THE ALGEBRAIC WEAK FACTORISATION SYSTEM FOR DELTA LENSES

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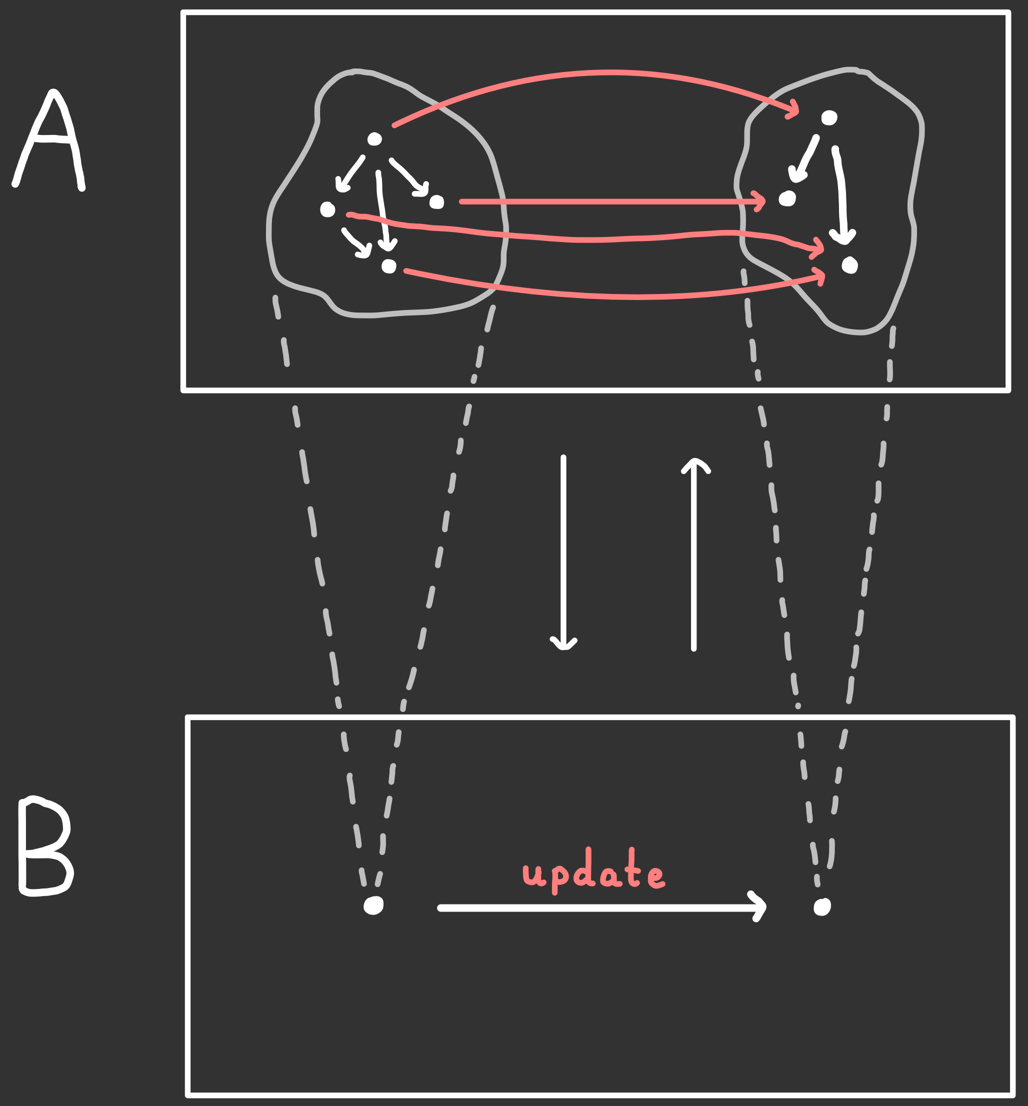
THE SETTING

A system is understood as a category whose:

- objects are the states;
- morphisms are the updates/transitions.

A bidirectional transformation between systems is modelled using a (delta) lens which consists of:

- a forwards component which determines when states are consistent;
- a backwards component which restores consistency when an update occurs.



HISTORY & CONTEXT

- 2005: The term lens first appears in the literature.
- 2006: Grandis & Tholen define algebraic weak factorisation systems.
- 2010: State-based lenses are shown to be algebras for a monad on Set/B.
- 2011: Diskin, Xiong, & Czarnecki introduce delta lenses.
- 2012: C-lenses (a.k.a. split opfibrations) defined as algebras for a monad on Cat/B.
- 2013: Delta lenses are shown to be certain algebras for a semi-monad on Cat/B.

MOTIVATION

Delta lenses are certain algebras for a semi-monad

PROBLEM

How do we sequentially compose delta lenses?

PROBLEM

Can we factorise through the free delta lens?

PROBLEM

Where does the lifting of a delta lens come from?

OVERVIEW

$$\begin{tikzcd} Ef \arrow[rr, "p"] \arrow[dr, "Rf"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

free lens

Algebras for a monad

$$\begin{tikzcd} A \arrow[r, "\shortmid" marking] \arrow[rr, bend right, "\shortmid" marking] & B \arrow[r, "\shortmid" marking] & C \end{tikzcd}$$

Composition

AWFS

Factorisation

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "Lf"'] & & B \\ & Ef \arrow[ur, "Rf"'] & \end{tikzcd}$$

free lens

Lifting

$$\begin{tikzcd} A \arrow[r] \arrow[d, "???"'] & C \arrow[d, "\text{lens}"] \\ B \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

STRUCTURE OF THE TALK

1. Delta Lenses
2. Algebraic Weak Factorisation Systems
3. The A.W.F.S. For Delta Lenses
4. Concluding Remarks

1. Delta Lenses

DEFINING DELTA LENSES

A delta lens (f,φ) is a functor $f\colon A \to B$ equipped with a lifting operation φ

$$\begin{tikzcd} A \arrow[d, "f"'] & a \arrow[r, "\varphi(a,u)"] & p(a,u) \\ B & fa \arrow[r, "u"] & b \end{tikzcd}$$

which satisfies the following axioms:

1. $f\varphi(a,u) = u$
2. $\varphi(a, 1_{fa}) = 1_a$
3. $\varphi(a, v \circ u) = \varphi(p(a,u), v) \circ \varphi(a,u)$

Let $\mathcal{L}ens$ be the category whose objects are delta lenses and whose morphisms

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{(f,\varphi)}"'] & C \arrow[d, "{(g,\psi)}"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

are pairs of functors $\langle h,k \rangle$ such that

$k \circ f = g \circ h$ and $h\varphi(a,u) = \psi(ha, ku)$.

<u>GOAL</u>: Show that $\mathcal{U}$ is monadic.

$$\mathcal{U}\colon \mathcal{L}ens \longrightarrow \mathcal{C}at^2$$

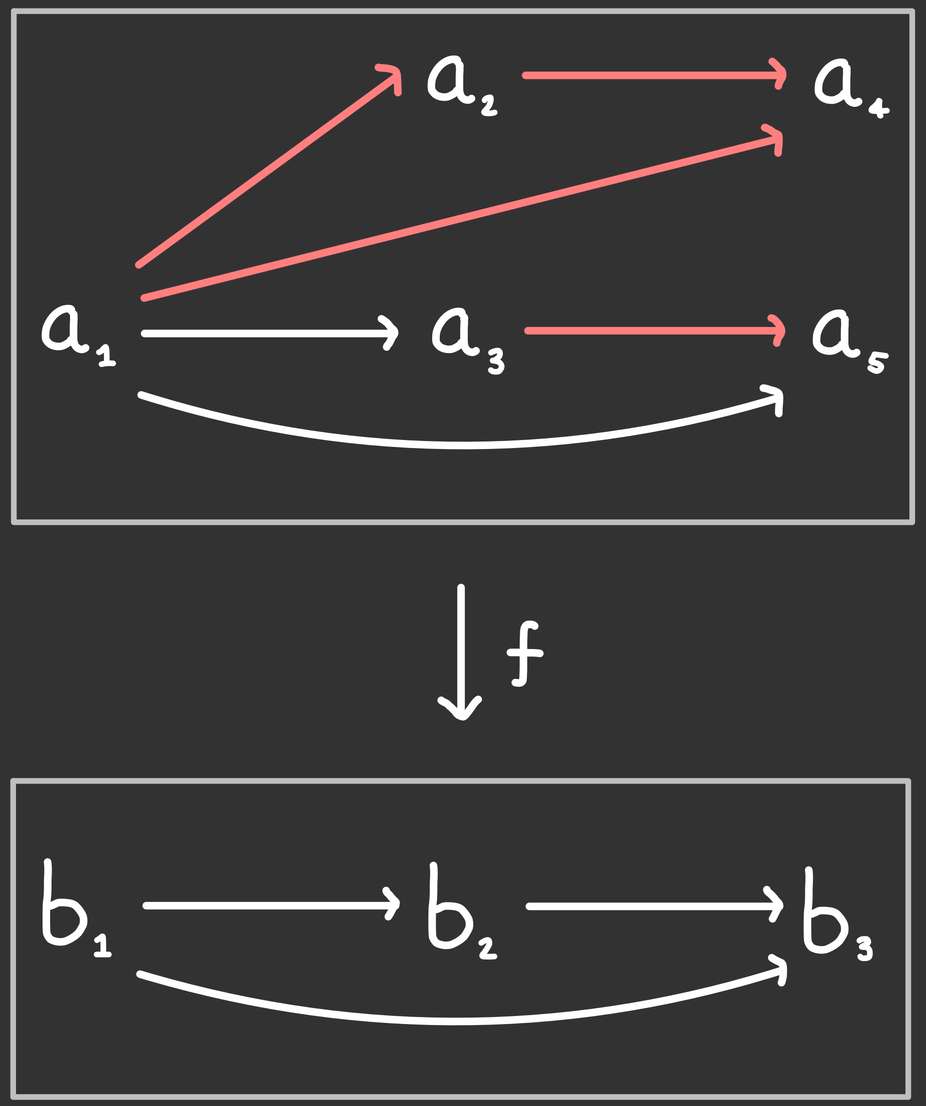
BASIC EXAMPLES

- State-based lenses are delta lenses between codiscrete categories.

$$f : A \longrightarrow B \qquad p : A \times B \longrightarrow A$$

- Discrete opfibrations are delta lenses such that $\varphi(a, fw) = w$.

- Split opfibrations are delta lenses such that the chosen lifts $\varphi(a, u)$ are opcartesian.



CERTAIN ALGEBRAS FOR A SEMI-MONAD

Given a functor $f\colon A \to B$ we define

$$Jf := \sum_{a \in A_0} {}^{fa}/B$$

the category whose:

- objects are $(a \in A,\ u\colon fa \to b \in B)$
- morphisms are commuting triangles:

$$\begin{tikzcd} a_1 \arrow[r, equal] & a_2 \\ fa_1 \arrow[r, equal] \arrow[d, "u_1"'] & fa_2 \arrow[d, "u_2"] \\ b_1 \arrow[r, "v"] & b_2 \end{tikzcd}$$

- Delta lens $Tf\colon Jf \to B$ given by codomain projection $(a, u) \mapsto \mathrm{cod}(u)$.
- The assignment $f \mapsto Tf$ defines a semi-monad (T, ν) on $\mathcal{C}at^2$.
- Delta lenses are certain algebras:

$$\begin{tikzcd} a \arrow[d, mapsto] & A_0 \arrow[r, "\eta_A"] \arrow[d] & A \arrow[d, "f"] \\ (a, 1_{fa}) & Jf \arrow[r, "Tf"'] \arrow[ur, "p"] & B \end{tikzcd}$$

$+$ multiplication law

2. Algebraic Weak Factorisation Systems

ORTHOGONAL FACTORISATION SYSTEMS

An OFS on a category $\mathcal{C}$ consists of two classes of morphisms $\mathcal{L}$ and $\mathcal{R}$, containing the isomorphisms & closed under composition, such that:

Factorisation:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\ell \in \mathcal{L}"'] & & B \\ & \bullet \arrow[ur, "r \in \mathcal{R}"'] & \end{tikzcd}$$

Orthogonality:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\ell \in \mathcal{L}"'] & C \arrow[d, "r \in \mathcal{R}"] \\ B \arrow[r, "k"'] \arrow[ur, dashed, "\exists ! \, j"] & D \end{tikzcd}$$

<u>Example</u>: The comprehensive factorisation system on $\mathbf{Cat}$ has left class the intial functors and right class the discrete opfibrations.

ALGEBRAIC WEAK FACTORISATION SYSTEMS

An algebraic weak factorisation system (AWFS) on a category $\mathcal{C}$ consists of:

- A functorial factorisation on $\mathcal{C}$;

$$\begin{tikzcd}
A \arrow[rrr, "h"] \arrow[ddd, "f"'] \arrow[dr, "Lf"] & & & C \arrow[dl, "Lg"'] \arrow[ddd, "g"] \\
& Ef \arrow[r, "E\langle h,k\rangle"] \arrow[ddl, "Rf"] & Eg \arrow[ddr, "Rg"] & \\
& & & \\
B \arrow[rrr, "k"'] & & & D
\end{tikzcd}$$

L-coalgebras & R-algebras replace the left & right class of morphisms

If L and R are idempotent, then OFS

- A comonad (L, ε, Δ) and a monad (R, η, μ) on $\mathcal{C}^2$;
- A distributative law $\delta : LR \Rightarrow RL$ of the comonad L over the monad R.

LIFTING L-COALGEBRAS AGAINST R-ALGEBRAS

Given an AWFS (L,R) on $\mathcal{C}$, consider the L-coalgebra (f,s) and R-algebra (g,p).

$$\begin{tikzcd} A \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[d, "Rf"] \\ B \arrow[ur, "s"] \arrow[r, equal] & B \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, "Lg"'] & C \arrow[d, "g"] \\ Eg \arrow[ur, "p"] \arrow[r, "Rg"'] & D \end{tikzcd}$$

Note that

$$\varphi(Lf, 1_B) = s$$

and

$$\varphi(1_C, Rg) = p$$

Construct a canonical diagonal filler, or lift, for each square:

$$\begin{tikzcd} A \arrow[rrr, "h"] \arrow[ddd, "f"'] \arrow[dr, "Lf"] & & & C \arrow[ddd, "g"] \arrow[dl, shift right, "Lg"'] \\ & Ef \arrow[r, "E\langle h,k\rangle"] \arrow[ddl, shift left, "Rf"] & Eg \arrow[ur, shift right, "p"'] \arrow[ddr, "Rg"'] & \\ & & & \\ B \arrow[uur, shift left, "s"] \arrow[rrr, "k"'] & & & D \end{tikzcd}$$

$$\varphi(h,k) = p \circ E\langle h,k\rangle \circ s : B \longrightarrow C.$$

COMPOSING R-ALGEBRAS VIA LIFTING

Cofree coalgebra

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[dd, "Lh"'] \arrow[dr, "f"] & A \arrow[d, "f"] \arrow[dd, bend left, "h"] \\
 & B \arrow[d, "g"] \\
Eh \arrow[r, "Rh"'] \arrow[ur, "\varphi(f,Rh)"'] \arrow[uur, "\varphi(1_A,\varphi(f,Rh))"] & C
\end{tikzcd}$$

Suppose we have R-algebras (f,p) and (g,r) as above.

How do we compose these R-algebras?

We construct lifts in two steps.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "Lh"'] & B \arrow[d, "g"] \\
Eh \arrow[r, "Rh"'] \arrow[ur, "\varphi(f,Rh)"'] & C
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "Lh"'] & A \arrow[d, "f"] \\
Eh \arrow[r, "\varphi(f,Rh)"'] \arrow[ur, "\varphi(1_A,\varphi(f,Rh))"] & B
\end{tikzcd}$$

EXAMPLES OF LENSES AS R-ALGEBRAS

There is an AWFS on **Set** which factors a function through the product:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\langle 1_A, f\rangle"'] & & B \\ & A \times B \arrow[ur, "\pi_B"'] & \end{tikzcd}$$

The R-algebras are **state-based lenses**.

- The L-coalgebras are **split monos**.
- Generalises to any category $\mathcal{C}$ with **finite products**.

There is an AWFS on **Cat** which factors a functor through the comma category:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\langle 1_A, f\rangle"'] & & B \\ & f/1_B \arrow[ur, "\pi_B"'] & \end{tikzcd}$$

The R-algebras are **split opfibrations**. (c-lenses)

- The L-coalgebras are **LALIs**.
- Generalises to any 2-category $\mathcal{K}$ with **comma objects**.

3. The A.W.F.S. For Delta Lenses

THE FREE DELTA LENS (1)

The free delta lens $Rf : Ef \longrightarrow B$ on a functor $f : A \longrightarrow B$ has domain whose:

- objects are pairs $(a \in A,\ u : fa \longrightarrow b \in B)$
- morphisms are generated by the following:

chosen lifts

$$\begin{tikzcd} a_1 \arrow[rr, equal] & & a_2 \\ fa_1 \arrow[rr, equal] \arrow[d, "u_1"'] & & fa_2 \arrow[d, "u_2"] \\ b_1 \arrow[rr, "v"] & & b_2 \end{tikzcd} \qquad \begin{tikzcd} a_1 \arrow[rr, "w"] & & a_2 \\ fa_1 \arrow[rr, "fw"] \arrow[d, "1"'] & & fa_2 \arrow[d, "1"] \\ fa_1 \arrow[rr, "fw"'] & & fa_2 \end{tikzcd}$$

morphisms of A

The functor Rf sends these generators to $v : b_1 \longrightarrow b_2$ and $fw : fa_1 \longrightarrow fa_2$, respectively.

The (fully faithful) functor Lf sends a morphism $w : a_1 \to a_2$ to the 2nd generator.

THE FREE DELTA LENS (2)

The **free delta lens** $Rf : Ef \longrightarrow B$ on a functor $f : A \longrightarrow B$ has domain whose:

- objects are pairs $(a \in A,\ u : fa \longrightarrow b \in B)$
- morphisms $(a_1, u_1) \longrightarrow (a_2, u_2)$ are given by the following two sorts:

$$
\begin{tikzcd}
a_1 \arrow[r, equal] & a_2 \\
fa_1 \arrow[r, equal] \arrow[d, "u_1"'] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[r, "v"] & b_2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a_1 \arrow[r, equal] & a_1 \arrow[r, "w"] & a_2 \arrow[r, equal] & a_2 \\
fa_1 \arrow[r, equal] \arrow[d, "u_1"'] & fa_1 \arrow[r, "fw"] \arrow[d, "1"'] & fa_2 \arrow[r, equal] \arrow[d, "1"] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[r, "v"] & fa_1 \arrow[r, "fw"] & fa_2 \arrow[r, "u_2"] & b_2
\end{tikzcd}
$$

The functor **Rf** sends these to $v : b_1 \longrightarrow b_2$ and $u_2 \circ fw \circ v : b_1 \longrightarrow b_2$, respectively.

THE FREE DELTA LENS (3)

The free delta lens $Rf : Ef \longrightarrow B$ on a functor $f : A \longrightarrow B$ has domain whose:

- objects are pairs $(a \in A,\ u : fa \longrightarrow b \in B)$
- morphisms $(a_1, u_1) \longrightarrow (a_2, u_2)$ are given by the following two sorts:

$$\begin{tikzcd}
a_1 \arrow[r, equal] & a_2 \\
fa_1 \arrow[r, equal] \arrow[d, "u_1"'] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[r, "v"] & b_2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a_1 \arrow[r, "w"] & a_2 \\
fa_1 \arrow[r, "fw"] \arrow[d, shift right, "u_1"'] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[u, shift right, "v"'] \arrow[r, "u_2 \circ fw \circ v"'] & b_2
\end{tikzcd}
\qquad
\boxed{v \circ u_1 = 1}$$

COMPACT VERSION

The functor Rf sends these to $v : b_1 \longrightarrow b_2$ and $u_2 \circ fw \circ v : b_1 \longrightarrow b_2$, respectively.

THE AWFS FOR DELTA LENSES

FACTORISATION

initial functor

$$\begin{tikzcd} A_0 \arrow[r, "\iota_A"] \arrow[d] & A \arrow[d, "Lf"] \arrow[dd, bend left, "f"] \\ \sum_{a \in A_0} fa/B = Jf \arrow[r] \arrow[d, "Tf"] \arrow[ur, phantom, "\ulcorner"] & Ef \arrow[d, dashed, "Rf"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

discrete opfibration

Generalises to category $\mathcal{C}$ with pushouts, an orthogonal factorisation system, and suitable idempotent comonad.

ALGEBRAS FOR A MONAD

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "Lf"'] & A \arrow[d, "f"] \\ Ef \arrow[r, "Rf"'] \arrow[ur, "\hat{p}"] & B \end{tikzcd} \simeq \begin{tikzcd} A_0 \arrow[r, "\iota_A"] \arrow[d] & A \arrow[d, "f"] \\ Jf \arrow[r, "Tf"'] \arrow[ur, "p"] & B \end{tikzcd}$$

IDEA: To obtain the lifting operation φ

$$\begin{tikzcd} a \arrow[r, equal] & a \\ fa \arrow[r, equal] \arrow[d, "1"'] & fa \arrow[d, "u"] \\ fa \arrow[r, "u"'] & b \end{tikzcd} \overset{p}{\longmapsto} a \xrightarrow{\varphi(a,u)} p(a,u)$$

L-COALGEBRAS & LIFTING

A L-coalgebra is an adjunction

$$\begin{tikzcd} A \arrow[r, hook, shift right, "f"'] \arrow[r, phantom, "\top"] & B \arrow[l, shift right, "q"'] \end{tikzcd} \qquad \begin{array}{l} q \circ f = id_A \\ \varepsilon : f \circ q \Rightarrow id_B \end{array}$$

such that if $q(u : b_1 \to b_2) \neq 1$, there is a specified $\bar{q}u : b_1 \to fqb_1$ such that:

$$\boxed{\begin{array}{r} \bar{q}u \circ \varepsilon_{b_1} = 1 \\ \varepsilon_{b_2} \circ fqu \circ \bar{q}u = u \end{array}}$$

$$\begin{tikzcd} fqb_1 \arrow[r, "fqu"] \arrow[d, shift right, "\varepsilon_{b_1}"'] & fqb_2 \arrow[d, "\varepsilon_{b_2}"] \\ b_1 \arrow[r, "u"'] \arrow[u, shift right, "\bar{q}u"'] & b_2 \end{tikzcd}$$

L-coalgebra

$$\begin{tikzcd} \{0\} \arrow[r, "a"] \arrow[d, hook] & A \arrow[d, "f"] \\ \{0 \to 1\} \arrow[r, "u"'] \arrow[ur, "\varphi(a,u)"] & B \end{tikzcd}$$

delta lens

<u>IDEA</u>: A L-coalgebra is the most general structure that a delta lens can lift.

L-coalgebra

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, hook, "{(f,q)}"'] & C \arrow[d, "{(g,\varphi)}"] \\ B \arrow[r, "k"'] \arrow[ur, "j"] & D \end{tikzcd}$$

delta lens

COMPOSITION

Delta lenses compose as follows:

$$\begin{tikzcd}
A \arrow[d, "{(f,\varphi)}"'] & a \arrow[r, "{\varphi(a,\psi(fa,u))}"] \arrow[d, dotted, no head] & \bullet \\
B \arrow[d, "{(g,\psi)}"'] & fa \arrow[r, "{\psi(fa,u)}"] \arrow[d, dotted, no head] & \bullet \\
C & gfa \arrow[r, "u"] & c
\end{tikzcd}$$

Can obtain same formula via AWFS.

$$\begin{tikzcd}
\{0\} \arrow[r, "a"] \arrow[dd, hook] \arrow[dr, "fa"] & A \arrow[d, "f"] \arrow[dd, bend left, "h"] \\
& B \arrow[d, "g"] \\
\{0 \to 1\} \arrow[uur, "{\varphi(a,\psi(fa,u))}"] \arrow[ur, "{\psi(fa,u)}"'] \arrow[r, "u"'] & C
\end{tikzcd}$$

Considering delta lenses as R-algebras, obtain a unitial and associative composition using the AWFS.

4. Concluding Remarks

A DOUBLE-CATEGORICAL STORY

A **double category** $\mathbb{D}$ consists of:

- objects $A, B, C, D, \ldots$
- horizontal morphisms $\bullet \longrightarrow \bullet$
- vertical morphisms $\bullet \nrightarrow \bullet$
- cells

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\
B \arrow[r, "k"'] & D
\end{tikzcd}
$$

that compose horizontally & vertically.

There is a double category $\mathbb{L}\mathrm{ens}$ of:

- categories
- functors $\Big\}$ $\mathcal{C}at$
- delta lenses
- morphisms of delta lenses $\Big\}$ $\mathcal{L}ens$

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "{(f, \varphi)}"', "\shortmid" marking] & C \arrow[d, "{(g, \psi)}", "\shortmid" marking] \\
B \arrow[r, "k"'] & D
\end{tikzcd}
$$

Delta lenses are **objects** <u>and</u> **morphisms**.

FUTURE WORK

Lots of examples of "lawful" lenses arising as R-algebras for an AWFS:

- state-based lenses
- discrete opfibrations
- split opfibrations (c-lenses)
- delta lenses

$\boxed{\text{Do } \underline{\text{all}} \text{ examples fit into this framework?}}$

free delta lens

$$\begin{tikzcd}
\mathcal{L}ens \arrow[dr, "cod"'] \arrow[rr, shift right=2, "U"'] \arrow[rr, phantom, "\perp"] & & \mathcal{C}at^2 \arrow[ll, shift right=2, "R"'] \arrow[dl, "cod"] \\
& \mathcal{C}at &
\end{tikzcd}$$

We have shown that U is monadic.

- What is the Kleisli category?
- What are the RU-coalgebras?
- What are the opcartesian lifts?

SUMMARY OF THE TALK

Delta lenses are algebras for a monad

$\mathcal{L}ens \xrightarrow{\text{monadic}} \mathcal{C}at^2$

Refines the semi-monad approach

Double category of delta lenses

Composition of delta lenses as R-algebras is canonically induced

AWFS for delta lenses

Useful perspective on backwards part of lens

Every functor factors through a free delta lens

New method for constructing examples

Lifts of delta lenses against L-coalgebras are canonically induced