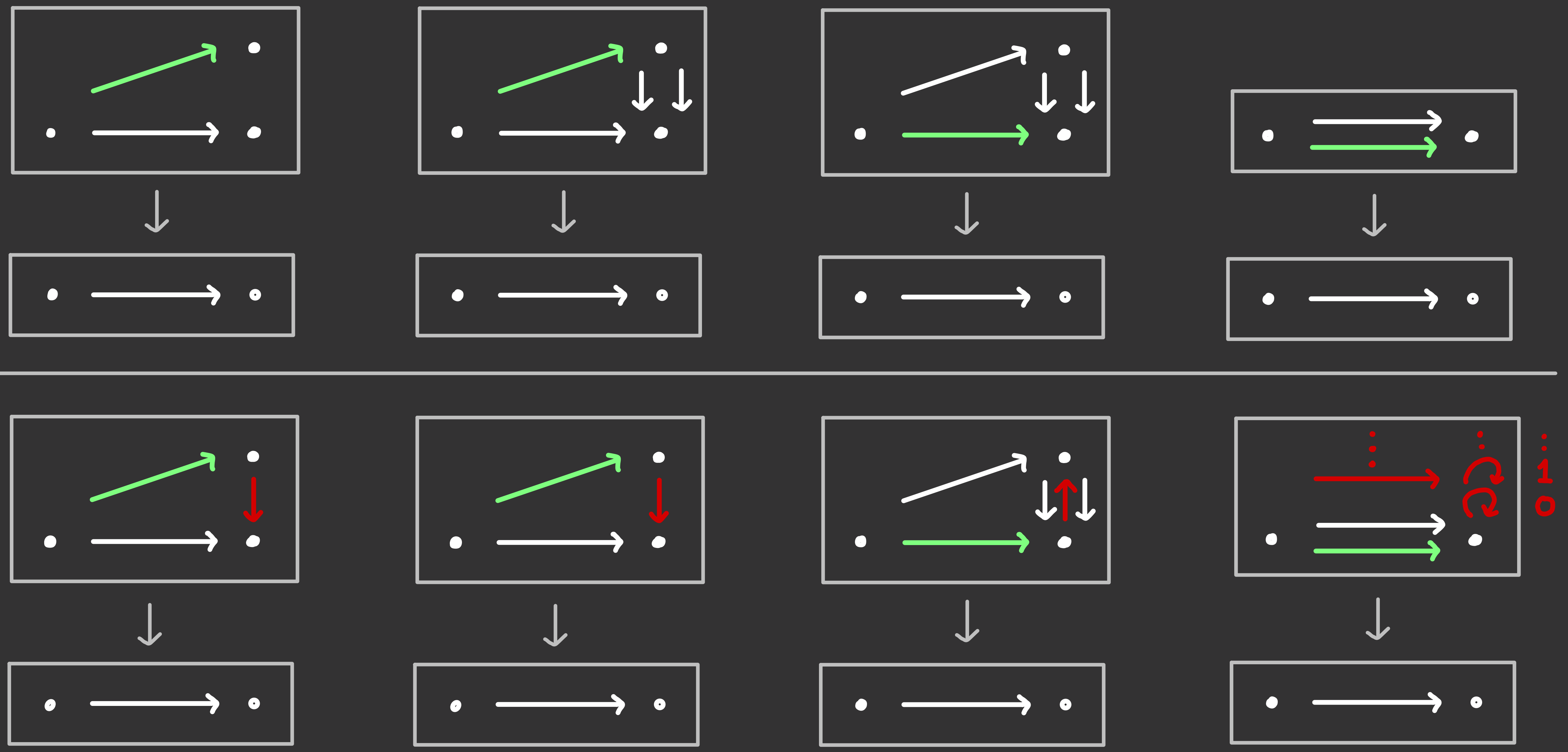
$$\begin{tikzcd} \mathrm{SOpf} \arrow[r, hook, shift left, "V"] \arrow[r, phantom, "\perp"] & \mathcal{L}\mathrm{ens} \arrow[l, shift left, "R"] \end{tikzcd}$$

There exists a right adjoint R which is induced by the right adjoint

$$\begin{tikzcd} \mathrm{Cat}_\perp \arrow[r, hook, shift left, "V"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_* \arrow[l, dashed, shift left, "R"] \end{tikzcd}$$

The previous approach fails, as domain of (f,φ) must change.

SIMPLE EXAMPLES



A POTENTIAL SYNTAX FOR L

$$\begin{tikzcd} \mathsf{SOpf} \arrow[r, hook, shift right, "V"'] \arrow[r, phantom, "\perp"] & \mathcal{L}\mathsf{ens} \arrow[l, shift right, "L"'] \end{tikzcd}$$

For a delta lens $(f,\varphi) : A \rightarrow\!\!\!\!\!\!+\, B$,

define $\pi : L(f,\varphi) \rightarrow\!\!\!\!\!\!+\, B$ to have:

- same objects as A.
- morphisms generated by

$$\frac{w : a \to a' \in A}{w : a \to a' \in L(f,\varphi)} \qquad \frac{m : x \to y \in L(f,\varphi)}{\chi(m) : \overline{\varphi}(x, fm) \to y \in L(f,\varphi)}$$

with chosen lifts same as A.

subject to the conditions

(1) $\pi w = fw \qquad \pi\chi(w) = id_{cod(w)}$

(2) $\chi(w) \circ \varphi(a, fw) = w$

(3) If $\pi m = id_b$ for $m \in L(f,\varphi)$,

$$\chi(m \circ \varphi(a,u)) = \chi(m)$$

We obtain a strict factorisation system on $L(f,\varphi)$ of chosen lifts followed by vertical morphisms.

SUMMARY & FUTURE WORK

$$\begin{tikzcd}
\mathsf{Cat}_\perp \arrow[rr, hook, "V" description] \arrow[dr, shift left] & & \mathsf{Cat}_* \arrow[ll, dashed, bend right, "L"'] \arrow[ll, dashed, bend left, "R"] \arrow[dl, shift left] \\
& \mathsf{Cat} \arrow[ul, shift left, "\top"] \arrow[ur, shift left, "\top"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{SOpf} \arrow[rr, hook, "V"'] \arrow[dr, shift left, "U_2"] & \perp & \mathsf{Lens} \arrow[ll, dashed, bend right, "L"'] \arrow[dl, shift left, "U_1"'] \\
& \mathsf{Cat}^2 \arrow[ul, shift left, "F_2"] \arrow[ur, shift left, "F_1"'] &
\end{tikzcd}$$

- Find an equivalence between $\mathcal{L}\mathsf{ens}$ and a category whose objects are Cat_*-valued functors.
- Can we prove that $\mathsf{SOpf} \hookrightarrow \mathcal{L}\mathsf{ens}$ has a right adjoint, or find a counterexample?
- Apply our results to construct a left adjoint to $\mathsf{SSEpi} \hookrightarrow \mathsf{SEpi}(\mathsf{Mon})$.