A new perspective on comodules of polynomial comonads

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Let $\mathcal{P}oly := \mathrm{Fam}(\mathrm{Set}^{op})$ be the category whose:

- objects are polynomials/containers

discrete category

$$I \xrightarrow{P} \mathrm{Set}$$

- morphisms are natural transformations

$$\begin{tikzcd} I \arrow[dd, "\varphi"'] \arrow[dr, "P"] & \\ \arrow[r, phantom, "\varphi^{\#}\Uparrow"] & \mathrm{Set} \\ J \arrow[ur, "Q"'] & \end{tikzcd}$$

i.e. function $\varphi : I \to J$ and a function

$\varphi^{\#}_i : Q(\varphi i) \longrightarrow P(i)$ for each $i \in I$.

There is a monoidal structure $(\mathcal{P}oly, \triangleleft, *)$ where $(P : I \to \mathrm{Set}) \triangleleft (Q : J \to \mathrm{Set})$ is

$$\sum_{i \in I} [P(i), J] \longrightarrow \mathrm{Set}$$

$$(i, \alpha : P(i) \to J) \longmapsto \sum_{x \in P(i)} Q(\alpha x)$$

with unit the polynomial $* : \{*\} \longrightarrow \mathrm{Set}$ that selects the singleton set.

For a monoidal category $(\mathcal{V}, \otimes, I)$ with reflexive coequalisers that are preserved by $A \otimes (-)$ and $(-) \otimes A$ for each $A \in \mathcal{V}$, we may construct a double category $\mathbb{C}\mathrm{omod}(\mathcal{V}, \otimes, I)$ whose:

- objects are comonoids
- tight morphisms are comonoid homomorphisms
- loose morphisms are comodules

GOAL: Construct $\mathbb{C}\mathrm{omod}(\mathcal{P}oly, \triangleleft, *)$.

Prop (Ahman-Uustalu, 2016): Comonoids in $(\mathbf{Poly}, \triangleleft, *)$ are precisely small categories.

Given a category $\mathcal{A}$ with set of objects A_0, its underlying polynomial is

$$
\begin{aligned}
A_0 &\xrightarrow{A[-]} \mathbf{Set} \\
a &\longmapsto \sum_{x \in A_0} \mathcal{A}(a,x) := A[a]
\end{aligned}
$$

together with counit and comultiplication maps:

$$
\begin{tikzcd}
A_0 \arrow[rr, bend left, "A[-]"] \arrow[d, "!"'] & \arrow[d, phantom, "\varepsilon\Uparrow"] & \mathbf{Set} \\
\{*\} \arrow[rru, bend right, "*"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A_0 \arrow[rr, bend left, "A[-]"] \arrow[d] & \arrow[d, phantom, "\delta\Uparrow"] & \mathbf{Set} \\
\sum_{a \in A_0} [A[a], A_0] \arrow[rru, bend right] & &
\end{tikzcd}
$$

Prop (Ibid.): Comonoid homomorphisms in $(\mathbf{Poly}, \triangleleft, *)$ are precisely retrofunctors (cofunctors).

$$
\begin{tikzcd}
A_0 \arrow[rr, bend left, "A[-]"] \arrow[d, "\varphi"'] & \arrow[d, phantom, "\varphi^{\#}\Uparrow"] & \mathbf{Set} \\
B_0 \arrow[rru, bend right, "B[-]"'] & &
\end{tikzcd}
$$

i.e. a function $\varphi : A_0 \longrightarrow B_0$ and a function $\varphi^{\#}_a : B[\varphi a] \longrightarrow A[a]$ for each $a \in A_0$.

$$
\frac{\displaystyle\sum_{b \in B_0} \mathcal{B}(\varphi a, b) \xrightarrow{\varphi^{\#}_a} \sum_{x \in A_0} \mathcal{A}(a,x)}{\displaystyle \mathcal{B}(\varphi a, b) \xrightarrow{\varphi^{\#}_{a,b}} \sum_{\cancel{x \in A_0}\ x \in \varphi^{-1}\{b\}} \mathcal{A}(a,x)}
$$

Respects the identities and composition in $\mathcal{B}$.

Prop: A retrofunctor $\mathcal{A} \longrightarrow \mathcal{B}$ is equivalent to a span of functors:

$$
\underset{\text{identity-on-objects}}{\mathcal{A} \longleftarrow} \bullet \underset{\text{discrete opfibration}}{\longrightarrow \mathcal{B}}
$$

$$
\begin{tikzcd}
A_0 \arrow[rr, bend left, "A[-]"] \arrow[d, equal] & \arrow[d, phantom, "\varphi^{\#}\Uparrow"] & \\
A_0 \arrow[rr, "B[\varphi -]"'] \arrow[d, "\varphi"'] & \arrow[d, phantom, "\circlearrowright"] & \mathbf{Set} \\
B_0 \arrow[rru, bend right, "B[-]"'] & &
\end{tikzcd}
$$

Induced by the (vertical, cartesian) O.F.S. on $\mathbf{Poly}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $P : I_0 \to \mathrm{Set}$ a polynomial.

$$\frac{(\mathcal{B}[-] : \mathcal{B}_0 \to \mathrm{Set}) \triangleleft (P : I_0 \to \mathrm{Set})}{\begin{aligned} \sum_{b \in \mathcal{B}_0} [\mathcal{B}[b], I_0] &\longrightarrow \mathrm{Set} \\ (b, \alpha : \mathcal{B}[b] \to I_0) &\longmapsto \sum_{u \in \mathcal{B}[b]} P(\alpha u) \end{aligned}}$$

$$\frac{(P : I_0 \to \mathrm{Set}) \triangleleft (\mathcal{A}[-] : \mathcal{A}_0 \to \mathrm{Set})}{\begin{aligned} \sum_{i \in I_0} [P(i), \mathcal{A}_0] &\longrightarrow \mathrm{Set} \\ (i, \alpha : P(i) \to \mathcal{A}_0) &\longmapsto \sum_{x \in P(i)} \mathcal{A}[\alpha x] \end{aligned}}$$

A right $\mathcal{A}$-comodule structure on $P : I_0 \to \mathrm{Set}$ amounts to the data:

- For each $i \in I_0$, a function $f_i : P(i) \longrightarrow \mathcal{A}_0$
- For each $x \in P(i)$, a function $\mathcal{A}[f_i x] \xrightarrow{\rho_x} P(i)$

A left $\mathcal{B}$-comodule structure on $P : I_0 \to \mathrm{Set}$ amounts to the data:

- function $g : I_0 \longrightarrow \mathcal{B}_0$
- For each $i \in I_0$, a function $\tau_i : \mathcal{B}[gi] \longrightarrow I_0$
- For each $u \in \mathcal{B}[gi]$, a function $P(\tau_i u) \xrightarrow{\lambda_u} P(i)$

A comodule $P : \mathcal{A} \nrightarrow \mathcal{B}$ is a compatible right $\mathcal{A}$-comodule and left $\mathcal{B}$-comodule structure on $P : I_0 \longrightarrow \mathrm{Set}$.

Lemma: A left $\mathcal{B}$-comodule structure on $P : I_0 \to \mathrm{Set}$ induces a category $\mathcal{I}$ with set of objects I_0, and a discrete opfibration $g : \mathcal{I} \longrightarrow \mathcal{B}$.

A morphism in $\mathcal{I}(i,j)$ is a morphism $u : gi \to b$ in $\mathcal{B}$ such that $\tau_i(u) = j$.

Let $\bar{P}(i,a) := f_i^{-1}\{a\}$ where $f_i : P(i) \longrightarrow A_0$ comes from a right $\mathcal{A}$-comodule $P : \mathcal{I} \to \mathbf{Set}$.

Lemma: A comodule $P : \mathcal{A} \nrightarrow \mathcal{B}$ induces a functor $\bar{P} : \mathcal{I}^{op} \times \mathcal{A} \longrightarrow \mathbf{Set}$, $(i,a) \longmapsto \bar{P}(i,a)$.

For $u : i \longrightarrow j$ in $\mathcal{I}$, arising from $u : gi \longrightarrow b$ in $\mathcal{B}$ we obtain

$$\begin{tikzcd} P(\underset{=j}{\tau_i u}) \arrow[rr, "\lambda_u"] \arrow[dr, "f_j"'] & & P(i) \arrow[dl, "f_i"] \\ & A_0 & \end{tikzcd}$$

commutes by compatibility

This induces a function $\bar{P}(j,a) \longrightarrow \bar{P}(i,a)$ for each $a \in A$ and $u : i \to j$ in $\mathcal{I}$.

For $v : a \to a'$ in $\mathcal{A}$, we obtain

$$f_i^{-1}\{a\} = \bar{P}(i,a) \xrightarrow{\rho_-(v)} P(i,a') = f_i^{-1}\{a'\}$$

Compatibility of λ and ρ by comodule axioms makes $\bar{P} : \mathcal{I}^{op} \times \mathcal{A} \longrightarrow \mathbf{Set}$ well-defined.

Prop (Garner, 2019): A comodule $\mathcal{A} \overset{P}{\nrightarrow} \mathcal{B}$ of comonoids in $(\mathbf{Poly}, \triangleleft, *)$ is equivalent to a span:

$$\begin{tikzcd} \mathcal{A} & \mathcal{I} \arrow[l, "|" marking] \arrow[r] & \mathcal{B} \end{tikzcd}$$

profunctor

discrete opfibration

By previous work of Weber, this is the same as a parametric right adjoint $[\mathcal{A}, \mathbf{Set}] \longrightarrow [\mathcal{B}, \mathbf{Set}]$.

Motivating Questions:

1. How do we compose "comodules as spans"?
2. What do cells in the double category $\mathbf{Comod}(\mathbf{Poly})$ look like?
3. What are the companion and conjoint of a retrofunctor?
4. Profunctors $A \nrightarrow B$ are functors $A^{op} \times B \to \mathbf{Set}$; does a similiar result hold for comodules?

The category of polynomials admits a canonical functor:

$$\mathcal{P}oly \xrightarrow{\mathcal{U}} \mathsf{Set}$$
$$(P : I \to \mathsf{Set}) \longmapsto I$$

Theorem: A commutative square

$$\begin{tikzcd} A^{op} \times B \arrow[r, "P"] \arrow[d, "\pi"'] & \mathcal{P}oly \arrow[d, "\mathcal{U}"] \\ B \arrow[r, "F"'] & \mathsf{Set} \end{tikzcd}$$

$\mathrm{Lan}_\pi \mathcal{U}P$ such that $id : \mathcal{U}P \Rightarrow F\pi$

is equivalent to a comodule $\mathcal{A} \nrightarrow \mathcal{B}$ in $(\mathcal{P}oly, \triangleleft, *)$.

PROOF: For each $b \in B$, we have a set $F(b)$.

For each $a \in A$, $x \in F(b)$, we have a set $P(x,a)$.

$$A^{op} \times B \xrightarrow{P} \mathcal{P}oly$$
$$(a,b) \longmapsto F(b) \xrightarrow{P(-,a)} \mathsf{Set}$$

For each $v : a' \to a$ we have

$$\begin{tikzcd} F(b) \arrow[r, bend left, "P(-,a)"] \arrow[r, bend right, "P(-,a')"'] \arrow[r, phantom, "v_* \Uparrow"] & \mathsf{Set} \end{tikzcd}$$

$v_* : P(x,a') \to P(x,a)$

For each $u : b \to b'$ we have

$$\begin{tikzcd} F(b) \arrow[dr, "P(-,a)"] \arrow[dd, "u_*"'] & \\ & \mathsf{Set} \arrow[l, phantom, "u^* \Uparrow"] \\ F(b') \arrow[ur, "P(-,a)"'] & \end{tikzcd}$$

$u_* : F(b) \longrightarrow F(b')$

$u^* : P(u_* x, a) \longrightarrow P(x,a)$

Commutativity requires that

$$\begin{tikzcd} F(b) \arrow[dr, "P(-,a)"] \arrow[dd, "u_*"'] & \\ & \mathsf{Set} \arrow[l, phantom, "u^* \Uparrow"] \\ F(b') \arrow[ur] \arrow[ur, bend right=50, "P(-,a')"'] \arrow[ur, phantom, bend right=25, "\Uparrow v_*"] & \end{tikzcd} = \begin{tikzcd} F(b) \arrow[dr] \arrow[dr, bend left=50, "P(-,a)"] \arrow[dr, phantom, bend left=25, "v_* \Uparrow"] \arrow[dd, "u_*"'] & \\ & \mathsf{Set} \arrow[l, phantom, "u^* \Uparrow"] \\ F(b') \arrow[ur, "P(-,a')"'] & \end{tikzcd}$$

Altogether, we obtain a span

$$A \xleftarrow{\quad + \quad} \textstyle\int F \longrightarrow B$$

profunctor

discrete opfibration

where $\int F$ is the category of elements of $F: B \longrightarrow \mathrm{Set}$. This is a comodule in $(\mathrm{Poly}, \triangleleft, *)$ by Garner's result.

Conversely, given a span

$$A \xleftarrow[\quad + \quad]{Q} X \xrightarrow{\quad g \quad} B$$

profunctor

discrete opfibration

we define

$$A^{op} \times B \longrightarrow \mathcal{P}oly$$

$$(a, b) \longmapsto g^{-1}\{b\} \longrightarrow \mathrm{Set}$$

$$x \longmapsto Q(x, a)$$

For $v: a' \to a$ and $u: b \to b'$ we have

$$\begin{tikzcd} g^{-1}\{b\} \arrow[d, "u_*"'] \arrow[dr, "Q(-,a)"] \arrow[d, phantom, shift left=6ex, "Q(\hat{u},v)\Uparrow"] & \\ g^{-1}\{b\} \arrow[r, "Q(-,a')"'] & \mathrm{Set} \end{tikzcd}$$

Since g is a discrete opfibration

where $\hat{u} : x \to u_* x$ is the opcartesian lift of (x, u).

This completes the proof. $\square$

We have a nice parallel between concepts:

module of monads in $\mathbb{S}\mathrm{pan}(\mathrm{Set})$	$\rightsquigarrow$	functor $\mathcal{A}^{op} \times \mathcal{B} \longrightarrow \mathrm{Set}$
comodule of comonoids in $(\mathrm{Poly}, \triangleleft, *)$	$\rightsquigarrow$	functor $\mathcal{A}^{op} \times \mathcal{B} \longrightarrow \mathrm{Poly}$

Every functor $f: \mathcal{A} \to \mathcal{B}$ induces a companion and conjoint profunctor given by:

$$\mathcal{A}^{op} \times \mathcal{B} \longrightarrow \mathrm{Set} \qquad\qquad \mathcal{B}^{op} \times \mathcal{A} \longrightarrow \mathrm{Set}$$

$$(a, b) \longmapsto \mathcal{B}(fa, b) \qquad\qquad (b, a) \longmapsto \mathcal{B}(b, fa)$$

Given a retrofunctor $\varphi : \mathcal{A} \to \mathcal{B}$ we have:

companion
$$\left\{\begin{array}{lll} \mathcal{A}^{op} \times \mathcal{B} & \longrightarrow & \mathcal{P}oly \\ (a\,,\,b) & \longmapsto & \varphi^{-1}\{b\} \longrightarrow \mathbf{Set} \\ & & \quad x \;\longmapsto \mathcal{A}(x,a) \end{array}\right.$$

conjoint
$$\left\{\begin{array}{lll} \mathcal{B}^{op} \times \mathcal{A} & \longrightarrow & \mathcal{P}oly \\ (b\,,\,a) & \longmapsto & \{*\} \longrightarrow \mathbf{Set} \\ & & \quad * \;\longmapsto \sum_{x \in \varphi^{-1}\{b\}} \mathcal{A}(a,x) \end{array}\right.$$

Suppose we have a comodule $\mathcal{A} \overset{(P,F)}{\longrightarrow\!\!\!\!\!\!+\;\;} \mathcal{B}$ and retrofunctors $\varphi : \mathcal{A} \to \mathcal{C}$ and $\psi : \mathcal{B} \to \mathcal{D}$.

Then we may construct a comodule $\mathcal{C} \longrightarrow\!\!\!\!\!\!+\;\; \mathcal{D}$ given by:
$$\mathcal{C}^{op} \times \mathcal{D} \xrightarrow{\varphi^* \triangleleft (P,F) \triangleleft \psi_*} \mathcal{P}oly$$
$$(c,d) \longmapsto \sum_{b \in \psi^{-1}\{d\}} F(b) \longrightarrow \mathbf{Set}$$
$$(b,x) \longmapsto \sum_{a \in \varphi^{-1}\{c\}} P(x,a)$$

$$\begin{tikzcd} \mathcal{A} \arrow[r, "{(P,F)}", "\shortmid"{marking}] \arrow[d, "\varphi"'] & \mathcal{B} \arrow[d, "\psi"] \\ \mathcal{C} \arrow[r, dashed, "\shortmid"{marking}] & \mathcal{D} \end{tikzcd}$$

Using this construction, a cell in $\mathbf{Comod}(\mathcal{P}oly, \triangleleft, *)$ denoted

$$\begin{tikzcd} \mathcal{A} \arrow[r, "{(P,F)}", "\shortmid"{marking}] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\Theta"] & \mathcal{B} \arrow[d, "\psi"] \\ \mathcal{C} \arrow[r, "{(Q,G)}"', "\shortmid"{marking}] & \mathcal{D} \end{tikzcd}$$

corresponds to a natural transformation

$$\begin{tikzcd} \mathcal{C}^{op} \times \mathcal{D} \arrow[r, bend left, "\varphi^* \triangleleft (P,F) \triangleleft \psi_*"{name=U}] \arrow[r, bend right, "{(Q,G)}"'{name=L}] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{P}oly \end{tikzcd}$$

whose component at (c,d) is given by:

For each $b \in \psi^{-1}\{d\}$

$$\begin{tikzcd} F(b) \arrow[d, "\theta_b"'] \arrow[r, phantom, "\simeq"] & \sum_{b \in \psi^{-1}\{d\}} F(b) \arrow[d, "\theta"'] \arrow[dr] \arrow[dr, phantom, "\theta^{\#}\Uparrow"] & \\ G(\psi b) & G(d) \arrow[r] & \mathbf{Set} \end{tikzcd}$$

$$\theta^{\#}_x : Q(\theta_b x\,,\, c) \longrightarrow \sum_{a \in \varphi^{-1}\{c\}} P(x\,,\,a)$$

We write $\mathbf{Cat}^{\#} := \mathbf{Comod}(\mathbf{Poly}, \triangleleft, *)$.

EXAMPLES:

- Restricting to the terminal category

$$(\mathbf{Poly}, \triangleleft, *) \hookrightarrow \mathbf{Cat}^{\#}$$

- Restricting to discrete categories we obtain multivariable polynomials

$$\begin{tikzcd}
A \arrow[dd] & X \arrow[l] \arrow[r] & Y \arrow[r] \arrow[dd] & B \arrow[dd] \\
& \bullet \arrow[u] \arrow[ur] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & & \\
C & W \arrow[l] \arrow[r] & Z \arrow[r] & D
\end{tikzcd}$$

- A comodule $1 \nrightarrow B$ is a functor $B \longrightarrow \mathbf{Poly}$ which is equivalent to:

$$X \xrightarrow[\substack{\text{discrete}\\ \text{fibration}}]{} Y \xrightarrow[\substack{\text{discrete}\\ \text{opfibration}}]{} B$$

This is a "left B-comodule".

- A comodule $A \nrightarrow 1$ is a diagram

$$\begin{tikzcd}
A^{op} \arrow[r] \arrow[d] & \mathbf{Poly} \arrow[d] \\
1 \arrow[r, "X"'] & \mathbf{Set}
\end{tikzcd}$$

which is the same as a functor

$$A \longrightarrow [X, \mathbf{Set}]$$

which is equivalent to a span

$$A \xleftarrow[\substack{\text{discrete}\\ \text{opfibration}}]{} \bullet \longrightarrow \mathrm{disc}(X)$$

This is a "right A-comodule".

- Every profunctor $A \nrightarrow B$ yields a comodule $B \nrightarrow A$ in $\mathbf{Poly}$.

- Let $\mathbb{R}\mathrm{et}$ be the double category of categories, functors, and retrofunctors.

CONJECTURE: $\mathbb{R}\mathrm{et} \longrightarrow \mathbf{Cat}^{\#}$ is fully faithful (depending on choice of cells in $\mathbb{R}\mathrm{et}$).

COMPOSITION OF COMODULES:

If we consider comodules in $(\mathbf{Poly}, \triangleleft, *)$ as certain spans, how do we compose them?

$$
\begin{tikzcd}
 & & K \arrow[dl, "\hat{Q}"', "|" marking] \arrow[dr, "\hat{f}"] & & \\
 & I \arrow[dl, "P"', "|" marking] \arrow[dr, "f"] & & J \arrow[dl, "Q"', "|" marking] \arrow[dr, "g"] & \\
A & & B & & C
\end{tikzcd}
$$

like a distributivity pullback

$$P : I^{op} \times A \longrightarrow \mathcal{S}et$$

$$Q : J^{op} \times B \longrightarrow \mathcal{S}et$$

f, g - discrete opfibrations

Define K to be the category whose:

- objects are pairs $(j \in J, \alpha : Q(j,-) \Rightarrow f^{-1}\{-\})$ where $f^{-1}\{-\} : B \to \mathcal{S}et$ is well-defined since f is a discrete opfibration.
- morphisms $(j, \alpha) \longrightarrow (j', \alpha')$ are $w : j \to j'$ in J such that $\alpha' = \alpha \circ Q(w,-)$.

Projection in the first component gives a discrete opfibration $\hat{f} : K \longrightarrow J$.

We also may construct a functor

$$\hat{Q} : K^{op} \times I \longrightarrow \mathcal{S}et$$

$$(j \in J, \alpha : Q(j,-) \Rightarrow f^{-1}\{-\}, i \in I) \longmapsto$$

$$\hat{Q}(j, \alpha, i) = \{ q \in Q(j, fi) \mid \alpha_{fi}(q) = i \}$$

Altogether, we may take the composite $g \circ \hat{f}$ to obtain a discrete opfibration and the composite

$$P \circ \hat{Q} : K^{op} \times A \longrightarrow \mathcal{S}et$$

$$(j, \alpha : Q(j,-) \Rightarrow f^{-1}\{-\}, a) \longmapsto \int^{i \in I} \hat{Q}(j, \alpha, i) \times P(i, a)$$

to get the composite of comodules in $(\mathbf{Poly}, \triangleleft, *)$ viewed as spans.