Lifting twisted coreflections against delta lenses

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MOTIVATION

Lenses have two components:

$$\begin{tikzcd} \boxed{A} \arrow[r, shift left, "\text{forwards}"] & \boxed{B} \arrow[l, shift left, "\text{backwards}"] \end{tikzcd}$$

$$\left.\begin{array}{l} A \xrightarrow{f} B \\ A \xleftarrow{p} A \times B \end{array}\right\} \text{Classical lens between sets}$$

Backwards component = lifting

$$\begin{tikzcd} \{\bullet\} \arrow[r, "a"] \arrow[d, hook] & A \arrow[d, "f"] \\ \{\bullet \quad \bullet\} \arrow[r, "{(fa,b)}"'] \arrow[ur, dashed, "{p(a,b)}"] & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d, hook, "\text{split mono}"'] & A \arrow[d, "\text{classical lens}"] \\ Y \arrow[r] \arrow[ur, dashed, "\text{lift}"] & B \end{tikzcd}$$

Two kinds of lenses between categories:

- Split opfibrations (JRW, 2012)
- Delta lenses (DXC, 2011)

What can be lifted?

$$\begin{tikzcd} \{\bullet\} \arrow[r, "a"] \arrow[d, hook] & A \arrow[d, "{(f,\varphi)}"] \\ \{\bullet \to \bullet\} \arrow[r, "u"'] \arrow[ur, dashed, "{\varphi(a,u)}"] & B \end{tikzcd}$$

Functors admit factorisations:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "Lf"'] & & B \\ & \bullet \arrow[ur, "Rf"'] & \end{tikzcd}$$

What is the left class?

OUTLINE OF THE TALK

1 Split opfibrations

2. Delta lenses

3. Double categories & lifting awfs

4. Concluding remarks

Main contributions:

- Introducing twisted coreflections
- Constructing lifting awfs with:

Left class	Right class
split coreflection	split opfibration
twisted coreflection	delta lens

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\ell"'] & \bullet \arrow[d, "r"] \\ \bullet \arrow[r] \arrow[ur, "\text{lift}"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[rr] \arrow[dr, "\ell"'] & & \bullet \\ & \bullet \arrow[ur, "r"'] & \end{tikzcd}$$

PART 1: SPLIT OPFIBRATIONS

SPLIT OPFIBRATIONS

A split opfibration is a functor equipped with a lifting operation (splitting)

$$
\begin{tikzcd}
A \arrow[d, "f"'] & a \arrow[r, "\varphi(a,u)"] \arrow[d, dotted, no head] & a' \arrow[d, dotted, no head] \\
B & fa \arrow[r, "u"] & b
\end{tikzcd}
$$

such that:

1. $f\varphi(a,u) = u$
2. $\varphi(a, 1_{fa}) = 1_a$
3. $\varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a,u)$
4. Each lift $\varphi(a,u)$ is opcartesian.

$$
\begin{tikzcd}
a \arrow[rr, "\varphi(a,u)"] \arrow[dr, "w"'] & & a' \arrow[dl, dashed, "\exists!"'] \\
& a'' &
\end{tikzcd}
$$

$$\downarrow f$$

$$
\begin{tikzcd}
fa \arrow[rr, "u"] \arrow[dr, "fw"'] & & b \arrow[dl, "v"] \\
& fa'' &
\end{tikzcd}
$$

SPLIT COREFLECTIONS

A split coreflection is a functor equipped with a right-adjoint-left-inverse (RALI).

$$\begin{tikzcd} A \arrow[r, hook, shift right=2, "f"'] \arrow[r, phantom, "\top"] & B \arrow[l, shift right=2, "q"'] \end{tikzcd}$$

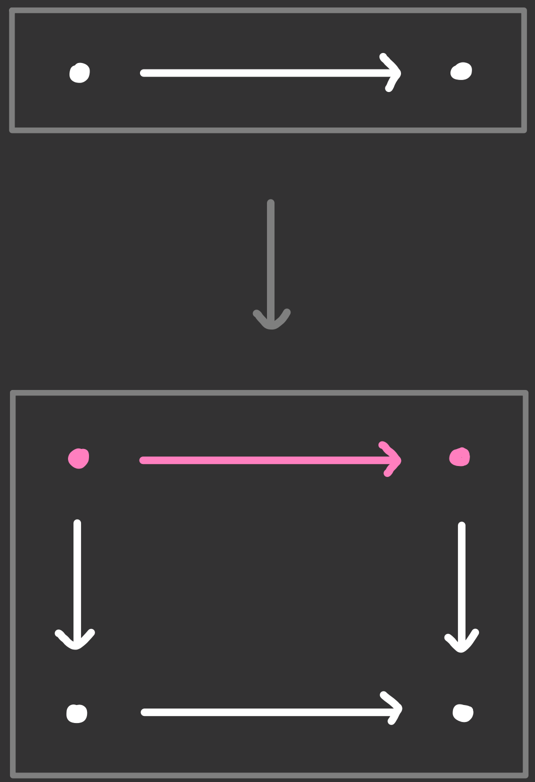
counit

$$\varepsilon : fq \Rightarrow 1_B$$

Three equations hold:

$$qf = 1_A \qquad \varepsilon \cdot f = 1_f \qquad q \cdot \varepsilon = 1_q$$

The simplest example of a split coreflection is given by:



LIFTING AGAINST SPLIT OPFIBRATIONS

$$qx \xrightarrow{qu} qy$$

$$\begin{tikzcd}
fqx \arrow[r, "fqu"] \arrow[d, "\varepsilon_x"'] & fqy \arrow[d, "\varepsilon_y"] \\
x \arrow[r, "u"] & y
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "{(f,q,\varepsilon)}"'] & C \arrow[d, "{(g,\psi)}"] \\
B \arrow[r, "k"'] \arrow[ur, "j"] & D
\end{tikzcd}$$

$$\begin{tikzcd}
hqx \arrow[r, "hqu"] \arrow[d, "{\psi(hqx, k\varepsilon_x)}"'] & hqy \arrow[d, "{\psi(hqy, k\varepsilon_y)}"] \\
jx \arrow[r, dashed, "ju"] & jy
\end{tikzcd}$$

$$\begin{tikzcd}
g(hqx) \arrow[d, equal] & g(hqy) \arrow[d, equal] \\
kfqx \arrow[r, "kfqu"] \arrow[d, "k\varepsilon_x"'] & kfqy \arrow[d, "k\varepsilon_y"] \\
kx \arrow[r, "ku"] & ky
\end{tikzcd}$$

FACTORISATION THROUGH THE COMMA CATEGORY

For $f : A \to B$, the category $f/1_B$ has:

- objects given by $(a \in A,\ u : fa \to b)$
- morphisms $\langle w, v \rangle : (a_1, u_1) \to (a_2, u_2)$ given by commutative squares.

$$a_1 \xrightarrow{\ w\ } a_2$$

$$\begin{tikzcd}
fa_1 \arrow[r, "fw"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\circlearrowright"] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[r, "v"'] & b_2
\end{tikzcd}$$

Every functor factorises through the comma category into a (cofree) split coreflection followed by a (free) split opfibration.

$$\begin{tikzcd}
& A \arrow[ddl, bend right, "1_A"'] \arrow[ddr, bend left, "f"] \arrow[d, dashed, "\eta_f"] & \\
& f/1_B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] \arrow[d, phantom, "\Rightarrow"] & \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

PART 2: DELTA LENSES

DELTA LENSES

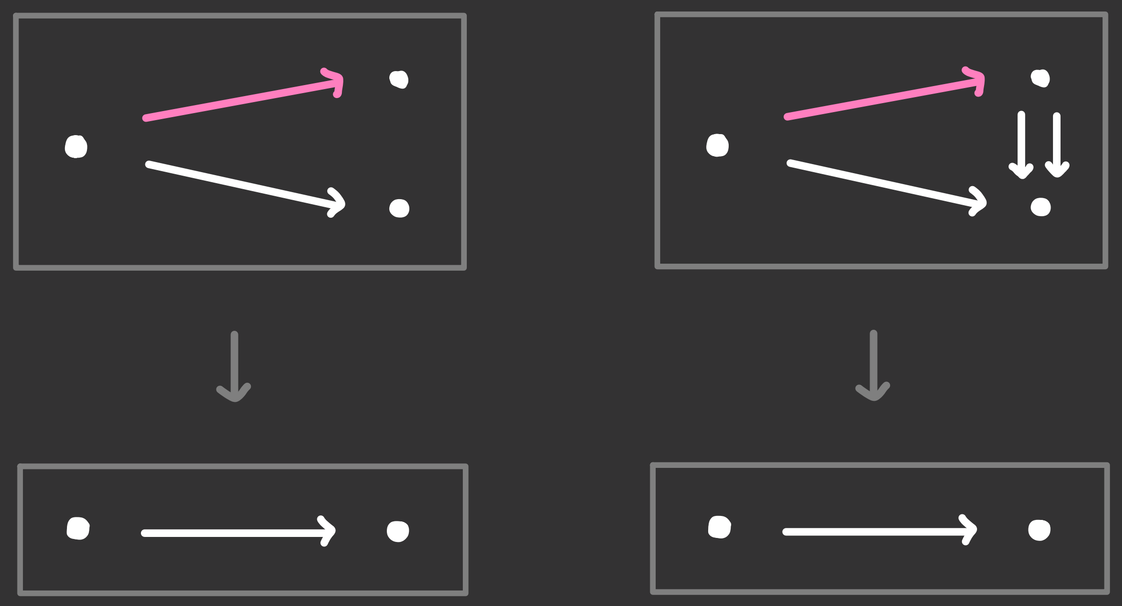
A **delta lens** is a functor equipped with a lifting operation

$$
\begin{tikzcd}
A \arrow[d, "f"'] & a \arrow[r, "\varphi(a,u)"] \arrow[d, dotted, no head] & a' \arrow[d, dotted, no head] \\
B & fa \arrow[r, "u"] & b
\end{tikzcd}
$$

that satisfies the following axioms:

1. $f\varphi(a,u) = u$
2. $\varphi(a, 1_{fa}) = 1_a$
3. $\varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a, u)$

Two simple examples of delta lenses which are <u>not</u> split opfibrations.



Motivating question: which lifting problems do delta lenses solve?

LIFTING PROBLEMS

Given a commutative square in Cat

Gives a "diagram" to lift

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] \arrow[ur, dashed, "j"] & D \end{tikzcd}$$

such that $g : C \to D$ is equipped with a delta lens structure, what are the conditions on $f : A \to B$ such that a canonical $j : B \to C$ exists?

Example:

$$\begin{tikzcd} \{\bullet\} \arrow[r, "a"] \arrow[d, hook, "d_1"'] & A \arrow[d, "{(f,\varphi)}"] \\ \{\bullet \to \bullet\} \arrow[r, "u"'] \arrow[ur, dashed, "{\varphi(a,u)}"] & B \end{tikzcd}$$

Non-example:

$$\begin{tikzcd} \{\bullet \to \bullet\} \arrow[r] \arrow[d, hook] & A \arrow[d, "{(f,\varphi)}"] \\ \left\{\begin{smallmatrix} \bullet & \to & \bullet \\ \downarrow & & \downarrow \\ \bullet & \to & \bullet \end{smallmatrix}\right\} \arrow[r] \arrow[ur, dashed, "???"] & B \end{tikzcd}$$

TWISTED COREFLECTIONS

A **twisted coreflection** is a split coreflection

$$\begin{tikzcd} A \arrow[r, hook, bend right, "f"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "q"'] \end{tikzcd} \qquad \varepsilon : fq \Rightarrow 1_B$$

equipped with a partial **twisting operation**

$$u : x \to y \quad \longmapsto \quad \tau u : x \to fqx \qquad \boxed{\text{if } qu \neq 1}$$

that satisfies the following axioms:

1. $\varepsilon_y \circ fqu \circ \tau u = u$

2. $\tau u \circ \varepsilon_x = 1_{fqx}$

3. $\tau(v \circ u) = \begin{cases} \tau v \circ u & \text{if } qu = 1 \\ \tau u & \text{otherwise} \end{cases}$

Two kinds of naturality square for counit.

$$\begin{tikzcd} fqx \arrow[r, equal, "fqu"] \arrow[d, "\varepsilon_x"'] & fqy \arrow[d, "\varepsilon_y"] \\ x \arrow[r, "u"'] & y \end{tikzcd}$$

$$\begin{tikzcd} fqx \arrow[r, "fqu"] \arrow[d, shift right, "\varepsilon_x"'] & fqy \arrow[d, "\varepsilon_y"] \\ x \arrow[u, shift right, "\tau u"'] \arrow[r, dash, "u"'] & y \end{tikzcd}$$

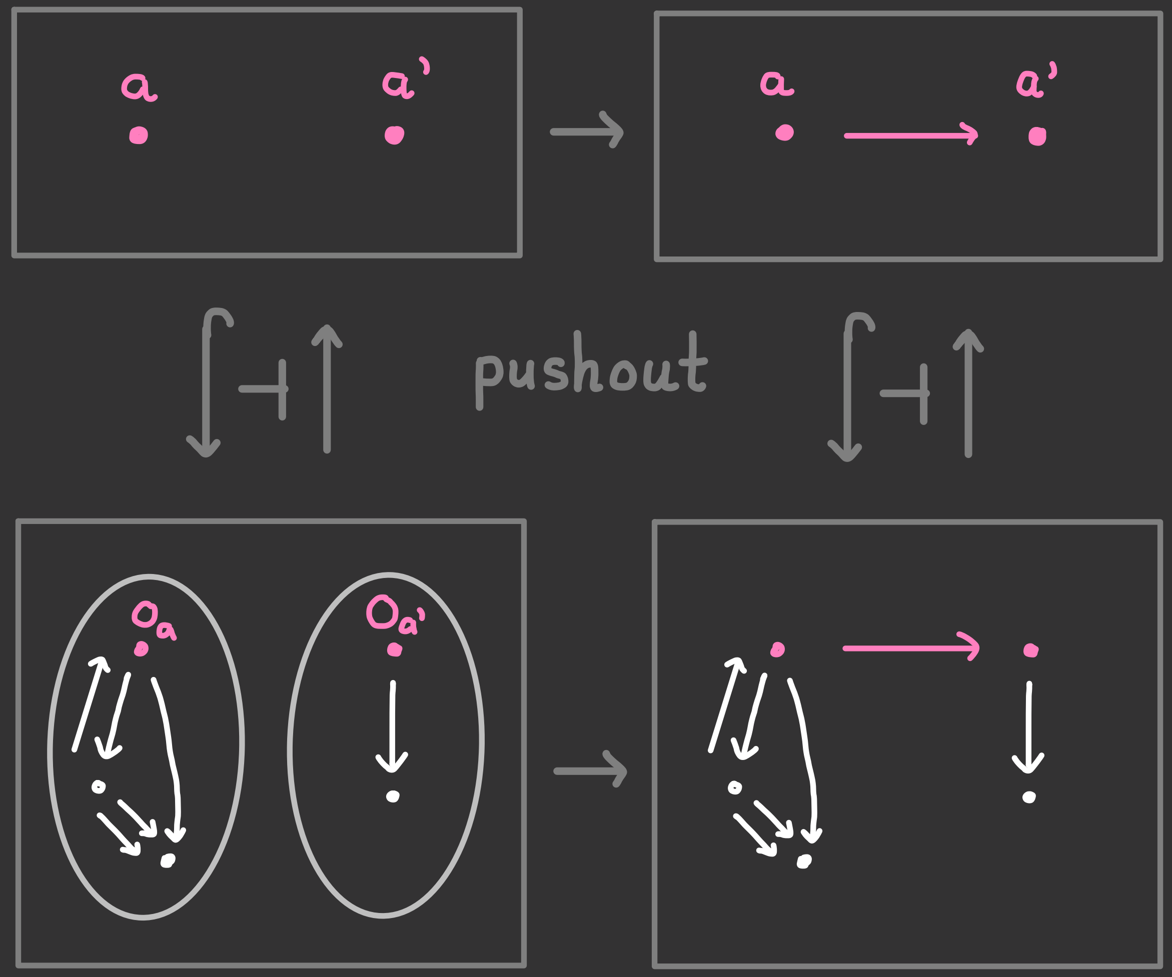
BUILDING A TWISTED COREFLECTION

1. Choose a category A for the domain of the twisted coreflection.

2. For each object $a \in A$, choose a category X_a with an initial object.

3. Glue each initial object $0_a \in X_a$ to the corresponding $a \in A$.

4. Close under composition.



Does every t.c. arise in this way?

EXAMPLES OF TWISTED COREFLECTIONS

$$\begin{tikzcd} \bullet \arrow[r, "u"] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, "u"] & \bullet \arrow[r, "v"] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, shift right, "r"'] & \bullet \arrow[l, shift right, "s"'] \arrow[r, "u"] & \bullet \arrow[r, "v"] & \bullet \end{tikzcd}$$

$rs = 1$

$$\begin{tikzcd} \bullet \arrow[r, shift right, "r"'] & \bullet \arrow[l, shift right, "s"'] \arrow[r, "u"] & \bullet \arrow[r, "v"] & \bullet \arrow[r, "w"] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[d, "t"'] & \bullet \arrow[l, "s"'] \arrow[r, "u"] & \bullet \arrow[r, "v"] & \bullet \\ \bullet \arrow[ur, "r"'] \end{tikzcd}$$

$rts = 1$

$$\begin{tikzcd} \bullet \arrow[r, shift right, "r_1"'] & \bullet \arrow[l, shift right, "s_1"'] \arrow[r, "u"] & \bullet \arrow[r, "v"] \arrow[d, shift right, "s_2"'] & \bullet \arrow[r, "w"] & \bullet \\ & & \bullet \arrow[u, shift right, "r_2"'] \end{tikzcd}$$

$r_n s_n = 1$

for $n = 1, 2$

$$\begin{tikzcd} \bullet \arrow[r, shift right, "r_1"'] & \bullet \arrow[l, shift right, "s_1"'] \arrow[r, shift left, "s_2"] \arrow[d, "u"] & \bullet \arrow[l, shift left, "r_2"] \arrow[r, shift left, "s_3"] & \bullet \arrow[l, shift left, "r_3"] \\ & \bullet \end{tikzcd}$$

$r_n s_n = 1$

for $n = 1, 2, 3$

LIFTING AGAINST DELTA LENSES (1)

$$qx \xrightarrow{qu} qy$$

$$\begin{tikzcd} hqx \arrow[r, "hqu"] \arrow[d, shift right, "\Psi(hqx, k\varepsilon_x)"'] & hqy \arrow[d, "\Psi(hqy, k\varepsilon_y)"] \\ jx \arrow[u, shift right, "\Psi(jx, k\tau u)"'] \arrow[r, "ju"'] & jy \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{(f,q,\varepsilon,\tau)}"'] & C \arrow[d, "{(g,\Psi)}"] \\ B \arrow[ur, "j"] \arrow[r, "k"'] & D \end{tikzcd}$$

$$\begin{tikzcd} fqx \arrow[r, "fqu"] \arrow[d, shift right, "\varepsilon_x"'] & fqy \arrow[d, "\varepsilon_y"] \\ x \arrow[u, shift right, "\tau u"'] \arrow[r, "u"] & y \end{tikzcd}$$

$$\begin{tikzcd} g(hqx) \arrow[d, equal] & g(hqy) \arrow[d, equal] \\ kfqx \arrow[d, shift right, "k\varepsilon_x"'] & kfqy \arrow[d, "k\varepsilon_y"] \\ kx \arrow[u, shift right, "k\tau u"'] & ky \end{tikzcd}$$

LIFTING AGAINST DELTA LENSES (2)

$$\begin{tikzcd} qx \arrow[r, "qu=1"] & qy \end{tikzcd}$$

$$\begin{tikzcd} hqx \arrow[r, equal] \arrow[d, "\psi(hqx,\, k\varepsilon_x)"'] & hqy \arrow[d, "\psi(hqy,\, k\varepsilon_y)"] \\ jx \arrow[r, "\psi(jx,\, ku)"'] & jy \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "(f,q,\varepsilon,\tau)"'] & C \arrow[d, "(g,\psi)"] \\ B \arrow[r, "k"'] \arrow[ur, "j"] & D \end{tikzcd}$$

$$\begin{tikzcd} fqx \arrow[r, equal] \arrow[d, "\varepsilon_x"'] & fqy \arrow[d, "\varepsilon_y"] \\ x \arrow[r, "u"] & y \end{tikzcd}$$

$$\begin{tikzcd} g(hqx) \arrow[d, equal] & g(hqy) \arrow[d, equal] \\ kfqx \arrow[r, equal] \arrow[d, "k\varepsilon_x"'] & kfqy \arrow[d, "k\varepsilon_y"] \\ kx \arrow[r, "ku"] & ky \end{tikzcd}$$

FACTORISATION, ABSTRACTLY

$$\begin{tikzcd}
A_0 \arrow[r, "\iota_A"] \arrow[d, swap, "\text{initial functor}"] & A \arrow[d, "Lf"] \arrow[dd, bend left=60, "f"] \\
Jf \arrow[r] \arrow[d, swap, "\text{discrete opfibration}"] & Ef \arrow[d, dashed, "Rf"] \arrow[ul, phantom, "\ulcorner", very near start] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

where $Jf = \sum_{a \in A_0} fa / B$

- We know that $Rf : Ef \to B$ is the (free) delta lens on the functor $f : A \to B$. (see arXiv:2305.02732)

- To show that $Lf : A \to Ef$ is a (cofree) twisted coreflection, we want an explicit characterisation of the pushout Ef.

Usually hard to do!

FACTORISATION, EXPLICITLY (1)

The category Ef (codomain of the cofree twisted coreflection) has:

- objects are pairs $(a \in A,\ u : fa \to b \in B)$
- morphisms are generated by the following:

$$
\begin{tikzcd}
a \arrow[r, equal] & a \\
fa \arrow[r, equal] \arrow[d, "u_1"'] & fa \arrow[d, "u_2"] \\
b_1 \arrow[r, "v"'] & b_2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a_1 \arrow[r, "w"] & a_2 \\
fa_1 \arrow[r, "fw"] \arrow[d, "1"'] & fa_2 \arrow[d, "1"] \\
fa_1 \arrow[r, "fw"'] & fa_2
\end{tikzcd}
$$

chosen lifts

morphisms of A

The functor $Lf : A \to Ef$ sends a morphism $w : a_1 \to a_2$ to the second generator.

FACTORISATION, EXPLICITLY (2)

The category Ef (codomain of the cofree twisted coreflection) has:

- objects are pairs $(a \in A,\ u : fa \to b \in B)$
- morphisms $(a_1, u_1) \longrightarrow (a_2, u_2)$ are given by the following two sorts:

$$\begin{tikzcd}
a_1 \arrow[r, equal] & a_2 \\
fa_1 \arrow[r, equal] \arrow[d, "u_1"'] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[r, "v"'] & b_2
\end{tikzcd}
\qquad
\begin{tikzcd}
a_1 \arrow[r, equal] & a_1 \arrow[r, "w"] & a_2 \arrow[r, equal] & a_2 \\
fa_1 \arrow[r, equal] \arrow[d, "u_1"'] & fa_1 \arrow[r, "fw"] \arrow[d, "1"'] & fa_2 \arrow[r, equal] \arrow[d, "1"] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[r, "v"'] & fa_1 \arrow[r, "fw"'] & fa_2 \arrow[r, "u_2"'] & b_2
\end{tikzcd}$$

The right adjoint $Ef \longrightarrow A$ sends these to 1 and w, respectively.

FACTORISATION, EXPLICITLY (3)

The category Ef (codomain of the cofree twisted coreflection) has:

- objects are pairs $(a \in A,\ u : fa \to b \in B)$
- morphisms $(a_1, u_1) \longrightarrow (a_2, u_2)$ are given by the following two sorts:

$$
\begin{tikzcd}
a_1 \arrow[r, equal] & a_2 \\
fa_1 \arrow[r, equal] \arrow[d, "u_1"'] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[r, "v"'] & b_2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a_1 \arrow[r, "w"] & a_2 \\
fa_1 \arrow[r, "fw"] \arrow[d, shift right, "u_1"'] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[u, shift right, "v"'] \arrow[r, "u_2 \circ fw \circ v"'] & b_2
\end{tikzcd}
\qquad
\boxed{v \circ u_1 = 1}
$$

COMPACT VERSION

The right adjoint $Ef \longrightarrow A$ sends these to 1 and w, respectively.

SUMMARY OF THE TALK SO FAR

For the classes

$\mathcal{L}$ = split coreflection / twisted coreflection

$\mathcal{R}$ = split opfibration / delta lens

we defined lifts of $\mathcal{L}$ against $\mathcal{R}$ and factorisations of functors into a $\mathcal{L}$ followed by a $\mathcal{R}$.

Questions:

- Are the classes $\mathcal{L}$ and $\mathcal{R}$ closed under composition?
- How can we show that $\mathcal{L}$ is the largest class which lift against $\mathcal{R}$?
- In what sense is the factorisations defined "universal"?

PART 3: DOUBLE CATEGORIES & LIFTING AWFS

DOUBLE CATEGORIES

A **double category** $\mathbb{D}$ consists of:

- objects
- horizontal morphisms
- vertical morphisms
- cells

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

+ unital & associative horizontal & vertical composition

- $\mathbb{D}$ is **thin** if cell = boundary.
- Example: For each category $\mathcal{C}$, the **double category of squares** $\mathbb{S}\mathrm{q}(\mathcal{C})$.
- $\mathbb{D}$ is **concrete** if thin & there is a double functor

$$\mathbb{D} \longrightarrow \mathbb{S}\mathrm{q}(\mathcal{C})$$

that is the identity on objects and horizontal morphisms.

THE DOUBLE CATEGORY OF DELTA LENSES

Delta lenses compose as follows:

$$\begin{array}{ccccc} A & & a & \xrightarrow{\varphi(a,\psi(fa,u))} & \bullet \\ \downarrow{\scriptstyle (f,\varphi)} & & \vdots & & \\ B & & fa & \xrightarrow{\psi(fa,u)} & \bullet \\ \downarrow{\scriptstyle (g,\psi)} & & \vdots & & \\ C & & gfa & \xrightarrow{u} & c \end{array}$$

Let $\mathbb{L}\mathrm{ens}$ denote the concrete double category of categories, functors, & delta lenses.

A cell with boundary

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{(f,\varphi)}"'] & C \arrow[d, "{(g,\psi)}"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

exists if $kf = gh$ & $h\varphi(a,u) = \psi(ha,ku)$.

$$\boxed{\mathbb{L}\mathrm{ens} \xrightarrow{V} \mathbb{S}\mathrm{q}(\mathcal{C}at)}$$

THE DOUBLE CATEGORY OF TWISTED COREFLECTIONS

Twisted coreflections compose:

$$\begin{tikzcd} A \arrow[d, shift right=2, "f"'] \arrow[d, phantom, "\dashv"] \\ B \arrow[u, shift right=2, "q"'] \arrow[d, shift right=2, "g"'] \arrow[d, phantom, "\dashv"] \\ C \arrow[u, shift right=2, "p"'] \end{tikzcd}
\qquad
\begin{tikzcd} gfqpx \arrow[r, "gfqpu"] \arrow[d, shift right=1, "g\varepsilon_{px}"'] & gfqpy \arrow[d, "g\varepsilon_{py}"] \\ gpx \arrow[r, "gpu"'] \arrow[u, shift right=1, "g\tau pu"'] \arrow[d, shift right=1, "\xi_x"'] & gpy \arrow[d, "\xi_y"] \\ x \arrow[r, "u"'] \arrow[u, shift right=1, "\sigma u"'] & y \end{tikzcd}$$

if $qpu \neq 1$, then $pu \neq 1$

$$\boxed{\mathbb{T}\mathrm{wCoRef} \xrightarrow{U} \mathbb{S}\mathrm{q}(\mathrm{Cat})}$$

Let $\mathbb{T}\mathrm{wCoRef}$ denote the concrete double cat. of categories, functors, & twisted coreflections. A cell with boundary

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{(f,q,\varepsilon,\tau)}"'] & C \arrow[d, "{(g,p,\xi,\sigma)}"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

exists if $kf = gh$, $hq = pk$, $k \cdot \varepsilon = \xi \cdot k$, and $k\tau u = \sigma k u$.

DOUBLE-CATEGORICAL LIFTING OPERATIONS

$$\mathbb{L} \xrightarrow{U} \mathbb{S}\mathrm{q}(\mathcal{C}) \xleftarrow{V} \mathbb{R}$$

A $(\mathbb{L},\mathbb{R})$-lifting operation is a family

$$\begin{tikzcd}
UA \arrow[r, "s"] \arrow[d, "Uj"'] & VC \arrow[d, "Vk"] \\
UB \arrow[r, "t"'] \arrow[ur, "\varphi_{j,k}(s,t)"] & VD
\end{tikzcd}$$

which satisfies certain horizontal and vertical compatibilities.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "U_i"'] & \bullet \arrow[r] \arrow[d, "U_j"'] & \bullet \arrow[d, "V_k"] \\
\bullet \arrow[r] & \bullet \arrow[r] \arrow[ur, "\varphi_{j,k}"] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "U_i"'] & \bullet \arrow[d, "V_k"] \\
\bullet \arrow[r] \arrow[ur, "\varphi_{i,k}"] & \bullet
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "U_i"'] & \bullet \arrow[dd, "V_k"] \\
\bullet \arrow[d, "U_j"'] \arrow[ur, "\varphi_{i,k}"] & \\
\bullet \arrow[r] \arrow[uur, "\varphi_{j,k}"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "U_i"'] & \bullet \arrow[dd, "V_k"] \\
\bullet \arrow[d, "U_j"'] & \\
\bullet \arrow[r] \arrow[uur, "\varphi_{ji,k}"'] & \bullet
\end{tikzcd}$$

+ dual compatibilities on right.

THE DOUBLE CATEGORY $\mathbb{RLP}(\mathbb{J})$

For each double functor $\mathbb{J} \xrightarrow{u} \mathbb{S}q(\mathcal{C})$

there is a concrete double cat.

$$\mathbb{RLP}(\mathbb{J}) \longrightarrow \mathbb{S}q(\mathcal{C})$$

whose:

- objects & horizontal mor. are from $\mathcal{C}$
- vertical mor. are pairs (f,φ) where

$$\begin{tikzcd} UA \arrow[r, "s"] \arrow[d, "Ui"'] & C \arrow[d, "f"] \\ UB \arrow[r, "t"'] \arrow[ur, "\varphi_i(s,t)"] & D \end{tikzcd}$$

f is morphism in $\mathcal{C}$

φ is a $(\mathbb{J},f)$-lifting operation

- cells $(f,\varphi) \longrightarrow (g,\psi)$ are given by:

$$\begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d, "Ui"'] & \cdot \arrow[r, "h"] \arrow[d, "f"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "t"'] \arrow[ur, "\varphi_i"] & \cdot \arrow[r, "k"'] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, "hs"] \arrow[d, "Ui"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "kt"'] \arrow[ur, "\psi_i"] & \cdot \end{tikzcd}$$

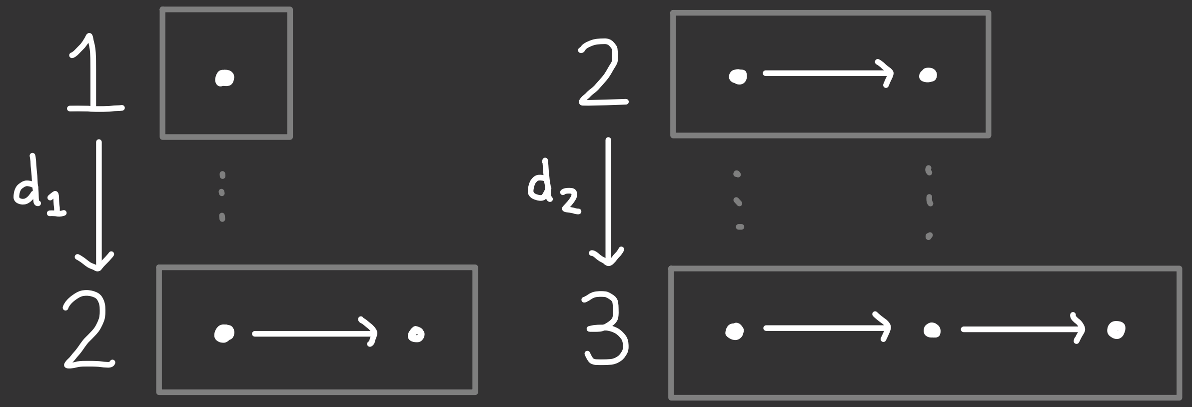
If $\mathbb{D} \cong \mathbb{RLP}(\mathbb{J})$, we say that $\mathbb{D}$ is **cofibrantly generated** by $\mathbb{J}$.

We can also define $\mathbb{LLP}(\mathbb{J})$ in a dual way.

CHARACTERISING DELTA LENSES VIA LIFTS

Consider the double cat $\mathbb{J}$ whose:

- objects are the ordinals $1, 2, 3$
- horizontal morphisms are all order-preserving maps
- vertical morphisms generated by



- cells are generated by

$$\begin{tikzcd} 1 \arrow[r, "id"] \arrow[d, "d_1"'] & 1 \arrow[d, "id"] \\ 2 \arrow[r, "!"'] & 1 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "d_0"] \arrow[d, "d_1"'] & 2 \arrow[d, "d_2"] \\ 2 \arrow[r, "d_0"'] & 3 \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] \arrow[dd, "d_1"'] & 1 \arrow[d, "d_1"] \\ & 2 \arrow[d, "d_2"] \\ 2 \arrow[r, "d_1"'] & 3 \end{tikzcd}$$

<u>Theorem</u>:

$$\mathbb{L}\mathrm{ens} \cong \mathbb{RLP}(\mathbb{J})$$

(Note: $\mathbb{J} \hookrightarrow \mathbb{T}\mathrm{wCoRef}$)

MAIN THEOREM

(Bourke, 2023): A lifting awfs is an $(\mathbb{L}, \mathbb{R})$-lifting operation φ where

$$\mathbb{L} \xrightarrow{U} \mathbb{S}q(\mathcal{C}) \xleftarrow{V} \mathbb{R}$$

such that the following axioms hold:

(i) induced double functors are iso

$$\mathbb{L} \longrightarrow \mathbb{LLP}(\mathbb{R}) \qquad \mathbb{R} \longrightarrow \mathbb{RLP}(\mathbb{L})$$

(ii) Each f in $\mathcal{C}$ admits a factorisation

$$\cdot \xrightarrow{U_1 g} \cdot \xrightarrow{V_1 h} \cdot \;=\; \cdot \xrightarrow{f} \cdot$$

which is U_1-couniversal & V_1-universal.

Theorem: There is a lifting awfs:

$$\mathbb{T}\mathrm{w}\mathbb{C}\mathrm{o}\mathbb{R}\mathrm{ef} \xrightarrow{U} \mathbb{S}q(\mathcal{C}at) \xleftarrow{V} \mathbb{L}\mathrm{ens}$$

- The lifts of twisted coreflections against delta lenses are compatible with horizontal/vertical composition.
- Delta lenses are determined by lifts against twisted coreflections.
- Every functor factorises into cofree twisted coreflection & free delta lens.

PART 4: CONCLUDING REMARKS

CONSEQUENCES & COROLLARIES

- Split coreflections & split opfibrations form a lifting awfs.

- Exists a morphism of lifting awfs:

$$\begin{tikzcd}
\mathbb{T}\mathrm{w}\mathbb{C}\mathrm{o}\mathbb{R}\mathrm{ef} \arrow[dr] \arrow[dd, hook] & & \mathbb{L}\mathrm{ens} \arrow[dl] \\
& \mathbb{S}\mathrm{q}(\mathbb{C}\mathrm{at}) & \\
\mathbb{C}\mathrm{o}\mathbb{R}\mathrm{ef} \arrow[ur] & & \mathbb{S}\mathbb{O}\mathrm{pf} \arrow[ul] \arrow[uu, hook]
\end{tikzcd}$$

- Delta lenses are algebras for a monad on $\mathrm{Sq}(\mathcal{C}at)$, and are stable under pullback.

- Twisted coreflections are coalgebras for a comonad on $\mathrm{Sq}(\mathcal{C}at)$, and are stable under pushouts.

- $\mathbb{J} \hookrightarrow \mathbb{T}\mathrm{w}\mathbb{C}\mathrm{o}\mathbb{R}\mathrm{ef} \cong \mathbb{LLP}(\mathbb{RLP}(\mathbb{J}))$

SUMMARY & FUTURE WORK

- We introduced twisted coreflections and showed that with delta lenses they form a lifting awfs.

- Showed that delta lenses are cofibrantly generated by small $\mathbb{J}$.

- Developed a better understanding on the similarities & differences between delta lenses & split opfibrations.

<u>Conjecture</u>: Twisted coreflections are precisely pushouts of the form:

$$\begin{tikzcd}
A_0 \arrow[r, "\iota_A"] \arrow[d, hook, shift right=2] \arrow[d, phantom, "\dashv"] & A \arrow[d, hook, shift right=2] \arrow[d, phantom, "\dashv"] \\
X \arrow[r] \arrow[u, shift right=2] \arrow[ur, phantom, very near end, "\ulcorner"] & B \arrow[u, shift right=2]
\end{tikzcd}$$

<u>Question</u>: How can we extend the notion of $(\mathbb{L},\mathbb{R})$-lifting operation to capture retrofunctors/cofunctors?