

A new perspective on comodules of polynomial comonads

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Let $\text{Poly} := \text{Fam}(\text{Set}^{\circ P})$ be the category

whose:

- objects are polynomials/containers

discrete category $\rightsquigarrow I \xrightarrow{P} \text{Set}$

- morphisms are natural transformations

$$\begin{array}{ccc}
 I & & \\
 \downarrow \varphi & \searrow P & \\
 & & \text{Set} \\
 & \nearrow Q & \\
 J & & \\
 & \uparrow \varphi^\# &
 \end{array}$$

i.e. function $\varphi: I \rightarrow J$ and a function

$$\varphi_i^\#: Q(\varphi i) \longrightarrow P(i) \text{ for each } i \in I.$$

There is a monoidal structure $(\text{Poly}, \triangleleft, *)$ where $(P: I \rightarrow \text{Set}) \triangleleft (Q: J \rightarrow \text{Set})$ is

$$\sum_{i \in I} [P(i), J] \longrightarrow \text{Set}$$

$$(i, \alpha: P(i) \rightarrow J) \longmapsto \sum_{x \in P(i)} Q(\alpha x)$$

with unit the polynomial $*: \{*\} \rightarrow \text{Set}$ that selects the singleton set.

For a monoidal category $(\mathcal{V}, \otimes, I)$ with reflexive coequalisers that are preserved by $A \otimes (-)$ and $(-) \otimes A$ for each $A \in \mathcal{V}$, we may construct a double category

$\text{Comod}(\mathcal{V}, \otimes, I)$ whose:

- objects are comonoids
- tight morphisms are comonoid homomorphisms
- loose morphisms are comodules

GOAL: Construct $\text{Comod}(\text{Poly}, \triangleleft, *)$.

Prop (Ahman-Uustalu, 2016): Comonoids in $(\text{Poly}, \triangleleft, *)$ are precisely small categories.

Given a category \mathcal{A} with set of objects A_0 , its underlying polynomial is

$$A_0 \xrightarrow{A[-]} \text{Set}$$

$$a \longmapsto \sum_{x \in A_0} A(a, x) := A[a]$$

together with counit and comultiplication maps:

$$\begin{array}{ccc} A_0 & \xrightarrow{A[-]} & \text{Set} \\ \downarrow ! & \varepsilon \uparrow & \\ \{*\} & \xrightarrow{*} & \end{array} \quad \begin{array}{ccc} A_0 & \xrightarrow{A[-]} & \text{Set} \\ \downarrow & \delta \uparrow & \\ \sum_{a \in A_0} [A[a], A_0] & \xrightarrow{\quad} & \end{array}$$

Prop (Ibid.): Comonoid homomorphisms in $(\text{Poly}, \triangleleft, *)$ are precisely retrofunctors (cofunctors).

$$\begin{array}{ccc} A_0 & \xrightarrow{A[-]} & \text{Set} \\ \varphi \downarrow & \varphi^\# \uparrow & \\ B_0 & \xrightarrow{B[-]} & \end{array}$$

i.e. a function $\varphi: A_0 \rightarrow B_0$ and a function $\varphi_a^\#: B[\varphi a] \rightarrow A[a]$ for each $a \in A_0$.

$$\frac{\sum_{b \in B_0} B(\varphi a, b) \xrightarrow{\varphi_a^\#} \sum_{x \in A_0} A(a, x)}{B(\varphi a, b) \xrightarrow{\varphi_{a,b}^\#} \sum_{\substack{x \in A_0 \\ x \in \varphi^{-1}\{b\}}} A(a, x)}$$

Respects the identities and composition in \mathcal{B} .

Prop: A retrofunctor $\mathcal{A} \rightarrow \mathcal{B}$ is equivalent to a span of functors:

$$A \xleftarrow{\quad} \bullet \xrightarrow{\quad} B$$

\uparrow
identity-on-objects
 \downarrow
discrete opfibration

$$\begin{array}{ccc} A_0 & \xrightarrow{A[-]} & \text{Set} \\ \parallel & \varphi^\# \uparrow & \\ A_0 & \xrightarrow{B[\varphi-]} & \text{Set} \\ \varphi \downarrow & \sim & \\ B_0 & \xrightarrow{B[-]} & \end{array}$$

Induced by the (vertical, cartesian) O.F.S. on Poly .

Let \mathcal{A} and \mathcal{B} be categories and $P: \mathcal{I}_0 \rightarrow \text{Set}$ a polynomial.

$$\frac{(\mathcal{B}[-]: \mathcal{B}_0 \rightarrow \text{Set}) \triangleleft (P: \mathcal{I}_0 \rightarrow \text{Set})}{\sum_{b \in \mathcal{B}_0} [\mathcal{B}[b], \mathcal{I}_0] \longrightarrow \text{Set}}$$

$$(b, \alpha: \mathcal{B}[b] \rightarrow \mathcal{I}_0) \longmapsto \sum_{u \in \mathcal{B}[b]} P(\alpha u)$$

$$\frac{(P: \mathcal{I}_0 \rightarrow \text{Set}) \triangleleft (\mathcal{A}[-]: \mathcal{A}_0 \rightarrow \text{Set})}{\sum_{i \in \mathcal{I}_0} [P(i), \mathcal{A}_0] \longrightarrow \text{Set}}$$

$$(i, \alpha: P(i) \rightarrow \mathcal{A}_0) \longmapsto \sum_{x \in P(i)} \mathcal{A}[\alpha x]$$

A **right \mathcal{A} -comodule** structure on $P: \mathcal{I}_0 \rightarrow \text{Set}$ amounts to the data:

- For each $i \in \mathcal{I}_0$, a function $f_i: P(i) \rightarrow \mathcal{A}_0$
- For each $x \in P(i)$, a function $\mathcal{A}[f_i x] \xrightarrow{P_x} P(i)$

A **left \mathcal{B} -comodule** structure on $P: \mathcal{I}_0 \rightarrow \text{Set}$ amounts to the data:

- function $g: \mathcal{I}_0 \rightarrow \mathcal{B}_0$
- For each $i \in \mathcal{I}_0$, a function $\tau_i: \mathcal{B}[gi] \rightarrow \mathcal{I}_0$
- For each $u \in \mathcal{B}[gi]$, a function $P(\tau_i u) \xrightarrow{\lambda_u} P(i)$

A **comodule** $P: \mathcal{A} \leftrightarrow \mathcal{B}$ is a compatible right \mathcal{A} -comodule and left \mathcal{B} -comodule structure on $P: \mathcal{I}_0 \rightarrow \text{Set}$.

Lemma: A left \mathcal{B} -comodule structure on $P: \mathcal{I}_0 \rightarrow \text{Set}$ induces a category \mathcal{I} with set of objects \mathcal{I}_0 , and a discrete opfibration $g: \mathcal{I} \rightarrow \mathcal{B}$.

A morphism in $\mathcal{I}(i, j)$ is a morphism $u: gi \rightarrow b$ in \mathcal{B} such that $\tau_i(u) = j$.

Let $\bar{P}(i, a) := f_i^{-1}\{a\}$ where $f_i: P(i) \rightarrow A_0$ comes from a right A -comodule $P: \mathcal{I} \rightarrow \text{Set}$.

Lemma: A comodule $P: A \rightrightarrows B$ induces a functor $\bar{P}: \mathcal{I}^{\text{op}} \times A \rightarrow \text{Set}, (i, a) \mapsto \bar{P}(i, a)$.

For $u: i \rightarrow j$ in \mathcal{I} , arising from $u: gi \rightarrow b$ in B we obtain

$$\begin{array}{ccc}
 P(\tau_i^j u) & \xrightarrow{\lambda_u} & P(i) \\
 f_j \searrow & & \swarrow f_i \\
 & & A_0
 \end{array}
 \quad \leftarrow \text{commutes by compatibility}$$

This induces a function $\bar{P}(j, a) \rightarrow \bar{P}(i, a)$ for each $a \in A$ and $u: i \rightarrow j$ in \mathcal{I} .

For $v: a \rightarrow a'$ in A , we obtain

$$f_i^{-1}\{a\} = \bar{P}(i, a) \xrightarrow{\rho_{-}(v)} P(i, a') = f_i^{-1}\{a'\}$$

Compatibility of λ and ρ by comodule axioms makes $\bar{P}: \mathcal{I}^{\text{op}} \times A \rightarrow \text{Set}$ well-defined.

Prop (Garner, 2019): A comodule $A \rightrightarrows B$ of comonoids in $(\text{Poly}, \triangleleft, *)$ is equivalent to a span:

$$\begin{array}{ccc}
 A & \longleftarrow & \mathcal{I} & \longrightarrow & B \\
 & \nearrow \text{profunctor} & & \nearrow \text{discrete opfibration} &
 \end{array}$$

By previous work of Weber, this is the same as a parametric right adjoint

$$[A, \text{Set}] \longrightarrow [B, \text{Set}].$$

MOTIVATING QUESTIONS:

1. How do we compose "comodules as spans"?
2. What do cells in the double category $\text{Comod}(\text{Poly})$ look like?
3. What are the companion and conjoint of a retrofunctor?
4. Profunctors $A \rightrightarrows B$ are functors $A^{\text{op}} \times B \rightarrow \text{Set}$; does a similar result hold for comodules?

The category of polynomials admits a canonical functor:

$$\text{Poly} \xrightarrow{u} \text{Set}$$

$$(P: I \rightarrow \text{Set}) \longmapsto I$$

Theorem: A commutative square

$$\begin{array}{ccc} A^{\text{op}} \times B & \xrightarrow{P} & \text{Poly} \\ \pi \downarrow & & \downarrow u \\ B & \xrightarrow{F} & \text{Set} \end{array}$$

$\text{Lan}_{\pi} uP$
 such that
 $\text{id}: uP \Rightarrow F\pi$

is equivalent to a comodule $A \rightrightarrows B$ in $(\text{Poly}, \triangleleft, *)$.

PROOF: For each $b \in B$, we have a set $F(b)$.

For each $a \in A$, $x \in F(b)$, we have a set $P(x, a)$.

$$\begin{array}{ccc} A^{\text{op}} \times B & \xrightarrow{P} & \text{Poly} \\ (a, b) & \longmapsto & F(b) \xrightarrow{P(-, a)} \text{Set} \end{array}$$

For each $v: a' \rightarrow a$ we have

$$F(b) \begin{array}{c} \xrightarrow{P(-, a)} \\ v_* \uparrow \\ \xrightarrow{P(-, a')} \end{array} \text{Set} \quad v_*: P(x, a') \rightarrow P(x, a)$$

For each $u: b \rightarrow b'$ we have

$$\begin{array}{ccc} F(b) & \xrightarrow{P(-, a)} & \text{Set} \\ u_* \downarrow & u^* \uparrow & \\ F(b') & \xrightarrow{P(-, a)} & \end{array} \quad \begin{array}{l} u_*: F(b) \rightarrow F(b') \\ u^*: P(u_*x, a) \rightarrow P(x, a) \end{array}$$

Commutativity requires that

$$\begin{array}{ccc} F(b) & \xrightarrow{P(-, a)} & \text{Set} \\ u_* \downarrow & u^* \uparrow & \\ F(b') & \xrightarrow{P(-, a)} & \end{array} = \begin{array}{ccc} F(b) & \xrightarrow{P(-, a)} & \text{Set} \\ u_* \downarrow & v_* \uparrow & \\ F(b') & \xrightarrow{P(-, a')} & \end{array}$$

Altogether, we obtain a span

$$A \leftarrow \int F \rightarrow B$$

↗ profunctor
↗ discrete opfibration

where $\int F$ is the category of elements of $F: B \rightarrow \text{Set}$. This is a comodule in $(\text{Poly}, \triangleleft, *)$ by Garner's result.

Conversely, given a span

$$A \leftarrow X \rightarrow B$$

↗ profunctor
↗ discrete opfibration

we define

$$A^{\text{op}} \times B \longrightarrow \text{Poly}$$

$$(a, b) \longmapsto g^{-1}\{b\} \longrightarrow \text{Set}$$

$$x \longmapsto Q(x, a)$$

For $v: a' \rightarrow a$ and $u: b \rightarrow b'$ we have

$$\begin{array}{ccc}
 g^{-1}\{b\} & \xrightarrow{Q(-, a)} & \text{Set} \\
 \downarrow Q(\hat{u}, v) & \uparrow & \uparrow \\
 g^{-1}\{b\} & \xrightarrow{Q(-, a')} & \text{Set}
 \end{array}$$

Since g is a discrete opfibration $\xrightarrow{u_*}$

where

$$\hat{u}: x \rightarrow u_* x$$

is the opcartesian lift of (x, u) .

This completes the proof. \square

We have a nice parallel between concepts:

$$\text{module of monads in } \mathcal{S}\text{pan}(\text{Set}) \rightsquigarrow \text{functor } A^{\text{op}} \times B \rightarrow \text{Set}$$

$$\text{comodule of comonoids in } (\text{Poly}, \triangleleft, *) \rightsquigarrow \text{functor } A^{\text{op}} \times B \rightarrow \text{Poly}$$

Every functor $f: A \rightarrow B$ induces a companion and conjoint profunctor given by:

$$\begin{array}{ll}
 A^{\text{op}} \times B \longrightarrow \text{Set} & B^{\text{op}} \times A \longrightarrow \text{Set} \\
 (a, b) \longmapsto B(fa, b) & (b, a) \longmapsto B(b, fa)
 \end{array}$$

Given a retrofunctor $\Psi : A \rightarrow B$ we have:

companion $\left\{ \begin{array}{l} A^{\text{op}} \times B \longrightarrow \text{Poly} \\ (a, b) \longmapsto \Psi^{-1}\{b\} \longrightarrow \text{Set} \\ x \longmapsto A(x, a) \end{array} \right.$

conjoint $\left\{ \begin{array}{l} B^{\text{op}} \times A \longrightarrow \text{Poly} \\ (b, a) \longmapsto \{*\} \longrightarrow \text{Set} \\ * \longmapsto \sum_{x \in \Psi^{-1}\{b\}} A(a, x) \end{array} \right.$

Suppose we have a comodule $A \xrightarrow{(P, F)} B$ and retrofunctors $\varphi : A \rightarrow C$ and $\psi : B \rightarrow D$.

Then we may construct a comodule $C \dashrightarrow D$

given by: $C^{\text{op}} \times D \xrightarrow{\varphi^* \Delta (P, F) \Delta \psi_*} \text{Poly}$
 $(c, d) \longmapsto \sum_{b \in \psi^{-1}\{d\}} F(b) \longrightarrow \text{Set}$

$$\begin{array}{ccc} A & \xrightarrow{(P, F)} & B \\ \varphi \downarrow & & \downarrow \psi \\ C & \dashrightarrow & D \end{array} \quad (b, x) \longmapsto \sum_{a \in \varphi^{-1}\{c\}} P(x, a)$$

Using this construction, a cell in

$\text{Comod}(\text{Poly}, \Delta, *)$ denoted

$$\begin{array}{ccc} A & \xrightarrow{(P, F)} & B \\ \varphi \downarrow & \Theta & \downarrow \psi \\ C & \xrightarrow{(Q, G)} & D \end{array}$$

corresponds to a natural transformation

$$C^{\text{op}} \times D \xrightarrow{\varphi^* \Delta (P, F) \Delta \psi_*} \text{Poly} \xrightarrow{(Q, G)} \text{Poly}$$

whose component at (c, d) is given by:

For each $b \in \psi^{-1}\{d\}$

$$\begin{array}{ccc} F(b) & & \sum_{b \in \psi^{-1}\{d\}} F(b) \\ \Theta_b \downarrow & \cong & \downarrow \Theta \\ G(\psi b) & & G(d) \end{array} \quad \begin{array}{ccc} & & \text{Set} \\ & \nearrow \Theta^* & \\ & \searrow & \end{array}$$

$$\Theta_x^\# : Q(\Theta_b x, c) \longrightarrow \sum_{a \in \varphi^{-1}\{c\}} P(x, a)$$

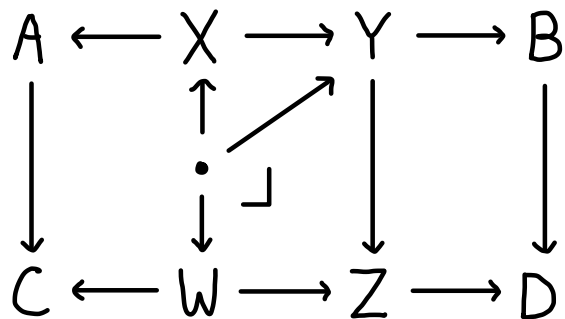
We write $\mathcal{C}at^\# := \mathcal{C}omod(\mathcal{P}oly, \triangleleft, *)$.

EXAMPLES:

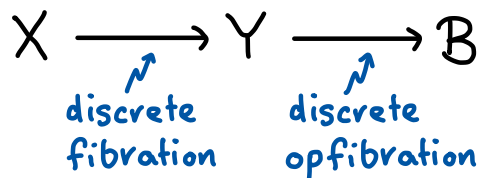
- Restricting to the terminal category

$$(\mathcal{P}oly, \triangleleft, *) \hookrightarrow \mathcal{C}at^\#$$

- Restricting to discrete categories we obtain multivariable polynomials

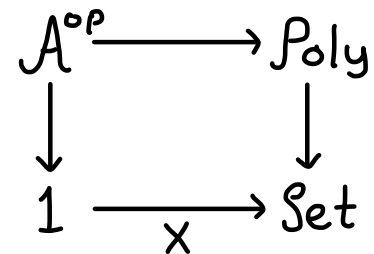


- A comodule $1 \rightrightarrows \mathcal{B}$ is a functor $\mathcal{B} \rightarrow \mathcal{P}oly$ which is equivalent to:



This is a "left \mathcal{B} -comodule".

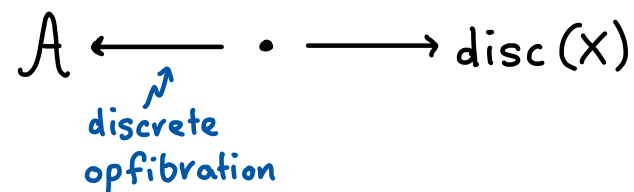
- A comodule $A \rightrightarrows 1$ is a diagram



which is the same as a functor

$$A \longrightarrow [X, \mathcal{S}et]$$

which is equivalent to a span

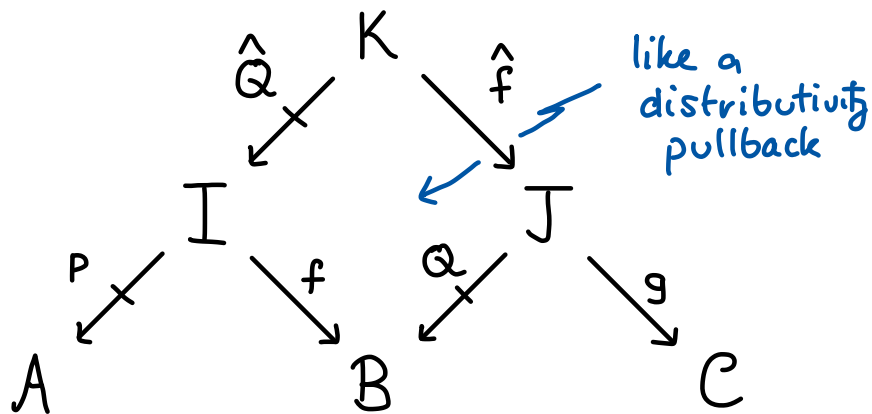


This is a "right A -comodule".

- Every profunctor $A \rightrightarrows B$ yields a comodule $B \rightrightarrows A$ in $\mathcal{P}oly$.
- Let $\mathbb{R}et$ be the double category of categories, functors, and retrofunctors.
CONJECTURE: $\mathbb{R}et \rightarrow \mathcal{C}at^\#$ is fully faithful (depending on choice of cells in $\mathbb{R}et$).

COMPOSITION OF COMODULES:

If we consider comodules in $(\text{Poly}, \Delta, *)$ as certain spans, how do we compose them?



$$P: I^{\text{op}} \times A \longrightarrow \text{Set}$$

$$Q: J^{\text{op}} \times B \longrightarrow \text{Set}$$

f, g - discrete opfibrations

Define K to be the category whose:

- objects are pairs $(j \in J, \alpha: Q(j, -) \Rightarrow f^{-1}\{-\})$ where $f^{-1}\{-\}: B \rightarrow \text{Set}$ is well-defined since f is a discrete opfibration.

- morphisms $(j, \alpha) \rightarrow (j', \alpha')$ are $w: j \rightarrow j'$ in J such that $\alpha' = \alpha \circ Q(w, -)$.

Projection in the first component gives a discrete opfibration $\hat{f}: K \rightarrow J$.

We also may construct a functor

$$\hat{Q}: K^{\text{op}} \times I \longrightarrow \text{Set}$$

$$(j \in J, \alpha: Q(j, -) \Rightarrow f^{-1}\{-\}, i \in I) \longmapsto$$

$$\hat{Q}(j, \alpha, i) = \{q \in Q(j, fi) \mid \alpha_{fi}(q) = i\}$$

Altogether, we may take the composite $g \circ \hat{f}$ to obtain a discrete opfibration and the composite

$$P \circ \hat{Q}: K^{\text{op}} \times A \longrightarrow \text{Set}$$

$$(j, \alpha: Q(j, -) \Rightarrow f^{-1}\{-\}, a) \longmapsto \int^{i \in I} \hat{Q}(j, \alpha, i) \times P(i, a)$$

to get the composite of comodules in $(\text{Poly}, \Delta, *)$ viewed as spans.