THE RIGHT-CONNECTED COMPLETION OF A DOUBLE CATEGORY

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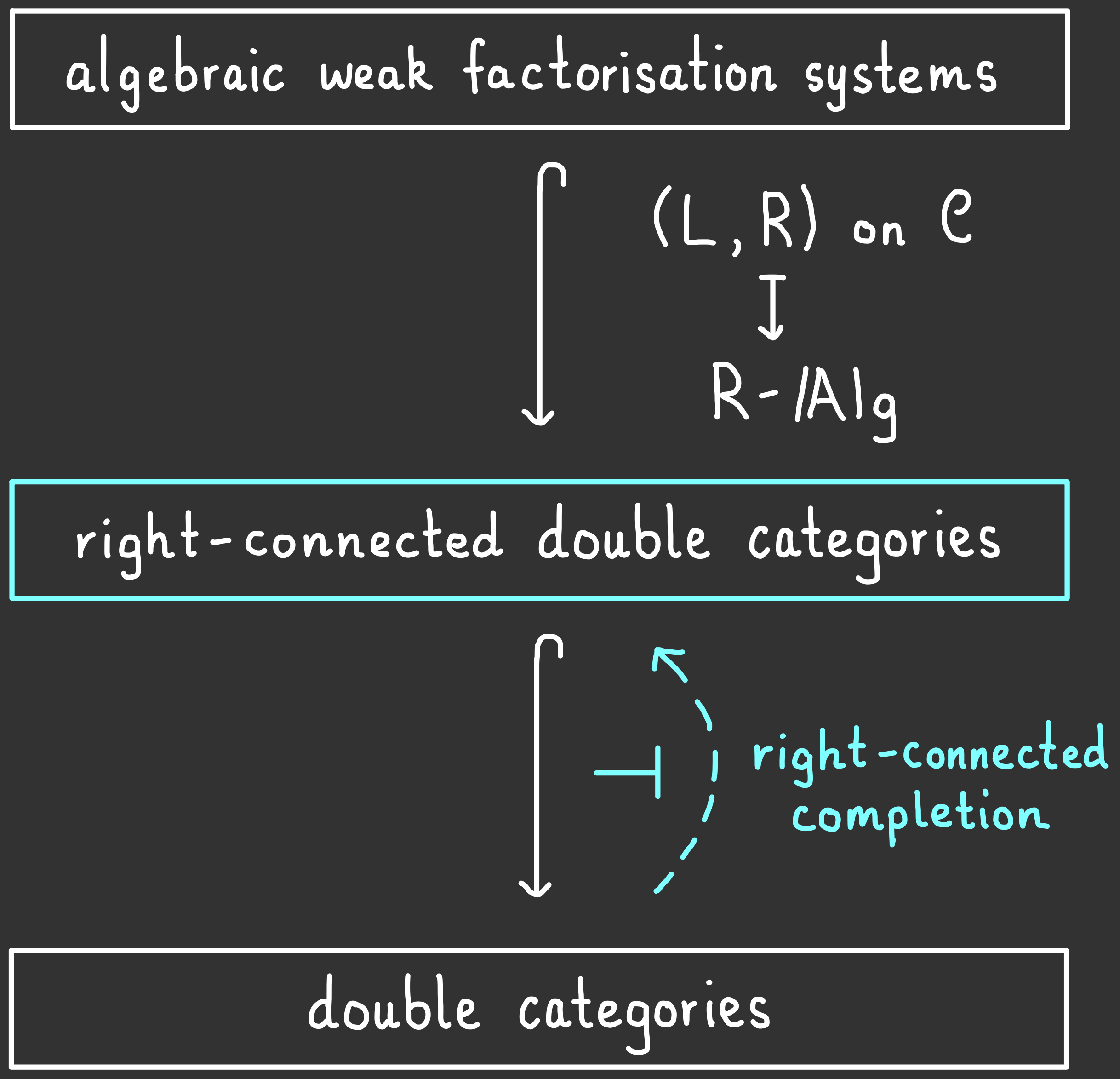
MOTIVATION & OVERVIEW



- A double category arises from AWFS if:
  - it is right-connected
  - satisfies a monadicity condition.

- Question: Can we construct an AWFS from a double category?

OUTLINE OF THE TALK

1. Three approaches to the R.C.C.
2. Examples + (co)monadicity conditions

DOUBLE CATEGORIES

A **double category** consists of:

- objects $A, B, C, D, \ldots$
- horizontal morphisms $\bullet \longrightarrow \bullet$
- vertical morphisms $\bullet \longrightarrow\!\!\!\!\!\!\!+\;\;\; \bullet$
- cells

$$\begin{array}{ccc} A & \xrightarrow{\;h\;} & C \\ f \big\downarrow & \alpha & \big\downarrow g \\ B & \xrightarrow[\;k\;]{} & D \end{array}$$

\+ identities & composition

A **double category** $\mathbb{D}$ is an internal category in the 2-category $\mathbf{CAT}$.

$$\mathbb{D}_0 \underset{\text{cod}}{\overset{\text{dom}}{\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{\text{id}}}}}} \mathbb{D}_1 \xleftarrow{\;\text{comp}\;} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$$

The **nerve** of a double category

$$\mathbf{DBL} \overset{N}{\hookrightarrow} [\Delta^{op}, \mathbf{CAT}]$$

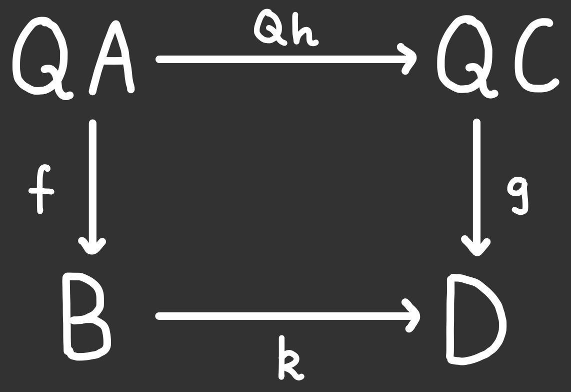
is 2-functor $N_{\mathbb{D}} \cong \mathbf{DBL}(\mathbb{V}(-), \mathbb{D})$ where:

$$\Delta \hookrightarrow \mathbf{CAT}_{\ell d} \xrightarrow{\;\mathbb{V}\;} \mathbf{DBL}$$

TWO RUNNING EXAMPLES

$\mathbb{K}\ell(\mathcal{C},Q)$ for comonad Q on $\mathcal{C}$.

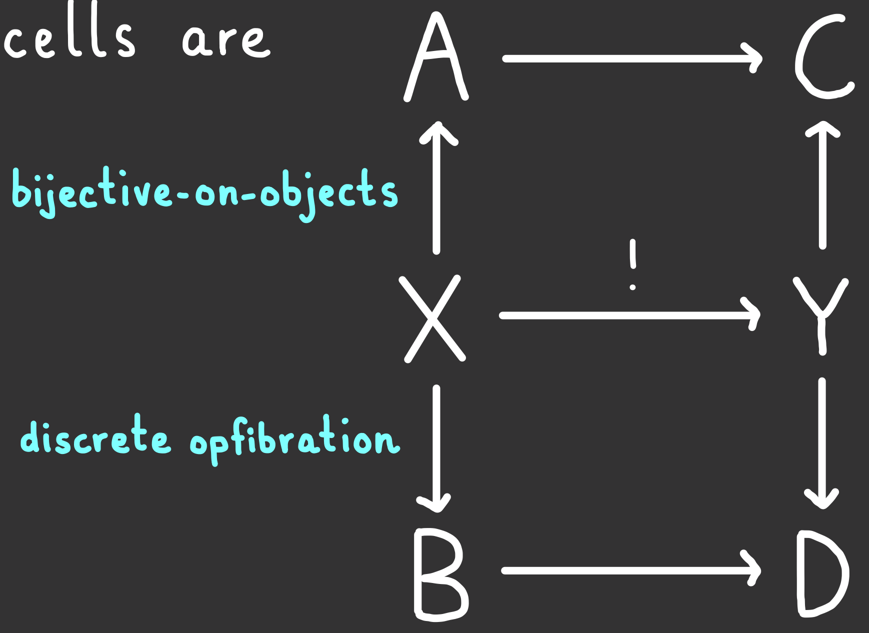
- objects are objects of $\mathcal{C}$
- horizontal morphisms are morphisms of $\mathcal{C}$
- vertical morphisms are cokleisli maps
- cells are commuting squares in $\mathcal{C}$



Interested in $\mathbb{K}\ell(\mathcal{C},Q)^{\vee}$; also $\mathbb{K}\ell(\mathcal{C},T)$.

$\mathbb{R}et$ (or $\mathbb{C}of$) (a.k.a. cofunctors)

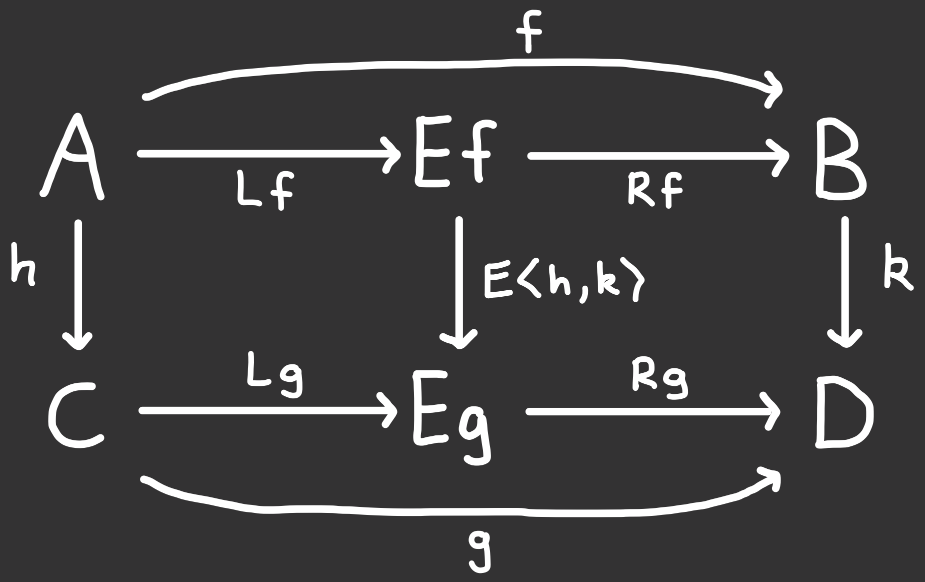
- objects are categories
- horizontal morphisms are functors
- vertical morphisms are retrofunctors
- cells are



ALGEBRAIC WEAK FACTORISATION SYSTEMS

An AWFS on a category $\mathcal{C}$ consists of:

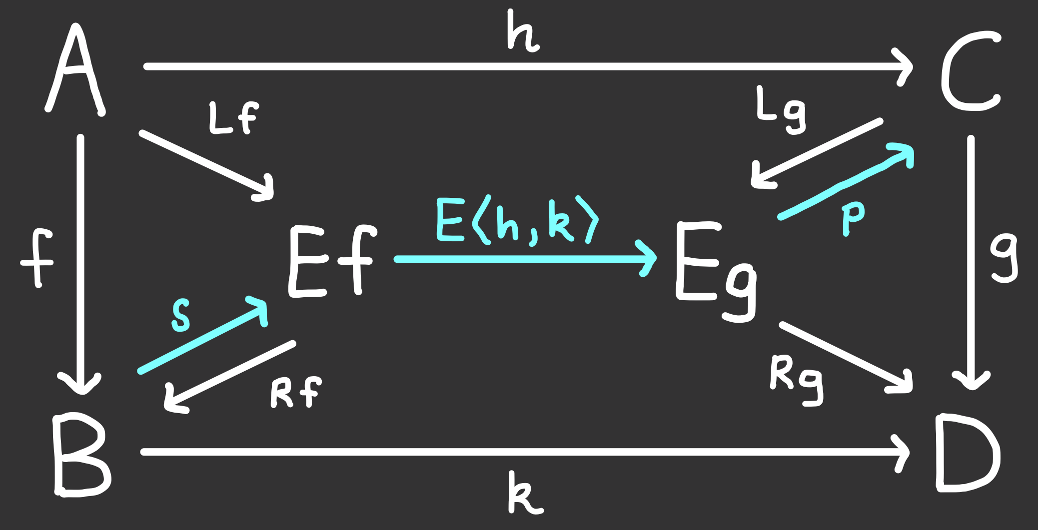
- a functorial factorisation (L, E, R)



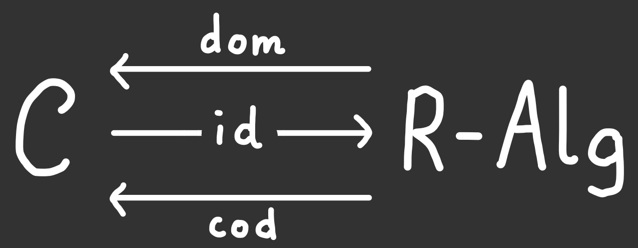
- comonad L & monad R on $\mathcal{C}^2$ + dist. law

If L and R are idempotent, then OFS on $\mathcal{C}$.

Lifts of L-coalgebras (f,s) against R-algebras (g,p):



We can use this to compose R-algebras, and define a double category R-𝔸lg:



RIGHT-CONNECTED DOUBLE CATEGORIES

A double cat. is **right-connected** if:

$$\mathbb{D}_0 \;\underset{\mathrm{cod}}{\overset{\mathrm{id}}{\rightleftarrows}}\; \mathbb{D}_1 \qquad (\mathrm{id} \dashv \mathrm{cod})$$

The unit ρ has components:

$$\begin{array}{c} A \\ f \downarrow \\ B \end{array} \quad \longmapsto \quad \begin{array}{ccc} A & \xrightarrow{Uf} & B \\ f \downarrow & \rho_f & \downarrow \mathrm{id}_B \\ B & \xrightarrow[1_B]{} & B \end{array}$$

Idea: Uf is the **underlying** hor. morph. of f.

$$\mathrm{RcDBL} \;\; \begin{array}{c} \xleftarrow{\;\mathbb{S}\mathrm{q}\;} \\ \top \\ \xrightarrow{\;\mathrm{obj}\;} \\ \top \\ \xleftarrow[\;\mathbb{H}\;]{} \end{array} \;\; \mathrm{CAT}$$

The unit has components $U : \mathbb{D} \longrightarrow \mathbb{S}\mathrm{q}(\mathbb{D}_0)$:

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ f \downarrow & \alpha & \downarrow g \\ B & \xrightarrow[k]{} & D \end{array} \quad \longmapsto \quad \begin{array}{ccc} A & \xrightarrow{h} & C \\ Uf \downarrow & & \downarrow Ug \\ B & \xrightarrow[k]{} & D \end{array}$$

Thm: $U_1 : \mathbb{D}_1 \longrightarrow \mathrm{Sq}(\mathbb{D}_0)$ is monadic

$\iff$ $\mathbb{D} \cong R\text{-}\mathbb{A}\mathrm{lg}$ for an AWFS (L,R) on $\mathbb{D}_0$.

RIGHT-CONNECTED COMPLETION

$$\mathrm{RcDBL} \;\underset{\hookrightarrow}{\overset{\mathbb{\Gamma}}{\underset{\top}{\longleftarrow}}}\; \mathrm{DBL}$$

The right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ has:

- same objects & horizontal morphisms as $\mathbb{D}$;
- vertical morphisms are cells in $\mathbb{D}$ of shape:

$$\begin{array}{c} A \\ {\scriptstyle (f,f',\alpha)}\downarrow \\ B \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{f'} & B \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle \mathrm{id}_B} \\ B & \xrightarrow[1_B]{} & B \end{array}$$

- cells $(f,f',\alpha) \longrightarrow (g,g',\beta)$ are cells θ in $\mathbb{D}$:

$$\begin{array}{ccccc} \bullet & \xrightarrow{h} & \bullet & \xrightarrow{g'} & \bullet \\ {\scriptstyle f}\downarrow & \theta & \downarrow{\scriptstyle g} & \beta & \downarrow{\scriptstyle \mathrm{id}} \\ \bullet & \xrightarrow[k]{} & \bullet & \xrightarrow[1]{} & \bullet \end{array} \;=\; \begin{array}{ccccc} \bullet & \xrightarrow{f'} & \bullet & \xrightarrow{k} & \bullet \\ {\scriptstyle f}\downarrow & \alpha & \downarrow & \mathrm{id}_k & \downarrow{\scriptstyle \mathrm{id}} \\ \bullet & \xrightarrow[1]{} & \bullet & \xrightarrow[k]{} & \bullet \end{array}$$

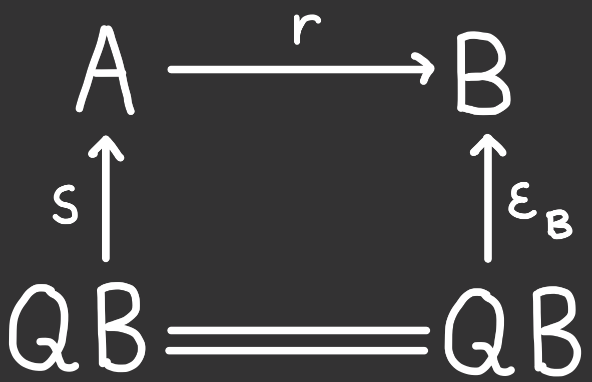
Composition of vertical morphisms is:

$$\begin{array}{c} \bullet \\ {\scriptstyle (f,f',\alpha)}\downarrow \\ \bullet \\ {\scriptstyle (g,g',\beta)}\downarrow \\ \bullet \end{array} \qquad\qquad \begin{array}{ccccc} \bullet & \xrightarrow{f'} & \bullet & \xrightarrow{g'} & \bullet \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ \bullet & \xrightarrow[1]{} & \bullet & \xrightarrow{g'} & \bullet \\ {\scriptstyle g}\downarrow & & {\scriptstyle g}\downarrow & \beta & \downarrow{\scriptstyle \mathrm{id}} \\ \bullet & \xrightarrow[1]{} & \bullet & \xrightarrow[1]{} & \bullet \end{array}$$

The counit has components $V : \mathbb{\Gamma}(\mathbb{D}) \longrightarrow \mathbb{D}$ with assignment $(f,f',\alpha) \mapsto f$.

EXAMPLES OF THE RIGHT-CONNECTED COMPLETION

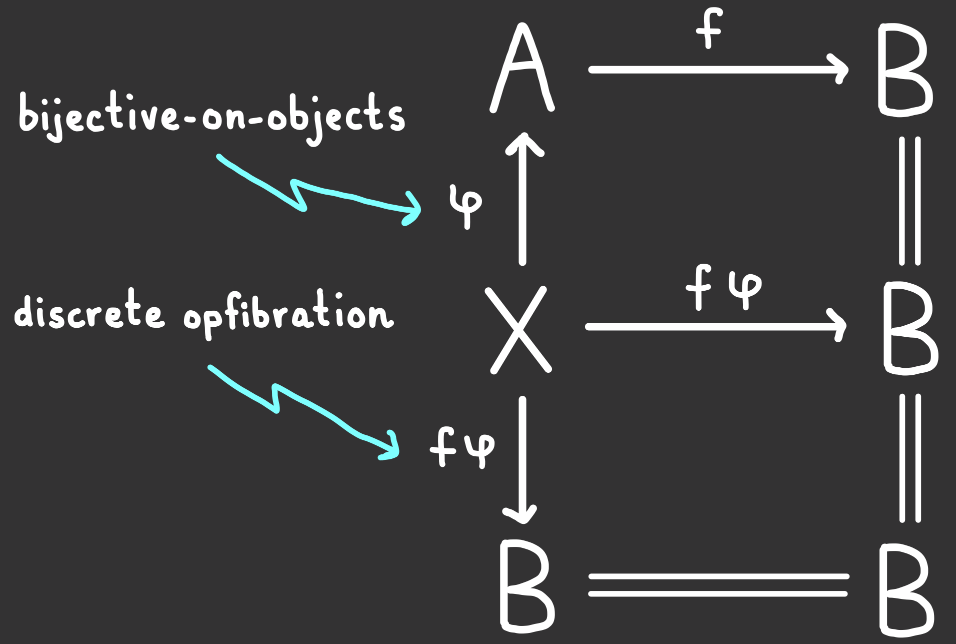
$\mathbb{\Gamma}(\mathbb{K}\ell(\mathcal{C},Q)^{\vee}) \cong \mathbb{S}\mathrm{pEpi}(\mathcal{C},Q)$



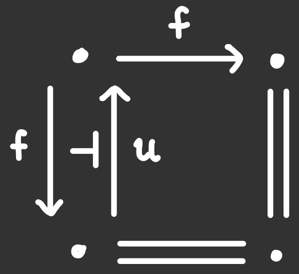
vertical morphism is a Q-split epi

Idea: Vertical morphisms in $\mathbb{\Gamma}(\mathbb{D}^{\vee})$ are generalised split epimorphisms.

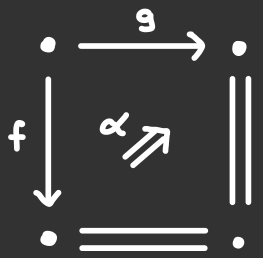
$\mathbb{\Gamma}(\mathbb{R}\mathrm{et}) \cong \mathbb{L}\mathrm{ens}$



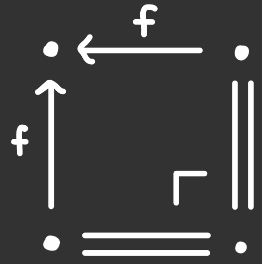
vertical morphism is a delta lens ⇝ generalised split opfibration



$\mathbb{\Gamma}(\mathbb{A}\mathrm{dj}(\mathcal{K}))$ ⇝ lali in $\mathcal{K}$

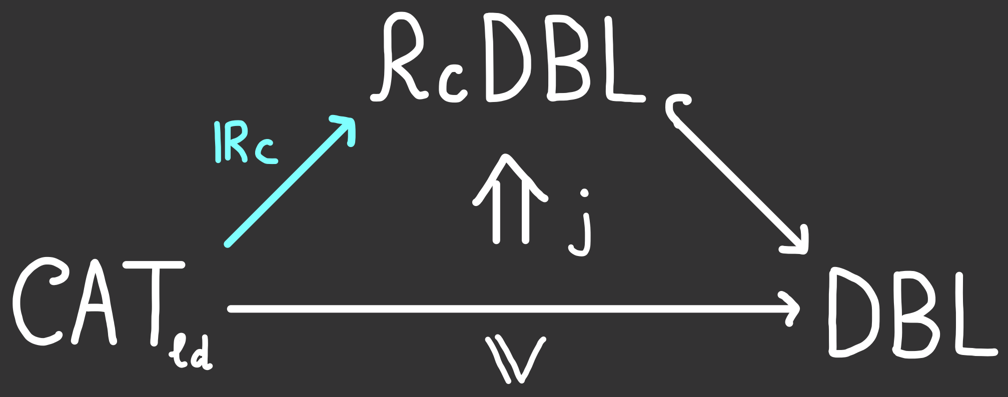


$\mathbb{\Gamma}(\mathbb{Q}(\mathcal{K}))$ ⇝ 2-cell in $\mathcal{K}$

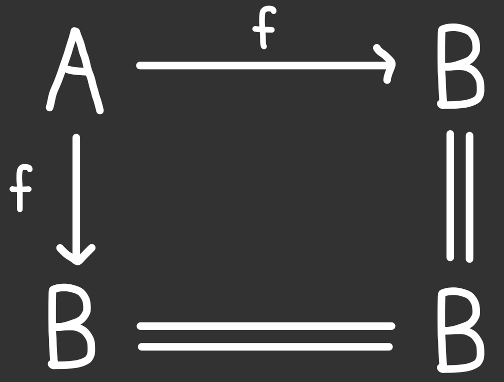


$\mathbb{\Gamma}(\mathbb{P}\mathrm{bSq}(\mathcal{C})^{vh})$ ⇝ mono in $\mathcal{C}$

APPROACH USING THE NERVE



The right-connected double category $\mathbb{R}c(\mathcal{C})$ is the restriction of $\mathbb{S}q(\mathcal{C})$ to cells of shape:



The unit has components $j_{\mathcal{C}} : \mathbb{V}(\mathcal{C}) \longrightarrow \mathbb{R}c(\mathcal{C})$.

We have a relative 2-adjunction:

$$\mathrm{RcDBL}(\mathbb{R}c(\mathcal{C}), \mathbb{D}) \cong \mathrm{DBL}(\mathbb{V}(\mathcal{C}), \mathbb{D})$$

The nerve of a r.c. double category

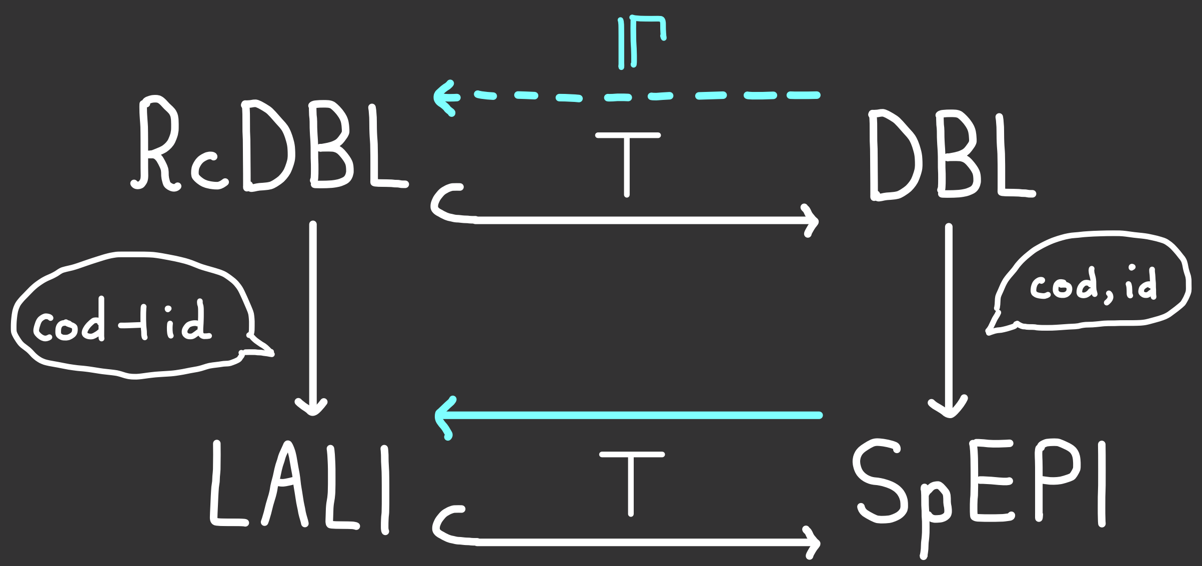
$$\mathrm{RcDBL} \xhookrightarrow{N} [\Delta^{op}, \mathrm{CAT}]$$

is 2-functor $N_{\mathbb{D}} \cong \mathrm{RcDBL}(\mathbb{R}c(-), \mathbb{D})$.

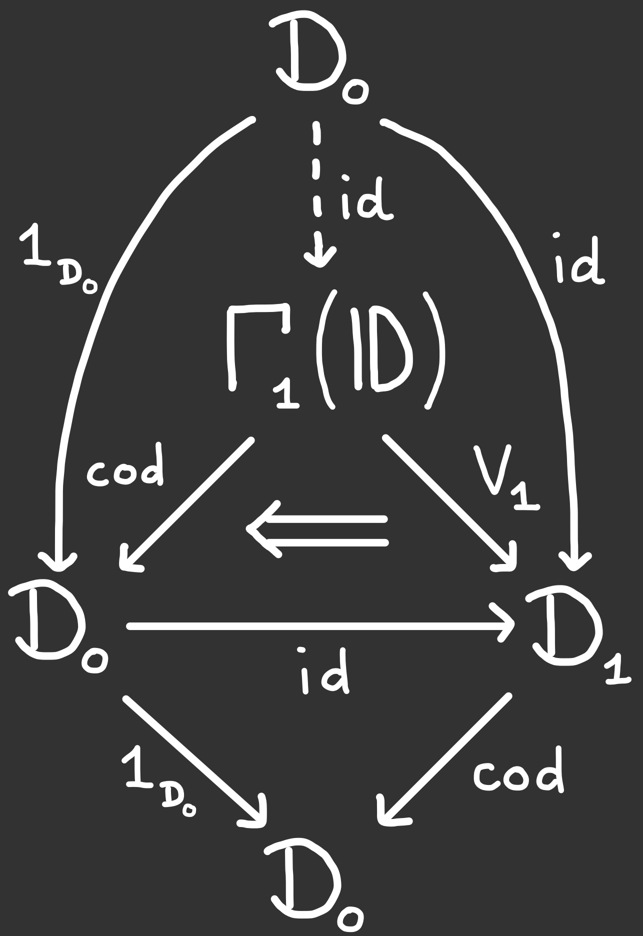
The right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ is determined by its nerve:

$$N_{\mathbb{\Gamma}(\mathbb{D})} \cong \mathrm{DBL}(\mathbb{R}c(-), \mathbb{D}) : \Delta^{op} \longrightarrow \mathrm{CAT}$$

APPROACH USING COMMA OBJECTS

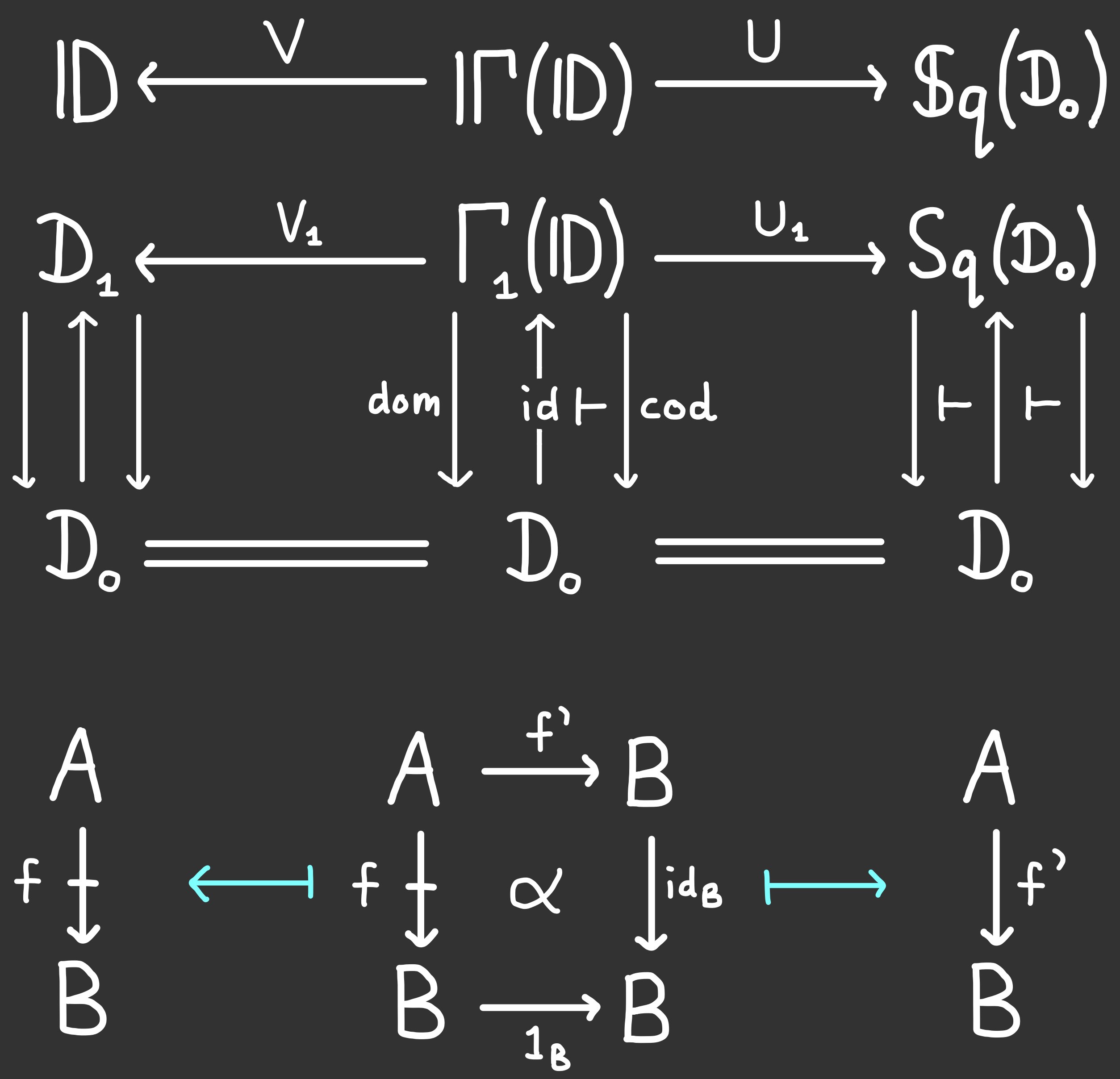


The right-connected completion ℾ(𝔻) arises by constructing the cofree lali on the section-retraction pair (id, cod) using comma objects in CAT/D_0.



This approach generalises to internal categories 𝔻 in any suitable 2-category.

(CO)MONADICITY CONDITION



Thm: $V_1 : \Gamma_1(\mathbb{D}) \longrightarrow \mathcal{D}_1$ is comonadic $\iff$ each fibre $\mathrm{cod}^{-1}\{B\}$ of the functor $\mathrm{cod} : \mathcal{D}_1 \longrightarrow \mathcal{D}_0$ admits products with the vertical identity morphism $\mathrm{id}_B : B \to B$.

Suppose that:

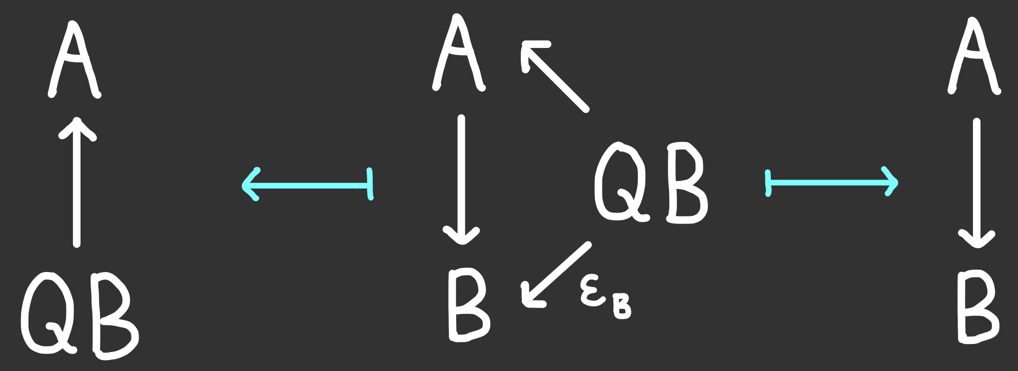
- $\mathrm{dom} : \mathcal{D}_1 \longrightarrow \mathcal{D}_0$ has a LARI,
- $\mathrm{cod} : \mathcal{D}_1 \longrightarrow \mathcal{D}_0$ is an opfibration

Then $U_1 : \Gamma_1(\mathbb{D}) \longrightarrow \mathrm{Sq}(\mathcal{D}_0)$ has left adjoint.

Open question: when is U_1 monadic?

EXAMPLES WHERE (CO)MONADICITY HOLDS

$$\mathbb{K}\ell(\mathcal{C}, Q)^{\vee} \xleftarrow{V} \mathbb{S}\mathrm{pEpi}(\mathcal{C}, Q) \xrightarrow{U} \mathbb{S}\mathrm{q}(\mathcal{C})$$

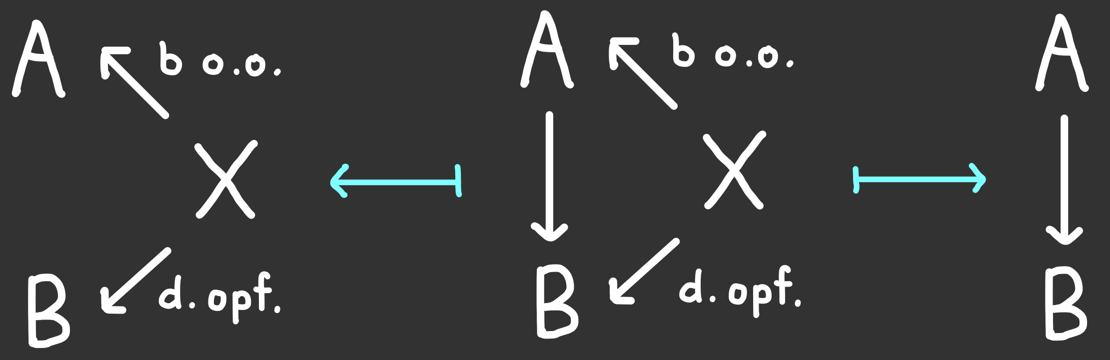


- $\mathcal{C}$ has products $\Rightarrow$ V_1 is comonadic
- $\mathcal{C}$ has coproducts $\Rightarrow$ U_1 is monadic

Other examples giving AWFS include:

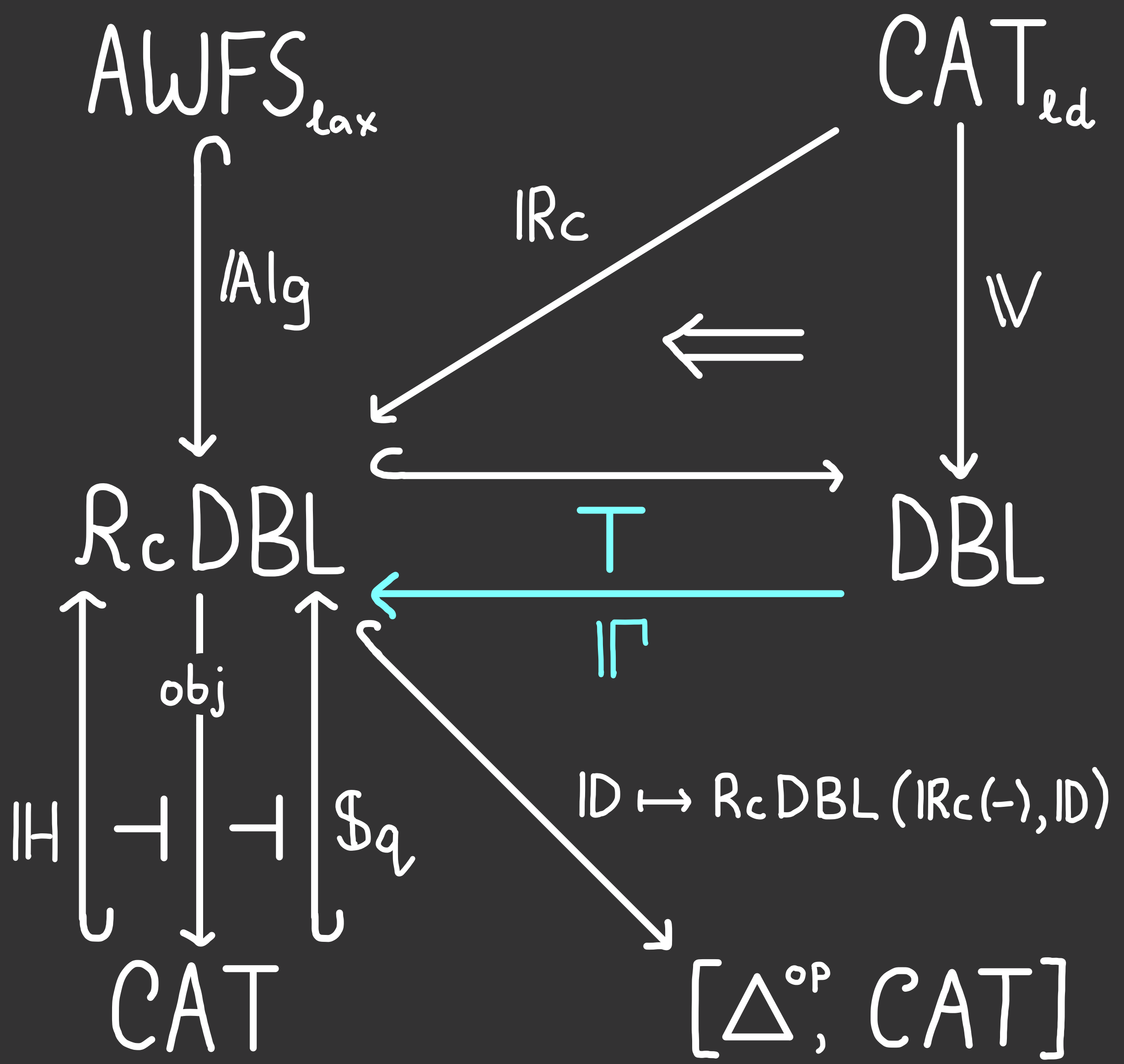
$\mathbb{\Gamma}(\mathbb{A}\mathrm{dj}(\mathcal{K}))$ $\qquad$ $\mathbb{\Gamma}(\mathbb{P}\mathrm{bSq}(\mathcal{C})^{vh})^{vh}$

$$\mathbb{R}\mathrm{et} \xleftarrow{V} \mathbb{L}\mathrm{ens} \xrightarrow{U} \mathbb{S}\mathrm{q}(\mathcal{C}\mathrm{at})$$



- V_1 is comonadic ⇝ lenses are retrofunctors with coalgebraic structure.
- U_1 is monadic ⇝ lenses are the R-algebras for an AWFS on $\mathcal{C}\mathrm{at}$.

SUMMARY & FUTURE WORK



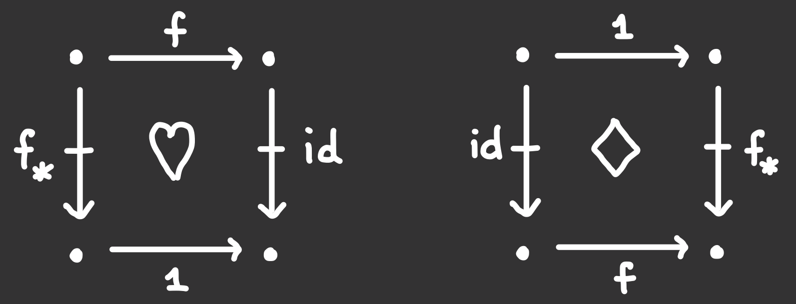
- Constructed the right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ of a double cat $\mathbb{D}$.
- In several examples, this gives an AWFS:

$$\mathbb{\Gamma}(\mathbb{R}\mathrm{et}) \cong \mathbb{L}\mathrm{ens}$$

- Can we extend $\mathbb{R}\mathrm{c}$ to a left 2-adjoint?
- When is $U_1 : \Gamma_1(\mathbb{D}) \longrightarrow Sq(\mathcal{D}_0)$ monadic?
- Is there a right 2-adjoint of $\mathsf{AWFS}_{lax} \hookrightarrow \mathsf{DBL}$?

BONUS: WHAT ABOUT COMPANIONS?

A double category has companions if for each horizontal morphism $f : A \to B$ there is a vertical morphism $f_* : A \;⇸\; B$ and cells

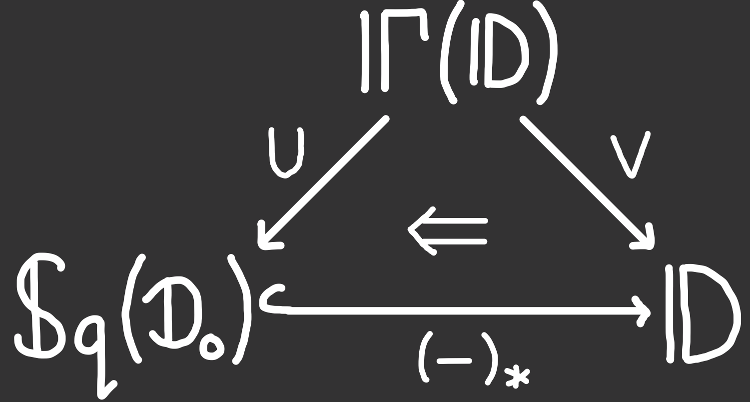


+ axioms

If $\mathbb{D}$ is right-connected, then:

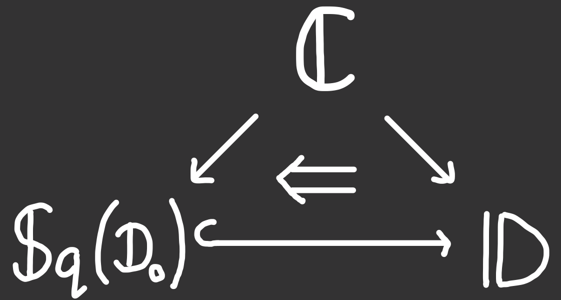
$$\mathbb{S}q(\mathbb{D}_0) \underset{(-)_*}{\overset{U}{\rightleftarrows}} \mathbb{D} \qquad (U \dashv (-)_*)$$

For $\mathbb{D}$ with companions we have



the universal colax globular cone over $(-)_*$.

$$\mathbb{C} \longrightarrow \mathbb{I}\Gamma(\mathbb{D}) \quad \text{lax double functor}$$



globular transformation