# The a.w.f.s. of twisted coreflections & delta lenses

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Category Theory Seminar

Université catholique de Louvain, 4 December 2023

# CONTEXT & MOTIVATION



$$
\begin{tikzcd}
\boxed{\text{O.F.S.}} \arrow[rr, "\text{remove uniqueness of lifts}"] & & \boxed{\text{W.F.S.}} \arrow[dl, "\text{require \underline{specified} lifts}"] \\
& \boxed{\text{A.W.F.S.}} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "\ell \in \mathcal{L}"'] & \bullet \arrow[d, "r \in \mathcal{R}"] \\
\bullet \arrow[r] \arrow[ur, dashed, "\exists !"] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "\ell \in \mathcal{L}"'] & \bullet \arrow[d, "r \in \mathcal{R}"] \\
\bullet \arrow[r] \arrow[ur, dashed, "\exists"] & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bullet \arrow[r, "s"] \arrow[d, "\ell \in \mathbb{L}"'] & \bullet \arrow[d, "r \in \mathbb{R}"] \\
\bullet \arrow[r, "t"'] \arrow[ur, "\varphi_{\ell,r}(s,t)"] & \bullet
\end{tikzcd}
$$

# OUTLINE OF THE TALK



1. Motivating AWFS via examples
2. Double categories & definition of AWFS
3. Delta lenses & twisted coreflections

Main goals for talk:

- Unpack the reformulation of AWFS due to Bourke (2023).
- Introduce the new notion of twisted coreflection.
- Construct an AWFS on Cat with:
  * left class = twisted coreflections
  * right class = delta lenses

# PART 1: MOTIVATING A.W.F.S. VIA EXAMPLES

# COPRODUCT INJECTIONS & SPLIT EPIMORPHISMS

Let $\mathcal{C}$ be a category with finite coproducts.

## FACTORISATION

$$
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & A \arrow[d, "j_A"] \arrow[dd, bend left=60, "f"] \\
B \arrow[r, tail, "j_B"'] \arrow[dr, bend right, "1_B"'] \arrow[ur, phantom, "\ulcorner", very near end] & A+B \arrow[d, "{[f,1_B]}"] \\
& B
\end{tikzcd}
$$

## LIFTING

$$
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & A \arrow[r, "h"] \arrow[d, "j_A"'] & C \arrow[d, two heads, shift right, "g"'] \\
B \arrow[r, "j_B"] \arrow[rr, bend right, "k"'] & A+B \arrow[ur, "{[h,k\psi]}"] \arrow[r, "{[gh,k]}"] & D \arrow[u, tail, shift right, "\psi"']
\end{tikzcd}
$$

# SPLIT MONOMORPHISMS & PRODUCT PROJECTIONS



Let $\mathcal{C}$ be a category with finite products.

## FACTORISATION

$$
\begin{tikzcd}
& A \arrow[dl, bend right, "1_A"'] \arrow[d, tail, "\langle 1_A, f\rangle"] \arrow[dd, bend left=60, "f"] \\
A \arrow[d] & A \times B \arrow[l, two heads, "\pi_A"'] \arrow[d, "\pi_B"] \arrow[dl, phantom, very near start, "\lrcorner"] \\
1 & B \arrow[l]
\end{tikzcd}
$$

## LIFTING

$$
\begin{tikzcd}
A \arrow[r, "\langle h, k\varphi\rangle"] \arrow[rr, bend left, "h"] \arrow[d, tail, shift right, "f"'] & C \times D \arrow[r] \arrow[d] & C \arrow[d] \\
B \arrow[u, two heads, shift right, "\varphi"'] \arrow[ur, "\langle h\varphi, k\rangle"'] \arrow[r, "k"'] & D \arrow[r] & 1
\end{tikzcd}
$$

# SPLIT OPFIBRATIONS

A split opfibration is a functor equipped with a lifting operation (splitting)

$$
\begin{tikzcd}
A \arrow[d, "f"'] & a \arrow[r, "\varphi(a,u)"] \arrow[d, dotted, no head] & a' \arrow[d, dotted, no head] \\
B & fa \arrow[r, "u"] & b
\end{tikzcd}
$$

such that:

1. $f\varphi(a,u) = u$
2. $\varphi(a, 1_{fa}) = 1_a$
3. $\varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a,u)$
4. Each lift $\varphi(a,u)$ is opcartesian.

$$
\begin{tikzcd}
a \arrow[rr, "\varphi(a,u)"] \arrow[dr, "w"'] & & a' \arrow[dl, dashed, "\exists!"] \\
& a'' &
\end{tikzcd}
$$

$$\downarrow f$$

$$
\begin{tikzcd}
fa \arrow[rr, "u"] \arrow[dr, "fw"'] & & b \arrow[dl, "v"] \\
& fa'' &
\end{tikzcd}
$$

# SPLIT COREFLECTIONS



A split coreflection is a functor $f$ equipped with a retraction $q$

$$\begin{tikzcd} A \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "q"'] \end{tikzcd} \qquad qf = 1_A$$

and a natural transformation

$$\begin{tikzcd} & A \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow \varepsilon"] & \\ B \arrow[ur, "q"] \arrow[rr, "1_B"'] & {} & B \end{tikzcd}$$

$$\varepsilon \cdot f = 1_f$$

$$q \cdot \varepsilon = 1_q$$

such that the equations hold:

Have a coreflective adjunction:

$$\begin{tikzcd} A \arrow[r, hook, shift right, "f"'] \arrow[r, phantom, "\top"] & B \arrow[l, shift right, "q"'] \end{tikzcd}$$

FACTORISATION

$$\begin{tikzcd} & A \arrow[d, dashed, "\eta_f"] \arrow[ddl, bend right, "1_A"'] \arrow[ddr, bend left, "f"] & \\ & f/1_B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ A \arrow[rr, "f"'] & \Rightarrow & B \end{tikzcd}$$

# LIFTING AGAINST SPLIT OPFIBRATIONS (1)



$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{(f,q,\varepsilon)}"'] & C \arrow[d, "{(g,\psi)}"] \\ B \arrow[r, "k"'] \arrow[ur, "j"] & D \end{tikzcd}$$

$$qx \xrightarrow{qu} qy$$

$$\begin{tikzcd} fqx \arrow[r, "fqu"] \arrow[d, "\varepsilon_x"'] & fqy \arrow[d, "\varepsilon_y"] \\ x \arrow[r, "u"] & y \end{tikzcd}$$

$$\begin{tikzcd} hqx \arrow[r, "hqu"] \arrow[d, "{\psi(hqx, k\varepsilon_x)}"'] & hqy \arrow[d, "{\psi(hqy, k\varepsilon_y)}"] \\ jx \arrow[r, dashed, "ju"] & jy \end{tikzcd}$$

$$\begin{tikzcd} g(hqx) \arrow[d, equal] & g(hqy) \arrow[d, equal] \\ kfqx \arrow[r, "kfqu"] \arrow[d, "k\varepsilon_x"'] & kfqy \arrow[d, "k\varepsilon_y"] \\ kx \arrow[r, "ku"] & ky \end{tikzcd}$$

# LIFTING AGAINST SPLIT OPFIBRATIONS (2)



<u>Useful fact</u>: $(f,\varphi) : A \longrightarrow B$ is a split opfibration iff $f_* : [X,A] \longrightarrow [X,B]$ is a split opfibration for all $X$.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[dr, bend right, "b"'] \arrow[dr, phantom, "\swArrow\theta"] & A \arrow[d, "f"] \\ & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "a'"'] \arrow[r, phantom, "\Downarrow\hat{\theta}"] & A \arrow[d, "f"] \\ \boxed{f\cdot\hat{\theta} = \theta} & B \end{tikzcd}$$

We can use this to construct lifts of split coreflections against split opfibrations.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{(f,q,\varepsilon)}"'] & C \arrow[d, "{(g,\psi)}"] \\ B \arrow[r, "k"'] \arrow[ur, "j"] & D \end{tikzcd}$$

We may define functor $j$ as follows:

$$\begin{tikzcd} B \arrow[r, "hq"] \arrow[dr, bend right, "k"'] \arrow[dr, phantom, "\swArrow k\cdot\varepsilon"] & C \arrow[d, "g"] \\ & D \end{tikzcd} \rightsquigarrow \begin{tikzcd} B \arrow[r, bend left, "hq"] \arrow[r, bend right, "j"'] \arrow[r, phantom, "\Downarrow\hat{\varepsilon}"] & C \arrow[d, "g"] \\ \boxed{g\cdot\hat{\varepsilon} = k\cdot\varepsilon} & D \end{tikzcd}$$

# THE STORY SO FAR



Several examples where we have factorisation and lifting:

| left class | right class |
|---|---|
| coproduct injections | split epis |
| split monos | product projections |
| split coreflections | split opfibrations |

Each class is closed under composition, but also have morphisms of morphisms!

<u>Example</u>: Morphisms of split epis

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, shift right, "f"'] & C \arrow[d, shift right, "g"'] \\ B \arrow[r, "k"'] \arrow[u, shift right, "\varphi"'] & D \arrow[u, shift right, "\psi"'] \end{tikzcd}$$

$$\boxed{\begin{aligned} kf &= gh \\ h\varphi &= \psi k \end{aligned}}$$

- How should lifting account for composition and "squares"?
- In what sense is factorisation universal?
- Does lifting against each class determine the other class?

# PART 2: DOUBLE CATEGORIES & A.W.F.S.

# DOUBLE CATEGORIES



A **double category** $\mathbb{D}$ consists of:

- objects
- horizontal morphisms
- vertical morphisms
- cells

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

+ unital & associative horizontal & vertical composition

- $\mathbb{D}$ is **thin** if cell $=$ boundary.

- Example: For each category $\mathcal{C}$, the **double category of squares** $\mathbb{S}\mathrm{q}(\mathcal{C})$.

| left class | right class |
|---|---|
| $\mathbb{I}\mathrm{nj}(\mathcal{C})$ | $\mathbb{S}\mathrm{pEpi}(\mathcal{C})$ |
| $\mathbb{S}\mathrm{pMono}(\mathcal{C})$ | $\mathbb{P}\mathrm{roj}(\mathcal{C})$ |
| $\mathbb{S}\mathrm{pCoref}$ | $\mathbb{S}\mathrm{pOpf}$ |

- Each have double functor to $\mathbb{S}\mathrm{q}(\mathcal{C})$.

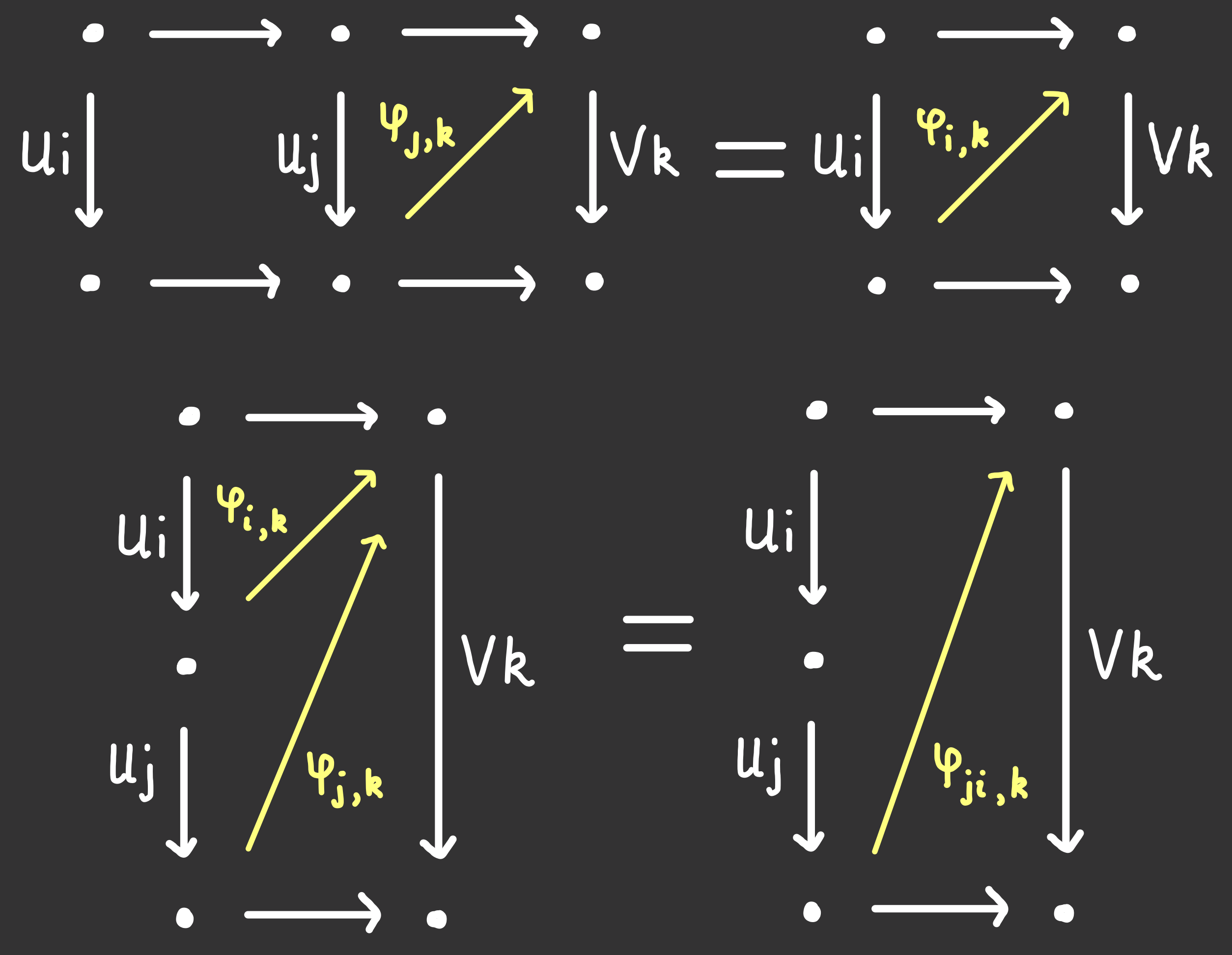
# DOUBLE-CATEGORICAL LIFTING OPERATIONS

$$\mathbb{L} \xrightarrow{U} \mathbb{S}\mathrm{q}(\mathcal{C}) \xleftarrow{V} \mathbb{R}$$

A $(\mathbb{L}, \mathbb{R})$-lifting operation is a family

$$\begin{tikzcd} UA \arrow[r, "s"] \arrow[d, "U_j"'] & VC \arrow[d, "V_k"] \\ UB \arrow[r, "t"'] \arrow[ur, "\varphi_{j,k}(s,t)"] & VD \end{tikzcd}$$

which satisfies certain horizontal and vertical compatibilities.




+ dual compatibilities on right.

<u>Example</u>: $\mathbb{S}\mathrm{pCoref} \longrightarrow \mathbb{S}\mathrm{q}(\mathcal{C}\mathrm{at}) \longleftarrow \mathbb{S}\mathrm{pOpf}$

# THE DOUBLE CATEGORY IRLP($\mathbb{J}$)

For each double functor $\mathbb{J} \xrightarrow{u} \mathbb{S}q(\mathcal{C})$ there is a double category

$$\mathbb{RLP}(\mathbb{J}) \longrightarrow \mathbb{S}q(\mathcal{C})$$

whose:

- objects & horizontal mor. are from $\mathcal{C}$
- vertical mor. are pairs $(f, \varphi)$ where

$$\begin{tikzcd} uA \arrow[r, "s"] \arrow[d, "ui"'] & C \arrow[d, "f"] \\ uB \arrow[r, "t"'] \arrow[ur, "\varphi_i(s,t)"] & D \end{tikzcd}$$

$f$ is morphism in $\mathcal{C}$

$\varphi$ is a $(\mathbb{J}, f)$-lifting operation

- cells $(f,\varphi) \longrightarrow (g,\psi)$ are given by:

$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "ui"'] & \bullet \arrow[r, "h"] \arrow[d, "f"] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "t"'] \arrow[ur, "\varphi_i"] & \bullet \arrow[r, "k"'] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, "hs"] \arrow[d, "ui"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "kt"'] \arrow[ur, "\psi_i"] & \bullet \end{tikzcd}$$

Dually, we can define $\mathbb{LLP}(\mathbb{J})$.

Given a $(\mathbb{L}, \mathbb{R})$-lifting operation we obtain canonical double functors:

$$\mathbb{L} \longrightarrow \mathbb{LLP}(\mathbb{R}) \qquad \mathbb{R} \longrightarrow \mathbb{RLP}(\mathbb{L})$$

# ALGEBRAIC WEAK FACTORISATION SYSTEMS



An algebraic weak factorisation system is a $(\mathbb{L}, \mathbb{R})$-lifting operation $\varphi$ where

$$\mathbb{L} \xrightarrow{U} \mathbb{S}\mathrm{q}(\mathcal{C}) \xleftarrow{V} \mathbb{R}$$

such that the following axioms hold:

(i) induced double functors are iso

$$\mathbb{L} \longrightarrow \mathbb{LLP}(\mathbb{R}) \qquad \mathbb{R} \longrightarrow \mathbb{RLP}(\mathbb{L})$$

(ii) Each $f$ in $\mathcal{C}$ admits a factorisation

$$\cdot \xrightarrow{U_1 g} \cdot \xrightarrow{V_1 h} \cdot \;=\; \cdot \xrightarrow{f} \cdot$$

which is $U_1$-couniversal & $V_1$-universal.

- Every OFS $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$ is an AWFS:

$$\mathbb{S}\mathrm{q}(\mathcal{C}, \mathcal{E}) \hookrightarrow \mathbb{S}\mathrm{q}(\mathcal{C}) \hookleftarrow \mathbb{S}\mathrm{q}(\mathcal{C}, \mathcal{M})$$

- Axiom (i) says that vert. morphisms in $\mathbb{L}$ and $\mathbb{R}$ are "orthogonal" w.r.t. $\varphi$.

- Axiom (ii) says that factorisation into $\mathbb{L}$-morphism followed by $\mathbb{R}$-morphism is "the most optimal" through each.

# Part 3: delta lenses & twisted coreflections

# CONTEXT FOR DELTA LENSES

**2011:** Delta lenses introduced as an algebraic model for "bidirectional transformations" in computer science.

**2012:** Split opfibrations studied as a competing model - "least change lifts".

**2013:** Delta lenses characterised as <u>certain</u> algebras for a semi-monad on $\mathcal{C}at^{2}$; comparison made with split opfibrations.

**2013-2021:** Long-running programme by Johnson & Rosebrugh to understand delta lenses using category theory.

**2022-2023:** I prove that delta lenses are algebras for a monad on $\mathcal{C}at^{2}$.

**Question:** What diagrams do lenses lift?

$$\begin{tikzcd} A \arrow[r] \arrow[d, "???"'] & C \arrow[d, "\text{delta lens}"] \\ B \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

# DELTA LENSES



A **delta lens** is a functor equipped with a lifting operation

$$\begin{tikzcd} A \arrow[d, "f"'] & a \arrow[r, "\varphi(a,u)"] \arrow[d, dotted, no head] & a' \arrow[d, dotted, no head] \\ B & fa \arrow[r, "u"] & b \end{tikzcd}$$

that satisfies the following axioms:

1. $f\varphi(a,u) = u$
2. $\varphi(a, 1_{fa}) = 1_a$
3. $\varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a, u)$

Let $\mathbb{L}\mathrm{ens}$ denote the double category of categories, functors, & delta lenses.

A cell with boundary

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{(f,\varphi)}"'] & C \arrow[d, "{(g,\psi)}"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

exists if $kf = gh$ & $h\varphi(a,u) = \psi(ha, ku)$.

$$\boxed{\mathbb{S}\mathrm{pOpf} \hookrightarrow \mathbb{L}\mathrm{ens} \longrightarrow \mathbb{S}\mathrm{q}(\mathcal{C}\mathrm{at})}$$

# TWISTED COREFLECTIONS



A twisted coreflection is a split coreflection

$$\begin{tikzcd} A \arrow[r, hook, bend right, "f"'] \arrow[r, phantom, "T"] & B \arrow[l, bend right, "q"'] \end{tikzcd}$$

such that if $q(u \cdot x \to y) \neq 1$, there exists a unique morphism $\bar{q}u : x \longrightarrow fqx$ such that:

$$\boxed{\begin{array}{c} \bar{q}u \circ \varepsilon_x = 1 \\ \varepsilon_y \circ fqu \circ \bar{q}u = u \end{array}}$$

$$\begin{tikzcd} fqx \arrow[r, "fqu"] \arrow[d, "\varepsilon_x"'] & fqy \arrow[d, "\varepsilon_y"] \\ x \arrow[u, dashed, bend right, "\exists!\, \bar{q}u"'] \arrow[r, "u"'] & y \end{tikzcd}$$

Let $\mathbb{T}\mathrm{wCoRef}$ denote the double cat. of categories, functors, & twisted coreflections. A cell with boundary

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{(f,q,\varepsilon)}"'] & C \arrow[d, "{(g,p,\xi)}"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

exists if $kf = gh$, $hq = pk$ & $k \cdot \varepsilon = \xi \cdot k$.

$$\boxed{\mathbb{T}\mathrm{wCoref} \hookrightarrow \mathbb{C}\mathrm{oref} \longrightarrow \mathbb{S}\mathrm{q}(\mathrm{Cat})}$$

# BUILDING A TWISTED COREFLECTION

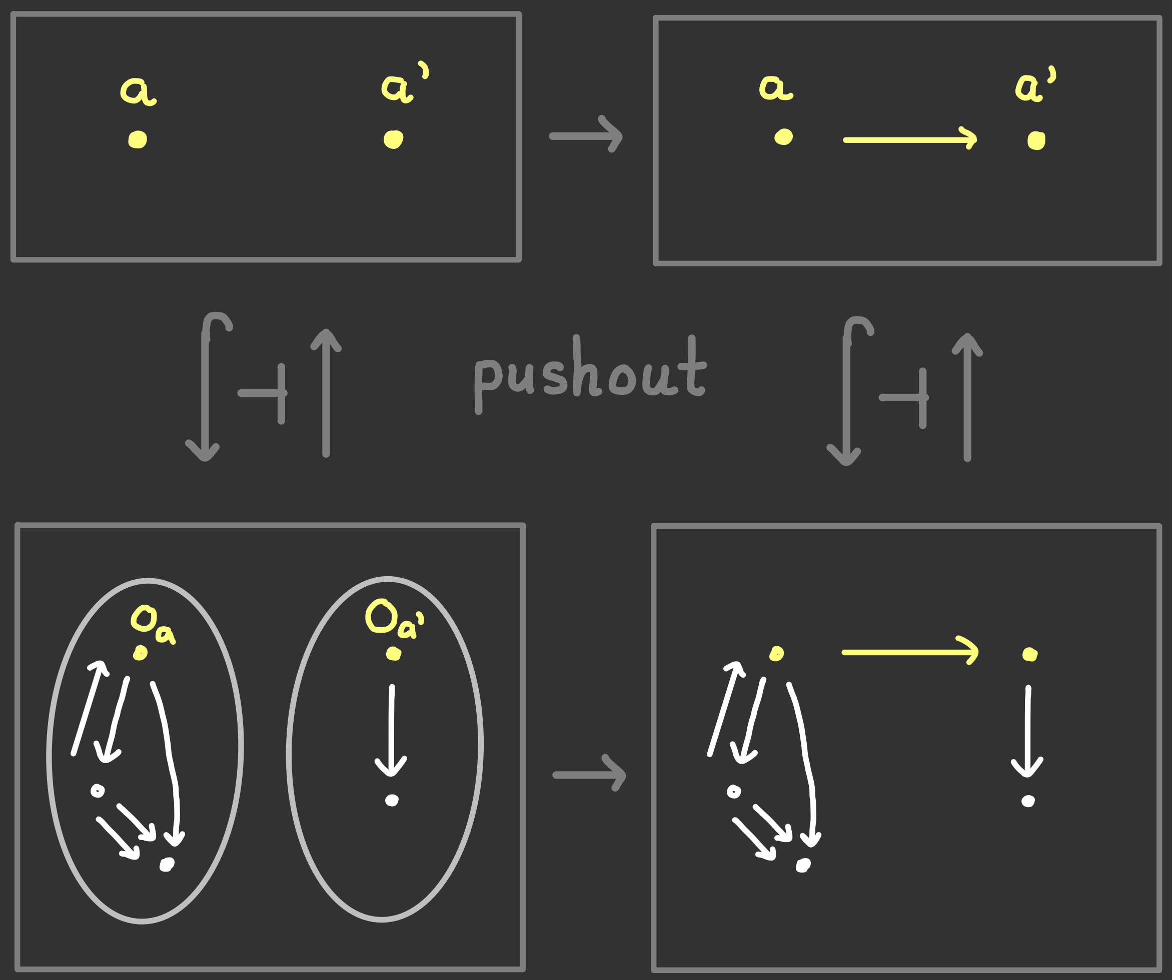


1. Choose a category $A$ for the domain of the twisted coreflection.

2. For each object $a \in A$, choose a category $X_a$ with an initial object.

3. Glue each initial object $0_a \in X_a$ to the corresponding $a \in A$.

4. Close under composition.



# DIAGRAMMATIC CHARACTERISATIONS



## DELTA LENS $(f, \varphi)$

bijective-on-objects

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "f\varphi"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

discrete opfibration

where the category $X$ has:

- same objects as $A$
- morphisms are the chosen lifts $\varphi(a, u)$

## TWISTED COREFLECTION $(f, q, \varepsilon)$

bijective-on-objects

discrete category

$$\begin{tikzcd} A_0 \arrow[r, "\iota_A"] \arrow[d, hook, shift right=2] \arrow[d, phantom, "\dashv"] & A \arrow[d, hook, shift right=2, "f"'] \arrow[d, phantom, "\dashv"] \\ \sum_{a \in A_0} q^{-1}\{a\} = X \arrow[r] \arrow[u, shift right=2] & B \arrow[u, shift right=2, "q"'] \end{tikzcd}$$

where morphisms in $B$ are either:

$$x \xrightarrow{u} y \qquad qu = id$$

OR

$$\begin{tikzcd} fa \arrow[r, "fw"] & fa' \arrow[d, "v"] \\ x \arrow[u, "u"] & y \end{tikzcd} \qquad \begin{matrix} qu = id \\ qv = id \end{matrix}$$

# FACTORISATION



$(-)_0 : \mathcal{C}at \to \mathcal{C}at$ – discrete category comonad

split coreflection

~~split~~ discrete opfibration

$$
\begin{tikzcd}
A_0 \arrow[r, "\iota_A"] \arrow[d, hook, shift right=2] \arrow[d, phantom, "\dashv"] \arrow[dr, phantom, very near end, "\ulcorner"] & A \arrow[d, hook, shift right=2, "Lf"'] \arrow[d, phantom, "\dashv"] \arrow[dd, bend left=50, "f"] \\
Jf \arrow[u, shift right=2] \arrow[r, "\text{b.o.o.}"'] \arrow[d] & Ef \arrow[u, shift right=2] \arrow[d, dashed, "Rf"'] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

where $Jf = \sum_{a \in A_0} fa / B$

- $Lf : A \longrightarrow Ef$ is the cofree twisted coreflection on $f : A \longrightarrow B$
- $Rf : Ef \longrightarrow B$ is the free delta lens on $f : A \longrightarrow B$ (see arXiv: 2305.02732).
- Close connection with (coproduct injection, split epimorphism) A.W.F.S.:
  * Initial object comonad $X \mapsto \emptyset$
  * (all morphisms, isomorphisms) O.F.S.

# LIFTING AGAINST DELTA LENSES

twisted coreflection

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "{(f,q,\varepsilon)}"'] & C \arrow[d, "{(g,\psi)}"] \\
B \arrow[r, "k"'] \arrow[ur, "j"] & D
\end{tikzcd}
$$

delta lens

Construct functor $j$ as follows:

① 

$$
\begin{tikzcd}
A_0 \arrow[rr, bend left, "\exists!\ \hat{h}"] \arrow[r, "\iota_A"] \arrow[d, shift right, "\hat{f}"'] \arrow[dr, phantom, "\ulcorner"] & A \arrow[r, "h"] \arrow[d, shift right, "f"'] & C \arrow[d, "g"] \\
X \arrow[u, shift right, "\hat{q}"'] \arrow[u, phantom, "\dashv"] \arrow[r, "\pi"'] & B \arrow[u, shift right, "q"'] \arrow[u, phantom, "\dashv"] \arrow[r, "k"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[d, "\varphi"] \arrow[dd, bend left, "g\varphi"] \\
C \arrow[d, "g"] \\
D
\end{tikzcd}
$$

②

$$
\begin{tikzcd}
X \arrow[r, "\hat{h}\hat{q}"] \arrow[dr, bend right, "k\pi"'] \arrow[r, phantom, shift right=6, "\swarrow k\pi\cdot\hat{\varepsilon}"] & Y \arrow[d, "g\varphi"] \\
& D
\end{tikzcd}
\overset{\exists!}{\rightsquigarrow}
\begin{tikzcd}
X \arrow[r, bend left, "\hat{h}\hat{q}"] \arrow[r, bend right, "\ell"'] \arrow[r, phantom, "\Downarrow\theta"] & Y \arrow[d, "g\varphi"] \\
& D
\end{tikzcd}
$$

$$
\boxed{g\varphi\cdot\theta = k\pi\cdot\hat{\varepsilon}}
$$

③

$$
\begin{tikzcd}
A_0 \arrow[r, "\iota_A"] \arrow[d, "\hat{f}"'] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d, "f"] \arrow[ddr, bend left, "h"] & \\
X \arrow[r, "\pi"'] \arrow[dr, "\ell"'] & B \arrow[dr, dashed, "j"] & \\
& Y \arrow[r, "\varphi"'] & C
\end{tikzcd}
$$

# COROLLARIES & FUTURE WORK

**Theorem:** There is a AWFS:

$$\mathbb{T}\mathrm{w}\mathbb{C}\mathrm{o}\mathbb{R}\mathrm{ef} \xrightarrow{\mathcal{U}} \mathbb{S}\mathrm{q}(\mathcal{C}\mathrm{at}) \xleftarrow{\mathcal{V}} \mathbb{L}\mathrm{ens}$$

- Delta lenses are algebras for a monad on $\mathcal{C}\mathrm{at}^2$, and are stable under pullback.
- Twisted coreflections are coalgebras for a comonad on $\mathcal{C}\mathrm{at}^2$, and are stable under pushouts.

$$\begin{tikzcd}
\mathrm{SchSpEpi}(\mathrm{Mon}) \arrow[rr, hook, shift right] \arrow[d, hook] \arrow[rr, phantom, "\perp"] & & \mathrm{SpEpi}(\mathrm{Mon}) \arrow[ll, dashed, shift right, "???"'] \arrow[dl, hook] \\
\mathrm{SpOpf} \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] \arrow[dr, "\mathrm{cod}"'] & \mathcal{L}\mathrm{ens} \arrow[l, shift right, "\text{USEFUL!}"'] \arrow[r, shift right] \arrow[r, phantom, "\perp"] \arrow[d, "\mathrm{cod}"] & \mathcal{C}\mathrm{at}^2 \arrow[l, shift right] \arrow[dl, "\mathrm{cod}"] \\
& \mathcal{C}\mathrm{at} &
\end{tikzcd}$$

**Idea:** Given an AWFS on $\mathcal{C}$ and an idempotent comonad on $\mathcal{C}$, we may construct a new AWFS on $\mathcal{C}$ if enough pushouts in $\mathcal{C}$ exist.

# SUMMARY OF THE TALK

- Unpacked the reformulation of AWFS due to Bourke using double categories:

$$\mathbb{L} \xrightarrow{U} \mathbb{S}\mathrm{q}(\mathcal{C}) \xleftarrow{V} \mathbb{R}$$

- Introduced twisted coreflections as split coreflections with a pushout property:

$$\begin{tikzcd}
A_0 \arrow[r, "\iota_A"] \arrow[d, shift right] \arrow[d, phantom, "\dashv"] & A \arrow[d, shift right, "f"'] \arrow[d, phantom, "\dashv"] \\
\sum\limits_{a \in A_0} q^{-1}\{a\} \arrow[r] \arrow[u, shift right] \arrow[ur, phantom, "\ulcorner"] & B \arrow[u, shift right, "q"']
\end{tikzcd}$$

- Constructed a new example of an AWFS on Cat consisting of twisted coreflections and delta lenses.

$$\begin{tikzcd}
A_0 \arrow[r, "\iota_A"] \arrow[d, shift right] \arrow[d, phantom, "\dashv"] & A \arrow[d, shift right, "Lf"'] \arrow[d, phantom, "\dashv"] \arrow[dd, bend left, "f"] \\
\sum\limits_{a \in A_0} fa/B = Jf \arrow[r] \arrow[u, shift right] \arrow[d] \arrow[ur, phantom, "\ulcorner"] & Ef \arrow[u, shift right] \arrow[d, dashed, "Rf"] \\
B \arrow[r, equal] & B
\end{tikzcd}$$