

THE ALGEBRAIC WEAK FACTORISATION SYSTEM OF TWISTED COREFLECTIONS & DELTA LENSES

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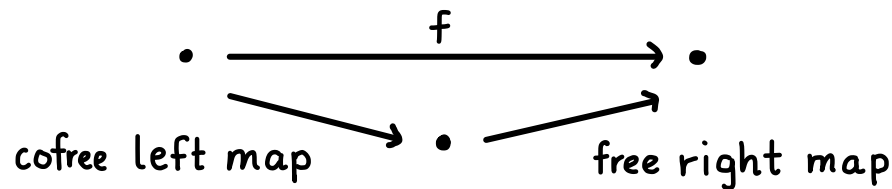
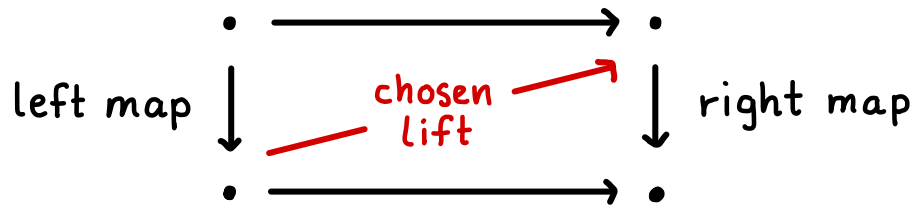
MOTIVATION & OVERVIEW

algebraic weak factorisation systems

GENERALISE

orthogonal factorisation systems

IDEA: Replace property with structure



delta lenses

GENERALISE

split opfibrations

IDEA: Drop requirement of opcartesianess

2011: Delta lenses introduced in comp. sci.

2013: Characterised as algebras for a semi-monad

TODAY:

left class	right class
split coreflection	split opfibration
twisted coreflection	delta lens

DELTA LENSES

A **delta lens** is a functor equipped with a **lifting operation**

$$\begin{array}{ccc} A & a & \xrightarrow{\varphi(a,u)} a' \\ f \downarrow & & \\ B & fa & \xrightarrow{u} b \end{array}$$

that satisfies the following axioms:

$$(L1) \quad f\varphi(a,u) = u$$

$$(L2) \quad \varphi(a, 1_{fa}) = 1_a$$

$$(L3) \quad \varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a, u)$$

A **split opfibration** is a delta lens such that:

$$(L4) \quad \text{Each } \varphi(a,u) \text{ is opcartesian.}$$

Let $\mathbb{L}ens$ denote the double category of categories, functors, & delta lenses.

A cell with boundary

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ (f, \varphi) \downarrow & & \downarrow (g, \psi) \\ B & \xrightarrow{k} & D \end{array}$$

exists if $kf = gh$ and $h\varphi(a,u) = \psi(ha, ku)$.

$$\mathbb{S}Opf \hookrightarrow \mathbb{L}ens \longrightarrow \mathbb{S}q(\mathbb{C}at)$$

DOUBLE-CATEGORICAL LIFTING OPERATIONS

$$\mathbb{L} \xrightarrow{u} \mathcal{S}_q(\mathcal{C}) \xleftarrow{v} \mathbb{R}$$

A (\mathbb{L}, \mathbb{R}) -lifting operation is a family

$$\begin{array}{ccc}
 \mathbb{U}A & \xrightarrow{s} & \mathbb{V}C \\
 \mathbb{U}i \downarrow & \nearrow \varphi_{i,j}(s,t) & \downarrow \mathbb{V}j \\
 \mathbb{U}B & \xrightarrow{t} & \mathbb{V}D
 \end{array}$$

which satisfies certain horizontal and vertical compatibilities.

$$\begin{array}{ccccc}
 \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\
 \mathbb{U}i \downarrow & & \mathbb{U}j \downarrow & \nearrow \varphi_{j,k} & \downarrow \mathbb{V}k \\
 \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot
 \end{array}
 =
 \begin{array}{ccccc}
 \cdot & \longrightarrow & \cdot & & \cdot \\
 \mathbb{U}i \downarrow & \nearrow \varphi_{i,j} & & & \downarrow \mathbb{V}k \\
 \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot
 \end{array}$$

$$\begin{array}{ccccc}
 \cdot & \longrightarrow & \cdot & & \cdot \\
 \mathbb{U}i \downarrow & \nearrow \varphi_{i,k} & & & \downarrow \mathbb{V}k \\
 \cdot & & \cdot & & \cdot \\
 \mathbb{U}j \downarrow & \nearrow \varphi_{j,k} & & & \downarrow \mathbb{V}k \\
 \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot
 \end{array}
 =
 \begin{array}{ccccc}
 \cdot & \longrightarrow & \cdot & & \cdot \\
 \mathbb{U}i \downarrow & \nearrow \varphi_{joi,k} & & & \downarrow \mathbb{V}k \\
 \cdot & & \cdot & & \cdot \\
 \mathbb{U}j \downarrow & & & & \downarrow \mathbb{V}k \\
 \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot
 \end{array}$$

+ dual compatibilities on right

THE DOUBLE CATEGORY IRLP(J)

$$\mathbb{J} \xrightarrow{u} \mathcal{S}_q(\mathcal{C})$$

Define a double category $IRLP(\mathbb{J})$ whose:

- objects & hor. morphisms are from \mathcal{C}
- vertical morphisms are pairs (f, φ) where

$$\begin{array}{ccc} UA & \xrightarrow{s} & C \\ \downarrow u_i & \nearrow \varphi_i(s,t) & \downarrow f \\ UB & \xrightarrow{t} & D \end{array}$$

f is a morphism in \mathcal{C}

φ is a (\mathbb{J}, f) -lifting operation

- cells $(f, \varphi) \rightarrow (g, \psi)$ are given by:

$$\begin{array}{ccc} \cdot & \xrightarrow{s} & \cdot & \xrightarrow{h} & \cdot & & \cdot & \xrightarrow{hs} & \cdot \\ u_i \downarrow & \nearrow \varphi_i & \downarrow f & & \downarrow g & = & u_i \downarrow & \nearrow \psi_i & \downarrow g \\ \cdot & \xrightarrow{t} & \cdot & \xrightarrow{k} & \cdot & & \cdot & \xrightarrow{kt} & \cdot \end{array}$$

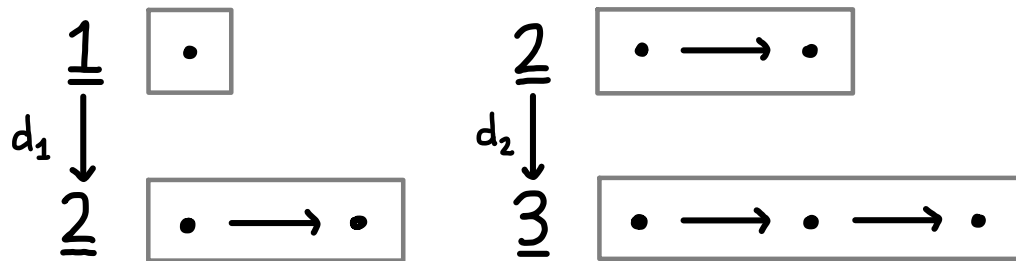
- Dually, we can define $ILLP(\mathbb{J})$.
- Given a (\mathbb{L}, \mathbb{R}) -lifting operation we obtain canonical double functors:

$$\mathbb{L} \longrightarrow ILLP(\mathbb{R}) \quad \mathbb{R} \longrightarrow IRLP(\mathbb{L})$$

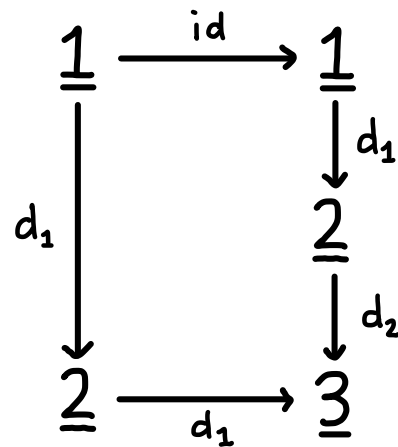
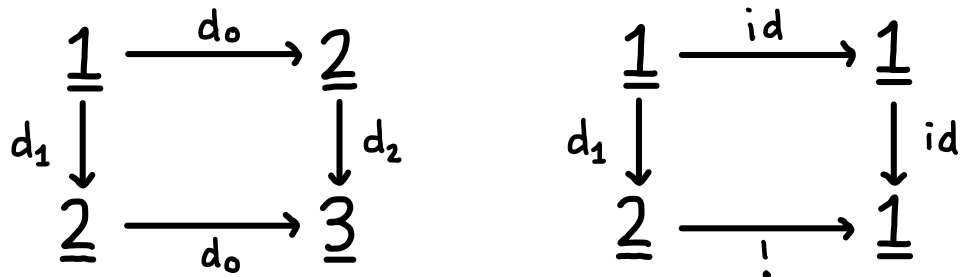
COFIBRANT GENERATION BY A SMALL DOUBLE CATEGORY

Let \mathbb{J}_{lens} be the double category whose:

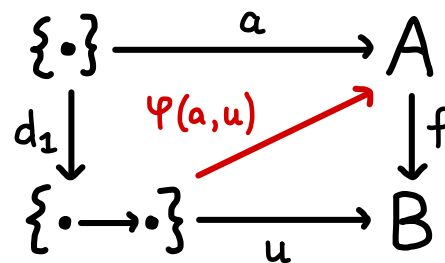
- objects are the posets $\underline{1}$, $\underline{2}$, and $\underline{3}$
- horizontal morphisms are monotone maps
- vertical morphisms are generated by



- cells are generated by



$$\mathbb{L}ens \cong \text{IRLP}(\mathbb{J}_{lens})$$



delta lens =
functor +
lifting operation

ALGEBRAIC WEAK FACTORISATION SYSTEMS

An **algebraic weak factorisation system** on \mathcal{C} is an (\mathbb{L}, \mathbb{R}) -lifting operation Ψ on

$$\mathbb{L} \xrightarrow{u} \mathcal{S}_q(\mathcal{C}) \xleftarrow{v} \mathbb{R}$$

such that the following axioms hold:

(i) the induced double functors are **iso**

$$\mathbb{L} \longrightarrow \mathbb{L}LP(\mathbb{R}) \quad \mathbb{R} \longrightarrow \mathbb{R}LP(\mathbb{L})$$

(ii) each f in \mathcal{C} admits a **factorisation**

$$\begin{array}{ccccc} \bullet & \xrightarrow{u_1 g} & \bullet & \xrightarrow{v_1 h} & \bullet \\ & \searrow & & \nearrow & \\ & & f & & \end{array}$$

which is U_1 -couniversal and V_1 -universal.

Theorem (Bourke-Garner): If \mathcal{C} is locally presentable & $\mathbb{J} \longrightarrow \mathcal{S}_q(\mathcal{C})$ is a small double category, then there exists an A.W.F.S. on \mathcal{C} with cospan:

$$\mathbb{L}LP(\mathbb{R}LP(\mathbb{J})) \longrightarrow \mathcal{S}_q(\mathcal{C}) \xleftarrow{\quad} \mathbb{R}LP(\mathbb{J})$$

Corollary: There exists an A.W.F.S. on $\mathcal{C}at$ whose right class is $\mathbb{L}ens$.

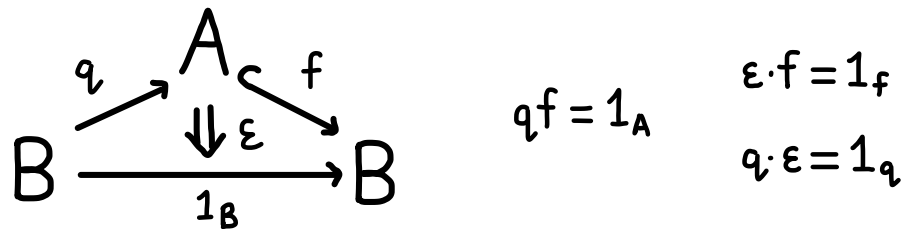
What is the left class $\mathbb{L}LP(\mathbb{L}ens)$?

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & D \end{array} \quad \text{delta lens}$$

???

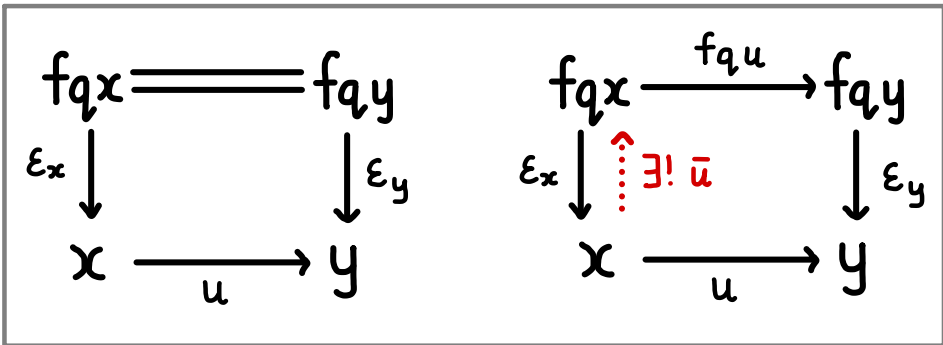
TWISTED COREFLECTIONS

A **twisted coreflection** is a split coreflection



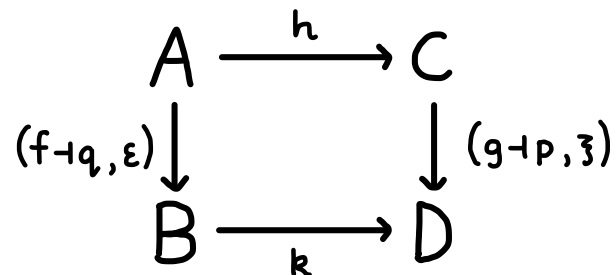
such that if $q(u: x \rightarrow y) \neq 1$, there exists a unique morphism $\bar{u}: x \rightarrow fqx$ such that:

- (i) $\bar{u} \circ \varepsilon_x = 1_{fqx}$
- (ii) $\varepsilon_y \circ fq u \circ \bar{u} = u$



Let $\Pi_w \text{Coref}$ denote the double category of categories, functors, & twisted coreflections.

A cell with boundary



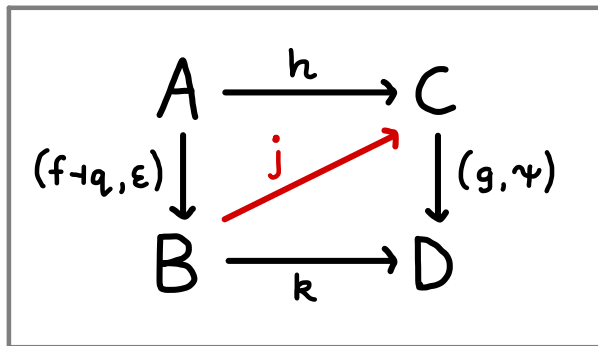
exists if $kf = gh$, $hq = pk$, and $k \cdot \varepsilon = \zeta \cdot k$.

$$\Pi_w \text{Coref} \hookrightarrow \mathcal{S} \text{Coref} \longrightarrow \mathcal{S}_q(\text{Cat})$$

LIFTING TWISTED COREFLECTIONS AGAINST DELTA LENSES

$$qx \xrightarrow{qu} qy$$

$$\begin{array}{ccc}
 hqx & \xrightarrow{hqu} & hqy \\
 \Psi(hqx, k\varepsilon_x) \downarrow & \uparrow \Psi(jx, k\bar{u}) & \downarrow \Psi(hqy, k\varepsilon_y) \\
 jx & \xrightarrow{j\bar{u}} & jy
 \end{array}$$

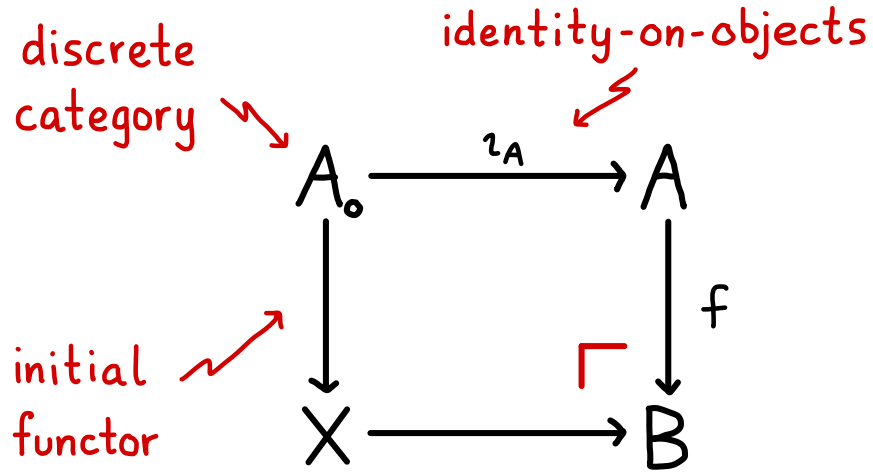


$$\begin{array}{ccc}
 fqx & \xrightarrow{fqu} & fqy \\
 \varepsilon_x \downarrow & \uparrow \exists! \bar{u} & \downarrow \varepsilon_y \\
 x & \xrightarrow{u} & y
 \end{array}$$

$$\begin{array}{ccc}
 g(hqx) & & g(hqy) \\
 \parallel & & \parallel \\
 kfqx & \longrightarrow & kfqy \\
 k\varepsilon_x \downarrow & \uparrow k\bar{u} & \downarrow k\varepsilon_y \\
 kx & \longrightarrow & ky
 \end{array}$$

DIAGRAMMATIC CHARACTERISATIONS

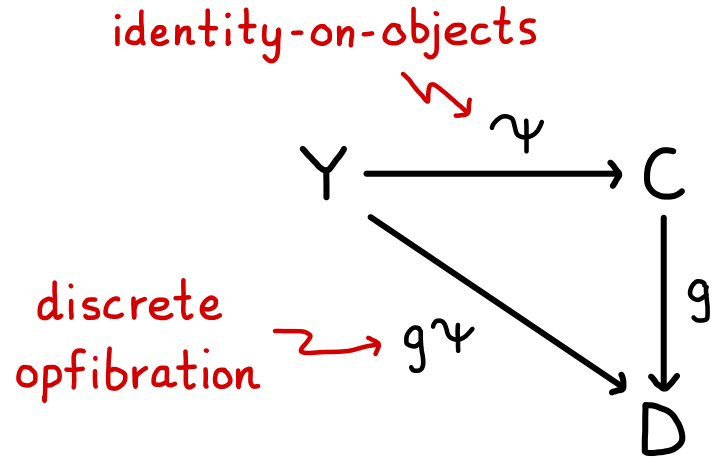
TWISTED COREFLECTION $(f \dashv q, \varepsilon)$



where the category X has:

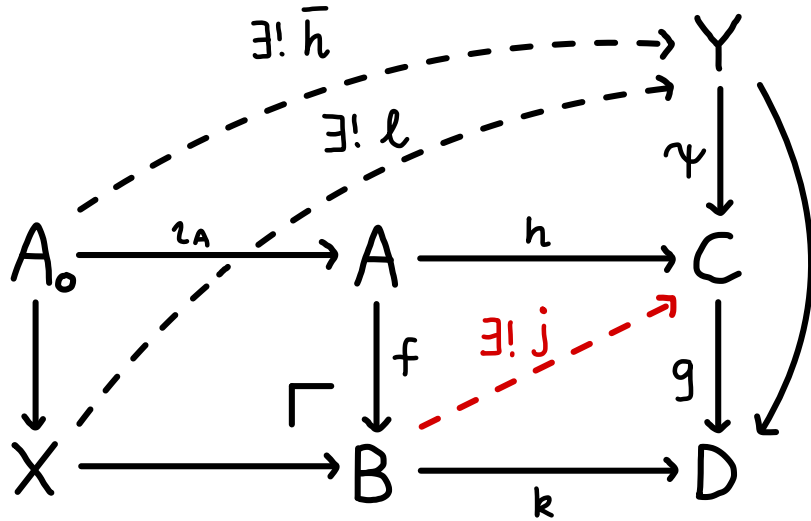
- same objects as B ;
- morphisms $u: x \rightarrow y$ in B such that $qu = 1$.

DELTA LENS (g, ψ)



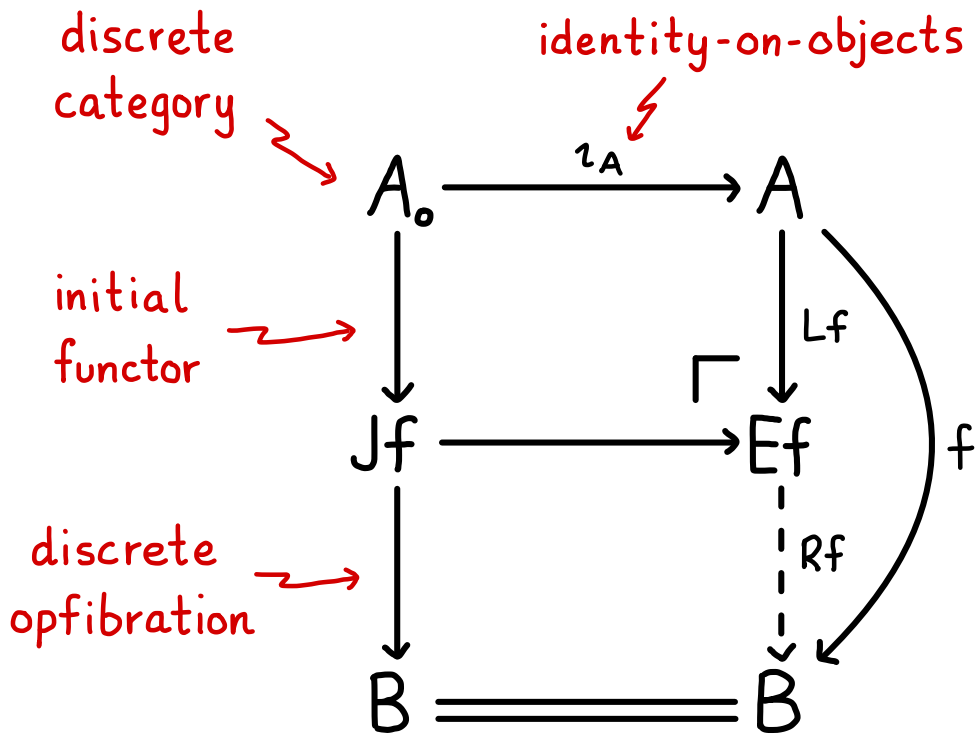
where the category Y has:

- same objects as C ;
- morphisms are the chosen lifts $\psi(a, u)$.



1. Construct unique $\bar{h}: A_0 \rightarrow Y$ by the universal property of **discrete categories** and **identity-on-objects** functors.
2. Construct unique $\ell: X \rightarrow Y$ by the universal property of the **comprehensive factorisation system** on Cat .
3. Construct unique $j: B \rightarrow C$ by the universal property of the **pushout**.

FACTORIZATION



where

$$Jf = \sum_{a \in A_0} f_a / B$$

- For a functor $f: A \rightarrow B$
 - the **cofree twisted coreflection** is $Lf: A \rightarrow Ef$
 - the **free delta lens** is $Rf: Ef \rightarrow B$
- There is a **comonad** L and a **monad** R on Cat^2 whose (co)algebras are twisted coreflections and delta lenses.

$$L\text{-Coalg} \cong \Pi \text{TwCoref}$$

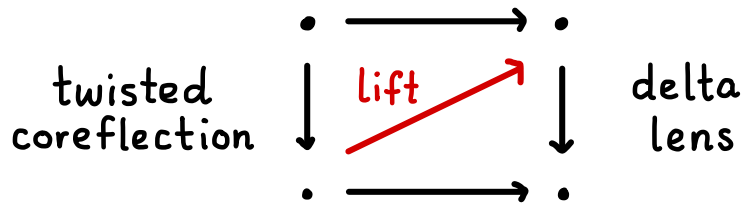
$$R\text{-Alg} \cong \Pi \text{Lens}$$

SUMMARY & FUTURE WORK

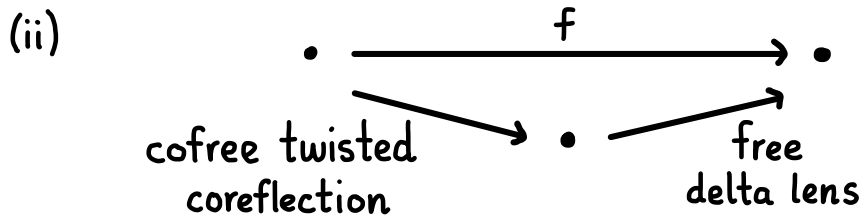
Theorem: There is an A.W.F.S. on Cat

$$\Pi\text{wCoref} \longrightarrow \mathcal{S}_q(\text{Cat}) \longleftarrow \text{Lens}$$

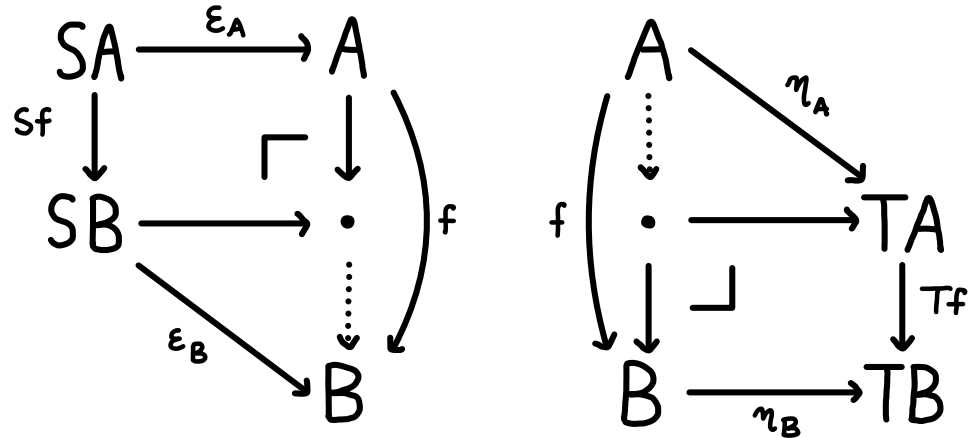
of twisted coreflections & delta lenses.



(i) $\Pi\text{wCoref} \cong \text{ILLP}(\text{Lens}) \cong L\text{-Coalg}$
 $\text{Lens} \cong \text{IRLP}(\Pi\text{wCoref}) \cong \text{IRLP}(\mathcal{J}_{\text{lens}}) \cong R\text{-Alg}$



- Can we generalise lifting operations by replacing $\mathcal{S}_q(\mathcal{C})$ with some ID?
- What is the relationship with (co)reflective factorisation systems?



(S, ε) idempotent comonad (T, η) idempotent monad