LIFTING & LENSES

BRYCE CLARKE

Inria Saclay, Palaiseau

bryceclarke.github.io

LHC days 2023

IRIF, Université Paris Cité

OVERVIEW & MOTIVATION

algebraic weak factorisation systems

GENERALISE

orthogonal factorisation systems

Informally, an AWFS on $\mathcal{C}$ consists of:

- compatible comonad L and monad R on $\mathcal{C}^{\mathbf{2}}$
- functorial factorisation of each $f \in \mathcal{C}$

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "Lf"'] & & B \\ & Ef \arrow[ur, "Rf"'] & \end{tikzcd}$$

cofree L-coalgebra

free R-algebra

- lifts of L-coalgebras against R-algebras

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] \arrow[ur, "\text{lift}"] & D \end{tikzcd}$$

L-coalgebra

R-algebra

This talk: 2 related examples of AWFS

L-coalgebra	R-algebra
lari	split opfibration
???	delta lens

SPLIT OPFIBRATIONS

A split opfibration is a functor equipped with a lifting operation (splitting)

$$
\begin{tikzcd}
A \arrow[d, "f"'] & a \arrow[r, "\varphi(a,u)"] & a' \\
B & fa \arrow[r, "u"] & b
\end{tikzcd}
$$

such that:

1. $f\varphi(a,u) = u$
2. $\varphi(a, 1_{fa}) = 1_a$
3. $\varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a,u)$
4. Each lift $\varphi(a,u)$ is opcartesian.

$$
\begin{tikzcd}
a \arrow[rr, "\varphi(a,u)"] \arrow[dr, "w"'] & & a' \arrow[dl, dashed, "\exists!"] \\
& a'' &
\end{tikzcd}
$$

$$\downarrow f$$

$$
\begin{tikzcd}
fa \arrow[rr, "u"] \arrow[dr, "fw"'] & & b \arrow[dl, "v"] \\
& fa'' &
\end{tikzcd}
$$

FREE SPLIT OPFIBRATIONS

The free split opfibration over B on a functor $f: A \longrightarrow B$ is given by:

$$\boxed{\begin{tikzcd} a_1 \arrow[r, "w"] & a_2 \\ fa_1 \arrow[r, "fw"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\circlearrowright"] & fa_2 \arrow[d, "u_2"] \\ b_1 \arrow[r, "v"'] & b_2 \end{tikzcd}}$$

$$\downarrow$$

$$\boxed{\begin{tikzcd} b_1 \arrow[r, "v"'] & b_2 \end{tikzcd}}$$

comma category $f/1_B$

The chosen opcartesian lifts are:

$$\begin{tikzcd} a_1 \arrow[r, equal] \arrow[rr, bend left, "w"] & a_1 \arrow[r, dashed, "w"] & a_2 \\ fa_1 \arrow[r, equal] \arrow[rr, bend left, "fw"] \arrow[d, "u_1"'] & fa_1 \arrow[r, dashed, "fw"] \arrow[d, "v \circ u_1"] & fa_2 \arrow[d, "u_2"] \\ b_1 \arrow[r, "v"'] & b_2 \arrow[r, "x"'] & b_3 \end{tikzcd}$$

$$\begin{tikzcd} b_1 \arrow[r, "v"'] & b_2 \arrow[r, "x"'] & b_3 \end{tikzcd}$$

THE AWFS FOR SPLIT OPFIBRATIONS

Factorisation

Every functor factorises into a (cofree) left-adjoint-right inverse followed by a (free) split opfibration.

$$\begin{tikzcd}
 & A \arrow[dl, bend right, "1_A"'] \arrow[d, dashed] \arrow[dr, bend left, "f"] & \\
 & f/1_B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] \arrow[d, phantom, "\Rightarrow"] & \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}$$

Lifting

Given a commutative square in Cat

lari

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "k"'] & D
\end{tikzcd}$$

split opfibration

there is a canonical functor $j : B \longrightarrow C$ such that $jf = h$ and $gj = k$.

E.g. Take $f : \{0 \to 2\} \longrightarrow \{0 \to 1 \to 2\}$.

DELTA LENSES

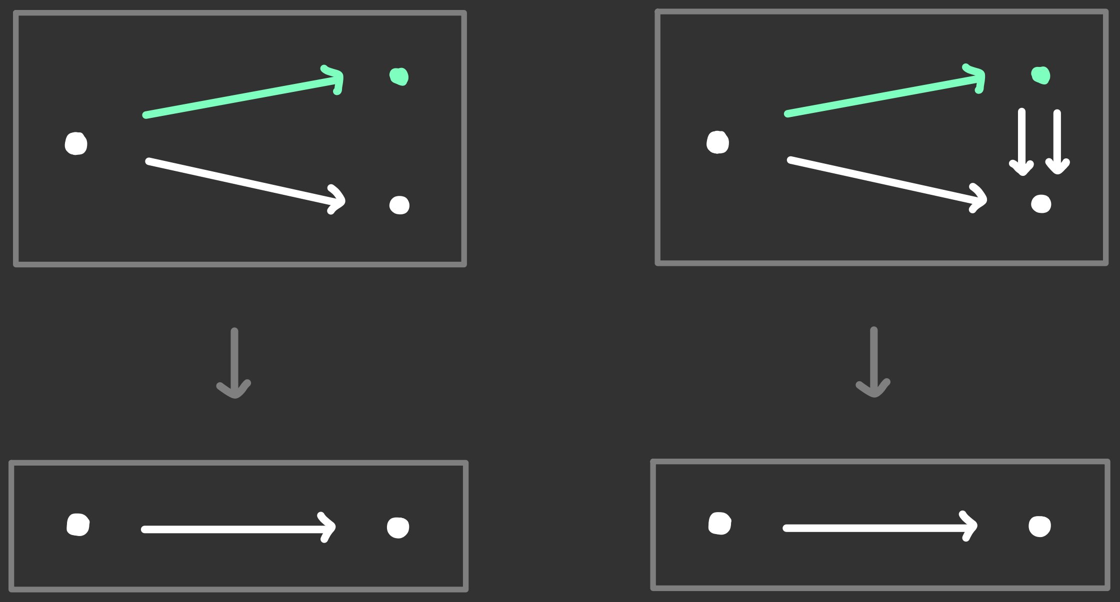
A delta lens is a functor equipped with a lifting operation

$$\begin{tikzcd} A \arrow[d, "f"'] & a \arrow[r, "\varphi(a,u)"] & a' \\ B & fa \arrow[r, "u"] & b \end{tikzcd}$$

such that:

1. $f\varphi(a,u) = u$
2. $\varphi(a, 1_{fa}) = 1_a$
3. $\varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a,u)$

Two simple examples of delta lenses which are <u>not</u> split opfibrations.



Q: What is the free delta lens?

FREE DELTA LENSES (1)

The free delta lens $Rf : Ef \longrightarrow B$ on a functor $f : A \longrightarrow B$ has domain whose:

- objects are pairs $(a \in A,\ u : fa \longrightarrow b \in B)$
- morphisms are generated by the following:

$$a = a$$

chosen lifts

$$\begin{tikzcd} fa \arrow[r, equal] \arrow[d, "u_1"'] \arrow[dr, phantom, "\circlearrowleft"] & fa \arrow[d, "u_2"] \\ b_1 \arrow[r, "v"'] & b_2 \end{tikzcd}$$

$$a_1 \xrightarrow{w} a_2$$

morphisms of A

$$\begin{tikzcd} fa_1 \arrow[r, "fw"] \arrow[d, "1"'] & fa_2 \arrow[d, "1"] \\ fa_1 \arrow[r, "fw"'] & fa_2 \end{tikzcd}$$

The functor Rf sends these generators to $v : b_1 \longrightarrow b_2$ and $fw : fa_1 \longrightarrow fa_2$, respectively.

FREE DELTA LENSES (2)

The free delta lens $Rf : Ef \longrightarrow B$ on a functor $f : A \longrightarrow B$ has domain whose:

- objects are pairs $(a \in A,\ u : fa \longrightarrow b \in B)$
- morphisms $(a_1, u_1) \longrightarrow (a_2, u_2)$ are given by the following two sorts:

$$
\begin{tikzcd}
a_1 \arrow[r, equal] & a_2 \\
fa_1 \arrow[r, equal] \arrow[d, "u_1"'] \arrow[dr, phantom, "\circlearrowright"] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[r, "v"'] & b_2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a_1 \arrow[r, equal] & a_1 \arrow[r, "w"] & a_2 \arrow[r, equal] & a_2 \\
fa_1 \arrow[r, equal] \arrow[d, "u_1"'] \arrow[dr, phantom, "\circlearrowright"] & fa_1 \arrow[r, "fw"] \arrow[d, "1"'] & fa_2 \arrow[r, equal] \arrow[d, "1"] & fa_2 \arrow[d, "u_2"] \\
b_1 \arrow[r, "v"'] & fa_1 \arrow[r, "fw"'] & fa_2 \arrow[r, "u_2"'] & b_2
\end{tikzcd}
$$

The functor Rf sends these to $v : b_1 \longrightarrow b_2$ and $u_2 \circ fw \circ v : fa_1 \longrightarrow fa_2$, respectively.

THE AWFS FOR DELTA LENSES

Factorisation

initial functor

discrete opfibration

$$
\begin{tikzcd}
A_0 \arrow[r, "\eta_A"] \arrow[d] \arrow[dr, phantom, very near end, "\ulcorner"] & A \arrow[d, "Lf"] \arrow[dd, bend left=60, "f"] \\
Jf \arrow[r] \arrow[d] & Ef \arrow[d, dashed, "Rf"] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

where $Jf = \sum_{a \in A_0} fa/B$

Lifting

Given a commutative square in Cat

L-coalgebra

delta lens (R-algebra)

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "k"'] & D
\end{tikzcd}
$$

there is a canonical functor $j : B \longrightarrow C$

such that $jf = h$ and $gj = k$.

Q: What are the L-coalgebras?

COALGEBRAS ARE "STRUCTURED" LARIs

A L-coalgebra is an adjunction

$$\begin{tikzcd} A \arrow[r, hook, shift right, "f"'] \arrow[r, phantom, "\top"] & B \arrow[l, shift right, "q"'] \end{tikzcd} \qquad q \circ f = id_A$$

such that if $q(u : b_1 \to b_2) \neq 1$, there is a specified $\bar{q}u : b_1 \to fqb_1$ such that:

$$\boxed{\begin{aligned} \bar{q}u \circ \varepsilon_{b_1} &= 1 \\ \varepsilon_{b_2} \circ fqu \circ \bar{q}u &= u \end{aligned}}$$

$$\begin{tikzcd} fqb_1 \arrow[r, "fqu"] \arrow[d, shift right, "\varepsilon_{b_1}"'] & fqb_2 \arrow[d, "\varepsilon_{b_2}"] \\ b_1 \arrow[r, "u"'] \arrow[u, shift right, "\bar{q}u"'] & b_2 \end{tikzcd}$$

The cofree L-coalgebra on $f : A \to B$ is

$$\begin{tikzcd} a & (a, u) \arrow[l, mapsto] \\ A \arrow[r, hook, shift right, "Lf"'] \arrow[r, phantom, "\top"] & Ef \arrow[l, shift right] \\ a \arrow[r, mapsto] & (a, 1_{fa}) \end{tikzcd}$$

with counit:

$$\begin{tikzcd} a \arrow[r, equal] & a \\ fa \arrow[r, equal] \arrow[d, "1_{fa}"'] & fa \arrow[d, "u"] \\ fa \arrow[r, "u"] & b \end{tikzcd}$$

HOW TO LIFT AGAINST DELTA LENSES (1)

$$\begin{tikzcd} qx \arrow[r, "qu"] & qy \end{tikzcd}$$

$$\begin{tikzcd} hqx \arrow[r, "hqu"] \arrow[d, shift right, "\varphi(hqx,k\varepsilon_x)"'] & hqy \arrow[d, "\varphi(hqy,k\varepsilon_y)"] \\ jx \arrow[r, "ju"'] \arrow[u, shift right, "\varphi(jx,k\bar{q}u)"'] & jy \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "(f,q)"'] & C \arrow[d, "(g,\varphi)"] \\ B \arrow[r, "k"'] \arrow[ur, "j"'] & D \end{tikzcd}$$

$$\begin{tikzcd} fqx \arrow[r, "fqu"] \arrow[d, shift right, "\varepsilon_x"'] & fqy \arrow[d, "\varepsilon_y"] \\ x \arrow[r, "u"] \arrow[u, shift right, "\bar{q}u"'] & y \end{tikzcd}$$

$$\begin{tikzcd} g(hqx) \arrow[d, equal] & g(hqy) \arrow[d, equal] \\ kfqx \arrow[d, shift right, "k\varepsilon_x"'] & kfqy \arrow[d, "k\varepsilon_y"] \\ kx \arrow[u, shift right, "k\bar{q}u"'] & ky \end{tikzcd}$$

HOW TO LIFT AGAINST DELTA LENSES (2)

$$\begin{tikzcd} qx \arrow[r, "qu=1"] & qy \end{tikzcd}$$

$$\begin{tikzcd} hqx \arrow[r, equal] \arrow[d, "{\varphi(hqx, k\varepsilon_x)}"'] & hqy \arrow[d, "{\varphi(hqy, k\varepsilon_y)}"] \\ jx \arrow[r, "{\varphi(jx, ku)}"'] & jy \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{(f,q)}"'] & C \arrow[d, "{(g,\varphi)}"] \\ B \arrow[r, "k"'] \arrow[ur, "j"'] & D \end{tikzcd}$$

$$\begin{tikzcd} fqx \arrow[r, equal] \arrow[d, "\varepsilon_x"'] & fqy \arrow[d, "\varepsilon_y"] \\ x \arrow[r, "u"] & y \end{tikzcd}$$

$$\begin{tikzcd} g(hqx) \arrow[d, equal] & g(hqy) \arrow[d, equal] \\ kfqx \arrow[r, equal] \arrow[d, "k\varepsilon_x"'] & kfqy \arrow[d, "k\varepsilon_y"] \\ kx \arrow[r, "ku"] & ky \end{tikzcd}$$

SUMMARY & FURTHER WORK

- We examined two examples of AWFS on Cat whose R-algebras were split opfibrations & delta lenses.
- We constructed explicitly the free delta lens on a functor.
- We characterised the coalgebras that delta lenses lift against as LARIs with extra structure.

- The AWFS for delta lenses generalises to any "nice" category with OFS and idempotent comonad.
- Can assemble a double category of categories, functors, and delta lenses.

Check out the preprint:

arXiv:2305.02732