THE ALGEBRAIC WEAK FACTORISATION SYSTEM OF TWISTED COREFLECTIONS & DELTA LENSES

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MOTIVATION & OVERVIEW

algebraic weak factorisation systems

GENERALISE

orthogonal factorisation systems

IDEA: Replace property with structure

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\text{left map}"'] & \bullet \arrow[d, "\text{right map}"] \\ \bullet \arrow[r] \arrow[ur, "\text{chosen lift}"] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "\text{cofree left map}"'] & & \bullet \\ & \bullet \arrow[ur, "\text{free right map}"'] & \end{tikzcd}$$

delta lenses

GENERALISE

split opfibrations

IDEA: Drop requirement of opcartesianess

2011: Delta lenses introduced in comp. sci.

2013: Characterised as algebras for a semi-monad

TODAY:

left class	right class
split coreflection	split opfibration
twisted coreflection	delta lens

DELTA LENSES

A **delta lens** is a functor equipped with a **lifting operation**

$$
\begin{tikzcd}
A \arrow[d, "f"'] & a \arrow[r, "\varphi(a,u)"] & a' \\
B & fa \arrow[r, "u"] & b
\end{tikzcd}
$$

that satisfies the following axioms:

(L1) $\quad f\varphi(a,u) = u$

(L2) $\quad \varphi(a, 1_{fa}) = 1_a$

(L3) $\quad \varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a,u)$

A **split opfibration** is a delta lens such that:

(L4) $\quad$ Each $\varphi(a,u)$ is opcartesian.

Let $\mathbb{L}\mathrm{ens}$ denote the double category of categories, functors, & delta lenses.

A cell with boundary

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "{(f,\varphi)}"'] & C \arrow[d, "{(g,\psi)}"] \\
B \arrow[r, "k"'] & D
\end{tikzcd}
$$

exists if $kf = gh$ and $h\varphi(a,u) = \psi(ha, ku)$.

$$\boxed{\mathbb{S}\mathrm{Opf} \hookrightarrow \mathbb{L}\mathrm{ens} \longrightarrow \mathbb{S}\mathrm{q}(\mathcal{C}\mathrm{at})}$$

DOUBLE-CATEGORICAL LIFTING OPERATIONS

$$\mathbb{L} \xrightarrow{U} \mathbb{S}q(\mathcal{C}) \xleftarrow{V} \mathbb{R}$$

A $(\mathbb{L}, \mathbb{R})$-lifting operation is a family

$$\begin{tikzcd} UA \arrow[r, "s"] \arrow[d, "Ui"'] & VC \arrow[d, "Vj"] \\ UB \arrow[r, "t"'] \arrow[ur, "\varphi_{i,j}(s,t)"] & VD \end{tikzcd}$$

which satisfies certain horizontal and vertical compatibilities.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "Ui"'] & \bullet \arrow[r] \arrow[d, "Uj"'] & \bullet \arrow[d, "Vk"] \\ \bullet \arrow[r] & \bullet \arrow[r] \arrow[ur, "\varphi_{j,k}"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r] \arrow[d, "Ui"'] & \bullet \arrow[d, "Vk"] \\ \bullet \arrow[r] \arrow[ur, "\varphi_{i,j}"] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "Ui"'] & \bullet \arrow[dd, "Vk"] \\ \bullet \arrow[d, "Uj"'] \arrow[ur, "\varphi_{i,k}"] & \\ \bullet \arrow[r] \arrow[uur, "\varphi_{j,k}"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r] \arrow[d, "Ui"'] & \bullet \arrow[dd, "Vk"] \\ \bullet \arrow[d, "Uj"'] & \\ \bullet \arrow[r] \arrow[uur, "\varphi_{j\circ i,k}"] & \bullet \end{tikzcd}$$

+ dual compatibilities on right

THE DOUBLE CATEGORY $\mathbb{RLP}(\mathbb{J})$

$$\mathbb{J} \xrightarrow{\;U\;} \mathbb{S}\mathrm{q}(\mathcal{C})$$

Define a double category $\mathbb{RLP}(\mathbb{J})$ whose:

- objects & hor. morphisms are from $\mathcal{C}$
- vertical morphisms are pairs (f, φ) where

$$\begin{tikzcd} UA \arrow[r, "s"] \arrow[d, "Ui"'] & C \arrow[d, "f"] \\ UB \arrow[r, "t"'] \arrow[ur, "\varphi_i(s,t)"] & D \end{tikzcd}$$

f is a morphism in $\mathcal{C}$

φ is a $(\mathbb{J}, f)$-lifting operation

- cells $(f, \varphi) \longrightarrow (g, \psi)$ are given by:

$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "Ui"'] & \bullet \arrow[r, "h"] \arrow[d, "f"] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "t"'] \arrow[ur, "\varphi_i"] & \bullet \arrow[r, "k"'] & \bullet \end{tikzcd} \quad = \quad \begin{tikzcd} \bullet \arrow[r, "hs"] \arrow[d, "Ui"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "kt"'] \arrow[ur, "\psi_i"] & \bullet \end{tikzcd}$$

- Dually, we can define $\mathbb{LLP}(\mathbb{J})$.

- Given a $(\mathbb{L}, \mathbb{R})$-lifting operation we obtain canonical double functors:

$$\mathbb{L} \longrightarrow \mathbb{LLP}(\mathbb{R}) \qquad \mathbb{R} \longrightarrow \mathbb{RLP}(\mathbb{L})$$

COFIBRANT GENERATION BY A SMALL DOUBLE CATEGORY

Let $\mathbb{J}_{\text{lens}}$ be the double category whose:

- objects are the posets $\underline{1}$, $\underline{2}$, and $\underline{3}$
- horizontal morphisms are monotone maps
- vertical morphisms are generated by

$$\begin{array}{ll} \underline{1} & \boxed{\bullet} \\ \downarrow d_1 & \\ \underline{2} & \boxed{\bullet \longrightarrow \bullet} \end{array} \qquad \begin{array}{ll} \underline{2} & \boxed{\bullet \longrightarrow \bullet} \\ \downarrow d_2 & \\ \underline{3} & \boxed{\bullet \longrightarrow \bullet \longrightarrow \bullet} \end{array}$$

- cells are generated by

$$\begin{tikzcd} \underline{1} \arrow[r, "d_0"] \arrow[d, "d_1"'] & \underline{2} \arrow[d, "d_2"] \\ \underline{2} \arrow[r, "d_0"'] & \underline{3} \end{tikzcd} \qquad \begin{tikzcd} \underline{1} \arrow[r, "id"] \arrow[d, "d_1"'] & \underline{1} \arrow[d, "id"] \\ \underline{2} \arrow[r, "!"'] & \underline{1} \end{tikzcd}$$

$$\begin{tikzcd} \underline{1} \arrow[r, "id"] \arrow[dd, "d_1"'] & \underline{1} \arrow[d, "d_1"] \\ & \underline{2} \arrow[d, "d_2"] \\ \underline{2} \arrow[r, "d_1"'] & \underline{3} \end{tikzcd}$$

$$\boxed{\mathbb{L}\mathrm{ens} \cong \mathbb{R}\mathrm{LP}(\mathbb{J}_{\text{lens}})}$$

$$\begin{tikzcd} \{\bullet\} \arrow[r, "a"] \arrow[d, "d_1"'] & A \arrow[d, "f"] \\ \{\bullet \to \bullet\} \arrow[r, "u"'] \arrow[ur, "\varphi(a{,}u)"] & B \end{tikzcd}$$

delta lens = functor + lifting operation

ALGEBRAIC WEAK FACTORISATION SYSTEMS

An algebraic weak factorisation system on $\mathcal{C}$ is an $(\mathbb{L},\mathbb{R})$-lifting operation φ on

$$\mathbb{L} \xrightarrow{U} \mathbb{S}\mathrm{q}(\mathcal{C}) \xleftarrow{V} \mathbb{R}$$

such that the following axioms hold:

(i) the induced double functors are iso

$$\mathbb{L} \longrightarrow \mathbb{LLP}(\mathbb{R}) \qquad \mathbb{R} \longrightarrow \mathbb{RLP}(\mathbb{L})$$

(ii) each f in $\mathcal{C}$ admits a factorisation

$$\begin{tikzcd} \bullet \arrow[r, "U_1 g"] \arrow[rr, bend right, "f"'] & \bullet \arrow[r, "V_1 h"] & \bullet \end{tikzcd}$$

which is U_1-couniversal and V_1-universal.

Theorem (Bourke-Garner): If $\mathcal{C}$ is locally presentable & $\mathbb{J} \longrightarrow \mathbb{S}\mathrm{q}(\mathcal{C})$ is a small double category, then there exists an A.W.F.S. on $\mathcal{C}$ with cospan:

$$\mathbb{LLP}(\mathbb{RLP}(\mathbb{J})) \longrightarrow \mathbb{S}\mathrm{q}(\mathcal{C}) \longleftarrow \mathbb{RLP}(\mathbb{J})$$

Corollary: There exists an A.W.F.S. on Cat whose right class is $\mathbb{L}\mathrm{ens}$.

What is the left class $\mathbb{LLP}(\mathbb{L}\mathrm{ens})$?

$$\begin{tikzcd} A \arrow[r] \arrow[d, "???"'] & C \arrow[d, "\text{delta lens}"] \\ B \arrow[r] \arrow[ur] & D \end{tikzcd}$$

TWISTED COREFLECTIONS

A twisted coreflection is a split coreflection

$$\begin{tikzcd} & A \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \\ B \arrow[ur, "q"] \arrow[rr, "1_B"'] & {} & B \end{tikzcd} \qquad qf = 1_A \qquad \begin{matrix} \varepsilon\cdot f = 1_f \\ q\cdot\varepsilon = 1_q \end{matrix}$$

such that if $q(u : x \to y) \neq 1$, there exists a unique morphism $\bar{u} : x \longrightarrow fqx$ such that:

(i) $\bar{u} \circ \varepsilon_x = 1_{fqx}$ (ii) $\varepsilon_y \circ fqu \circ \bar{u} = u$

$$\boxed{\begin{tikzcd} fqx \arrow[r, equal] \arrow[d, "\varepsilon_x"'] & fqy \arrow[d, "\varepsilon_y"] \\ x \arrow[r, "u"'] & y \end{tikzcd} \qquad \begin{tikzcd} fqx \arrow[r, "fqu"] \arrow[d, "\varepsilon_x"'] & fqy \arrow[d, "\varepsilon_y"] \\ x \arrow[r, "u"'] \arrow[u, dotted, bend right, "\exists!\,\bar{u}"'] & y \end{tikzcd}}$$

Let $\mathbb{T}\mathrm{wCoref}$ denote the double category of categories, functors, & twisted coreflections.

A cell with boundary

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "(f\dashv q,\,\varepsilon)"'] & C \arrow[d, "(g\dashv p,\,\zeta)"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

exists if $kf = gh$, $hq = pk$, and $k\cdot\varepsilon = \zeta\cdot k$.

$$\boxed{\mathbb{T}\mathrm{wCoref} \hookrightarrow \mathbb{S}\mathrm{Coref} \longrightarrow \mathbb{S}\mathrm{q}(\mathrm{Cat})}$$

LIFTING TWISTED COREFLECTIONS AGAINST DELTA LENSES

$$\begin{tikzcd} qx \arrow[r, "qu"] & qy \end{tikzcd}$$

$$\begin{tikzcd}[column sep=large, row sep=large] hqx \arrow[r, "hqu"] \arrow[d, shift right, "{\Psi(hqx,k\varepsilon_x)}"'] & hqy \arrow[d, "{\Psi(hqy,k\varepsilon_y)}"] \\ jx \arrow[r, "ju"'] \arrow[u, shift right, "{\Psi(jx,k\bar{u})}"'] & jy \end{tikzcd}$$

$$\begin{tikzcd}[column sep=large, row sep=large] A \arrow[r, "h"] \arrow[d, "{(f\dashv q,\varepsilon)}"'] & C \arrow[d, "{(g,\Psi)}"] \\ B \arrow[r, "k"'] \arrow[ur, "j"] & D \end{tikzcd}$$

$$\begin{tikzcd}[column sep=large, row sep=large] fqx \arrow[r, "fqu"] \arrow[d, shift right, "\varepsilon_x"'] & fqy \arrow[d, "\varepsilon_y"] \\ x \arrow[r, "u"'] \arrow[u, shift right, dotted, "\exists!\ \bar{u}"'] & y \end{tikzcd}$$

$$\begin{tikzcd}[column sep=large, row sep=large] g(hqx) \arrow[d, equals] & g(hqy) \arrow[d, equals] \\ kfqx \arrow[r] \arrow[d, shift right, "k\varepsilon_x"'] & kfqy \arrow[d, "k\varepsilon_y"] \\ kx \arrow[r] \arrow[u, shift right, "k\bar{u}"'] & ky \end{tikzcd}$$

DIAGRAMMATIC CHARACTERISATIONS

TWISTED COREFLECTION $(f \dashv q, \varepsilon)$

discrete category

identity-on-objects

$$\begin{tikzcd}
A_0 \arrow[r, "\iota_A"] \arrow[d] & A \arrow[d, "f"] \\
X \arrow[r] & B \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

initial functor

where the category X has:

- same objects as B;
- morphisms $u : x \to y$ in B such that $qu = 1$.

DELTA LENS (g, φ)

identity-on-objects

$$\begin{tikzcd}
Y \arrow[r, "\varphi"] \arrow[dr, "g\varphi"'] & C \arrow[d, "g"] \\
 & D
\end{tikzcd}$$

discrete opfibration

where the category Y has:

- same objects as C;
- morphisms are the chosen lifts $\varphi(a, u)$.

$$
\begin{tikzcd}
 & & Y \arrow[d, "\psi"] \arrow[dd, bend left] \\
A_0 \arrow[r, "\iota_A"] \arrow[d] \arrow[urr, dashed, "\exists!\, \bar{h}"] & A \arrow[r, "h"] \arrow[d, "f"] & C \arrow[d, "g"] \\
X \arrow[r] \arrow[uurr, dashed, "\exists!\, \ell"] & B \arrow[r, "k"'] \arrow[ur, dashed, "\exists!\, j"] \arrow[ul, phantom, "\ulcorner", very near start] & D
\end{tikzcd}
$$

1. Construct unique $\bar{h} : A_0 \longrightarrow Y$ by the universal property of discrete categories and identity-on-objects functors.

2. Construct unique $\ell : X \longrightarrow Y$ by the universal property of the comprehensive factorisation system on $\mathcal{C}at$.

3. Construct unique $j : B \longrightarrow C$ by the universal property of the pushout.

FACTORISATION

discrete category

identity-on-objects

initial functor

discrete opfibration

$$
\begin{tikzcd}
A_0 \arrow[r, "\iota_A"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & A \arrow[d, "Lf"] \arrow[dd, bend left=60, "f"] \\
Jf \arrow[r] \arrow[d] & Ef \arrow[d, dashed, "Rf"] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

where $$Jf = \sum_{a \in A_0} fa/B$$

- For a functor $f : A \to B$
  - the cofree twisted coreflection is $Lf : A \to Ef$
  - the free delta lens is $Rf : Ef \to B$
- There is a comonad L and a monad R on $\mathcal{C}at^2$ whose (co)algebras are twisted coreflections and delta lenses.

$$\boxed{\begin{gathered} L\text{-}\mathbb{C}\mathrm{oalg} \cong \mathbb{T}\mathrm{wCoref} \\ R\text{-}\mathbb{A}\mathrm{lg} \cong \mathbb{L}\mathrm{ens} \end{gathered}}$$

SUMMARY & FUTURE WORK

<ins>Theorem</ins>: There is an A.W.F.S. on $\mathcal{C}at$

$$\mathbb{T}\mathrm{wCoref} \longrightarrow \mathbb{S}\mathrm{q}(\mathcal{C}at) \longleftarrow \mathbb{L}\mathrm{ens}$$

of twisted coreflections & delta lenses.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "\text{twisted coreflection}"'] & \bullet \arrow[d, "\text{delta lens}"] \\
\bullet \arrow[r] \arrow[ur, "\text{lift}"] & \bullet
\end{tikzcd}$$

(i) $\mathbb{T}\mathrm{wCoref} \cong \mathbb{LLP}(\mathbb{L}\mathrm{ens}) \cong \mathsf{L}\text{-}\mathbb{C}\mathrm{oalg}$

$\mathbb{L}\mathrm{ens} \cong \mathbb{RLP}(\mathbb{T}\mathrm{wCoref}) \cong \mathbb{RLP}(\mathbb{J}_{\mathrm{lens}}) \cong \mathsf{R}\text{-}\mathbb{A}\mathrm{lg}$

(ii)

$$\begin{tikzcd}
\bullet \arrow[rr, "f"] \arrow[dr, "\text{cofree twisted coreflection}"'] & & \bullet \\
& \bullet \arrow[ur, "\text{free delta lens}"'] &
\end{tikzcd}$$

- Can we generalise lifting operations by replacing $\mathbb{S}\mathrm{q}(\mathcal{C})$ with some $\mathbb{D}$?
- What is the relationship with (co)reflective factorisation systems?

$$\begin{tikzcd}
SA \arrow[r, "\varepsilon_A"] \arrow[d, "Sf"'] & A \arrow[d] \arrow[dd, bend left, "f"] \\
SB \arrow[r] \arrow[dr, "\varepsilon_B"'] \arrow[ur, phantom, "\ulcorner"] & \bullet \arrow[d, dotted] \\
& B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, dotted] \arrow[dr, "\eta_A"] \arrow[dd, bend right, "f"'] & \\
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & TA \arrow[d, "Tf"] \\
B \arrow[r, "\eta_B"'] & TB
\end{tikzcd}$$

(S,ε) idempotent comonad $\qquad$ (T,η) idempotent monad